Parallel computation and canonicity

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Lattice expansions

A lattice expansion is a pair of an underlying lattice $\mathbb{L}$ and a set $\{f_1, f_2, \ldots\}$ of ϵ-operations on $\mathbb{L}$.

$$\langle \mathbb{L}, f_1, f_2, \ldots \rangle$$

An ϵ-operation f on $\mathbb{L}$ is a n-any monotone function wrt the order type $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, where each ϵ_i is either 1 or ∂.

$$f : \mathbb{L}^{\epsilon_1} \times \cdots \times \mathbb{L}^{\epsilon_n} \to \mathbb{L}$$

Example

The lattice operations $\vee$ and $\wedge$ are $(1, 1)$-operations.
The involution $\neg$ is a ∂-operation.
The implication $\to$ is a $(\partial, 1)$-operation.

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Lattice expansions in this talk

To get a syntactic description of canonical inequalities, we focus on lattice expansions only with ϵ-additive operations and ϵ-multiplicative operations.

An ϵ-additive operation f is a coordinate-wise join-preserving function wrt the order type ϵ. An ϵ-multiplicative operation g is a coordinate-wise meet-preserving function wrt the order type ϵ.

Example

The lattice operation $\vee$ is $(1, 1)$-additive.
The implication $\rightarrow$ is $(\partial, 1)$-multiplicative, because we have

- $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$, and
- $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$.

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Examples of our lattice expansions

- Boolean algebras
- Modal algebras
- Heyting algebras
- Distributive modal algebras
- FL-algebras
- $B.C_{\Box\Diamond}$-algebras

To avoid a possible complication, we consider a lattice expansion $\mathbf{L} = \langle \mathbb{L}, l, r, c \rangle$ only, where l is $(1,1)$-additive, r is $(\partial, 1)$-multiplicative and c is a constant.

The canonical extension

The canonical extension of $\mathbf{L} = \langle \mathbb{L}, l, r, c \rangle$ is $\overline{\mathbf{L}} = \langle \overline{\mathbb{L}}, l_\uparrow, r^\downarrow, c \rangle$, where

1. $\overline{\mathbb{L}}$ is the canonical extension of $\mathbb{L}$,
2. $l_\uparrow$, a.k.a. l^σ, is approximated from below by filters (closed elements),
3. $r^\downarrow$, a.k.a. r^π, is approximated from above by ideals (open elements),
4. c is the constant.

Approximation...? Let's recall the construction of canonical extensions. (on blackboards)

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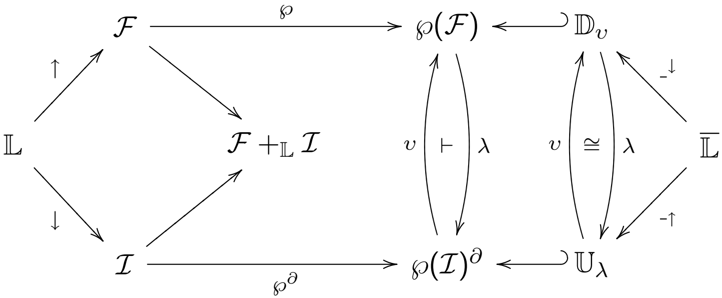
Canonical extensions of lattices



- $\lambda(\mathfrak{F}) := \{I \in \mathcal{I} \mid \forall F \in \mathfrak{F}.\ F \cap I \neq \emptyset\}$ approximated from below
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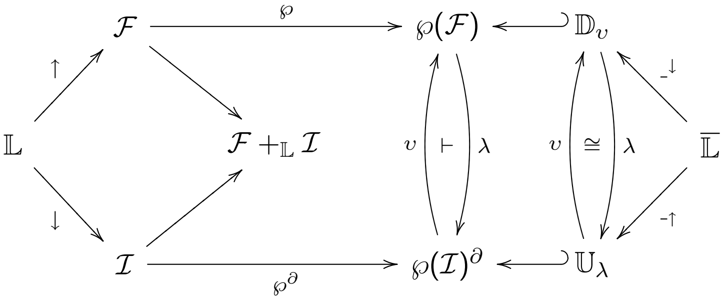

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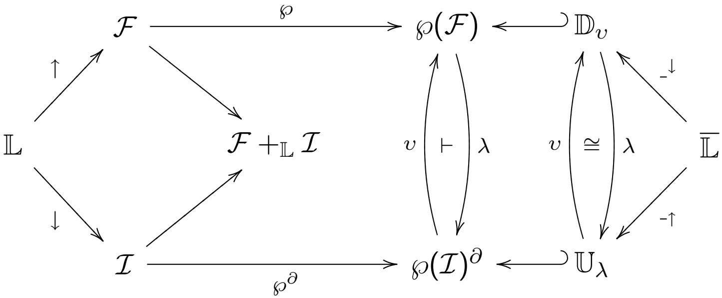

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Canonical extensions of ϵ-operations

We extend l and r as partial functions onto the intermediate level.

1. $l : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$,
   $l(F, G) := \{a \in L \mid f \in F, g \in G.\ l(f, g) \leq a\}$
2. $l : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$, $l(I, J) := \{a \in L \mid i \in I, j \in J.\ a \leq l(i, j)\}$
3. $r : \mathcal{I} \times \mathcal{F} \to \mathcal{F}$, $r(I, F) := \{a \in L \mid i \in I, f \in F.\ r(i, f) \leq a\}$
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We define $l_{\uparrow}$ and $r^{\downarrow}$ as approximations as follows.

1. $l_{\uparrow}(\alpha, \beta) := \lambda(\{l(F, G) \mid F \in \alpha^{\downarrow}, G \in \beta^{\downarrow}\})$
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Canonical inequalities

Definition (Canonical inequality)

Let s, t be terms. An inequality $s \leq t$ is *canonical* on a lattice expansion $\mathbf{L}$, if

$$\mathbf{L} \models s \leq t \iff \overline{\mathbf{L}} \models s \leq t.$$

Theorem

An inequality $s \leq t$ is canonical, if it has consistent variable occurrence.

Consistent variable occurrence...? (on blackboards)

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Consistent variable occurrence

Example

$l(r(x, l(y, z)), l(y, r(x, z)) \leq r(l(z, r(x, y)), r(l(y, x), z))$ has consistent variable occurrence.

Labelling and signing (on blackboards)

$$
\begin{aligned}
t_\cup &::= x \mid c \mid t_\cup \vee t_\cup \mid l(t_\cup, t_\cup) \mid t_\wedge \\
t_\cap &::= x \mid c \mid t_\cap \wedge t_\cap \mid r(t_\cup, t_\cap) \mid t_\vee \\
t_\vee &::= x \mid c \mid t_\vee \vee t_\vee \mid l(t_\vee, c) \mid l(c, t_\vee) \\
t_\wedge &::= x \mid c \mid t_\wedge \wedge t_\wedge \mid r(t_\vee, c) \mid r(c, t_\wedge)
\end{aligned}
$$

Ghilardi & Meloni's parallel computation

Their idea is simple.

Extend term functions on **L** *to the intermediate level.*

But, how?

The intermediate level is two-sorted (filters and ideals).

Their answer is

Let's compute a term function t both as a filter and as an ideal, in parallel.

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Intuitively speaking,

$$t: \quad \mathcal{F} \times \cdots \times \mathcal{F} \rightarrow \mathcal{F}$$
$$t: \quad \mathcal{I} \times \cdots \times \mathcal{I} \rightarrow \mathcal{I}$$

But, this is not really precise...

$$t: \quad (\mathcal{F} \| \mathcal{I}) \times \cdots \times (\mathcal{F} \| \mathcal{I}) \rightarrow \mathcal{F}$$
$$t(F_1 \| I_1, \ldots, F_n \| I_n)$$
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Outcomes of the parallel computation

Theorem (Rough basis)

Let t be each term. For all $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{L}}$, and all $F_i \leq \alpha_i$ and all $I_i \geq \alpha_i$ $(1 \leq i \leq n)$, we have

$$t(F_1 \| I_1, \ldots, F_n \| I_n) \leq t(\alpha_1, \ldots, \alpha_n) \leq t(I_1 \| F_1, \ldots, I_n \| F_n)$$

A very simple example

The inequality $c \leq I(r(r(x, y), x), x)$ is canonical.

Sketch.

For arbitrary $\alpha, \beta \in \overline{\mathbb{L}}$, we want to show

$$c \leq I(r(r(\alpha, \beta), \alpha), \alpha).$$

It suffices to show $c \leq Y$ for any ideal $Y \geq I(r(r(\alpha, \beta), \alpha), \alpha)$. Thanks to the parallel computation, for all $F \leq \alpha$ and $I \geq \beta$,

$$c \leq I(r(r(f, i), f), f) \leq a \leq Y$$

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Concluding remarks

- A connection to Jónsson's work t^σ and t^π.

$$t^\sigma \leq t \leq t^\pi$$