Canonical Extensions of Lattice Extensions: The Case of Finitely Generated Varieties

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The setting:

$$\mathcal{A} = \mathbb{HSP}(K)$$

where K is a FINITE lattice-based algebra (ie a lattice with, perhaps, additional operations).

What can we say about canonical extensions in this case?

Method 1 [Gehrke/Vosmaer]:

Specialize the general theory.

Technique: standard canonical extension methodology.

Method 2 [Davey/Priestley]:

Build in finite generation from the start.

Technique: topological algebra.

ALSO:

Method 3 [Harding, Gouveia, Vosmaer]: profinite completions

Topology

Now you see it . . .

Now you don't.

Topology

Now you see it . . .

Now you don't.

BUT IT'S THERE!

Exploiting finite generation: a black box

$\mathcal{A} = \mathbb{HSP}(K)$ $\quad$ (K finite lattice-based)

$\downarrow$

$$\boxed{\begin{array}{c}\text{Jónsson's Lemma}\\ + \\ \text{Birkhoff's Subdirect Product Theorem}\end{array}}$$

$\downarrow$

$\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ where $\mathcal{M} = \{M_1, \ldots, M_k\}$ is a FINITE set of FINITE lattice-based algebras.

We relate canonical extensions directly to $\mathcal{M}$.

Outline

$$\mathcal{A} = \mathbb{ISP}(\mathcal{M})$$

where $\mathcal{M} = \{M_1, \ldots, M_k\}$ is a FINITE set of FINITE lattice-based algebras.

- We introduce the **natural extension** $n_{\mathcal{A}}(A)$ of $A \in \mathcal{A}$: it is a subalgebra of powers of members of $\mathcal{M}$.
- **Reconciliation**: at the lattice level, $n_{\mathcal{A}}(A)$ is a dense and compact completions of A, so a canonical extension.
- As a topological algebra, $n_{\mathcal{A}}(A)$ is VERY NICE INDEED.
- **Reconciliation**: the operations of $n_{\mathcal{A}}(A)$ coincide with the σ- and π-extensions of those on A.
- **Reconciliation**: with duality approach.

At the lattice level

The **canonical extension** of a lattice L is a dense and compact completion $C = L^\delta$ of L: (Formally it is a pair (e, C) where $e\colon L \hookrightarrow C$, but we normally identify L with $e(L)$.)

Density: L is Δ_1-dense (that is, $\bigvee\bigwedge$- and $\bigwedge\bigvee$-dense) in C;

Compactness: for every filter F and every ideal I of L,

$$\bigwedge F \leqslant \bigvee I \implies F \cap I \neq \varnothing.$$

[equivalent conditions exist]

Classic examples, via duality

- Boolean algebras:

$$L \cong \text{clopen subsets of Boolean space } X \text{ (topologised ultrafilters)}$$

$$\hookrightarrow L^\delta = \wp(X) \qquad \textsf{(complete atomic BA)}$$

B. Jónsson (1993 survey on BAOs)

'This [density + compactness] is an algebraic way of describing the extension which arises from Stone's Duality Theorem.'

- Bounded distributive lattices:

$$L \cong \text{clopen up-sets of Priestley space } X \text{ (topologised prime filters, with } \subseteq)$$

$$\hookrightarrow L^\delta = \mathrm{Up}(X) \qquad \textsf{(complete dl satisfying \ldots)}$$

BUT this won't work for the variety of (bounded) lattices.

Canonical extensions do exist for bounded lattices

Method 1 [Gehrke/Harding 2001]: as Galois-closed sets for the Galois connection coming from a polarity.

Method 2 [Gehrke/Priestley 2008]: by combining free join- and meet-completions and then taking MacNeille completion.

Uniqueness: Any concretely constructed dense and compact completion serves as the canonical extension.

Finitely generated varieties of lattice-based algebras

A few examples:

- De Morgan algebras
- Any variety of MV-algebras generated by a finite chain
- Any variety generated by a finite subdirectly irreducible Heyting algebra, $L \oplus \mathbf{1}$, where L is a finite lattice.

  In particular; $\mathbb{HSP}(\mathbf{4}) = \mathbb{ISP}(\{\mathbf{4}, \mathbf{3}\})$
- $\dots$

The natural extension

Topological conventions

Any finite algebra, when topologised, carries the discrete topology.
Any product of finite algebras then carries the product topology.

Given: a quasivariety $\mathcal{A} = \mathbb{ISP}(\{M_1, \ldots, M_k\})$.
Here the M_i are finite, but need not be lattice-based for now.

Let $A \in \mathcal{A}$. **Define**

$$Z_i := \mathcal{A}(A, M_i) \quad \text{(homomorphisms from } A \text{ into } M_i\text{)}$$

Then Z_i is a closed subspace of M_i^A, hence compact.

We can embed A homomorphically into $M_1^{Z_1} \times \cdots \times M_k^{Z_k}$ by **multisorted evaluation**:

$$e_A \colon A \to \prod_{1 \leqslant i \leqslant k} M_i^{Z_i} \quad \text{where } e_A(a)(i)(x) = x(a), \\ \text{for } i \in \{1, \ldots, k\} \text{ and } x \in Z_i = \mathcal{A}(A, M_i).$$

(In fact $e_A(A) \subseteq \prod_{1 \leqslant i \leqslant k} \mathcal{C}(Z_i, M_i)$.)

A key definition:

The **natural extension** of A is

$$n_{\mathcal{A}}(A) := \overline{e_A(A)} \text{ in } \prod M_i^{\mathcal{A}(A,M_i)},$$

where $\overline{e_A(A)}$ denotes the **topological closure**. SO e_A embeds A as a topologically dense subalgebra of its natural extension.

Theorem: *Let $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ where $\mathcal{M}$ is a finite set of finite lattice-based algebras. $A \in \mathcal{A}$. Then, at the lattice level, $n_{\mathcal{A}}(A)$ is a dense and compact completion, and so a canonical extension.*

Contextualising, I

A wealth of dense and compact completions

Let $\{N_s\}_{s\in S}$ be a non-empty family of finite lattices.

Closed and complete sublattices

Let C be a sublattice of $\prod_{s\in S} N_s$. Then C is topologically closed iff C is a complete sublattice ($\equiv$ closed under arbitrary non-empty joins and meets).

Dense completions

Let L be a sublattice of $\prod_{s\in S} N_s$. Then $\overline{L}$ is a Δ_1-completion of L.

Compact completions

Let $Z_1,\dots,Z_k$ be compact spaces, let $M_1,\dots,M_k$ be finite lattices. Let L be a sublattice of $\mathrm{C}(Z_1,M_1)\times\cdots\times\mathrm{C}(Z_k,M_k)$. Then $\overline{L}$ (taken in $M_1^{Z_1}\times\cdots\times M_k^{Z_k}$) is a compact completion of L.

More about the natural extension

[illustration for special case $\mathbb{ISP}(M)$]

Extending the operations Let f be a basic operation (unary for simplicity). Then f lifts **pointwise** from M to $\widehat{f}$ on $n_{\mathcal{A}}(A)$ so as to extend its action on $A \cong e_A(A)$:

$$(\widehat{f}(\varphi))(x) := \varphi(f \circ x) \quad \text{for } x \in \mathcal{A}(A, M) \text{ and } \varphi \in n_{\mathcal{A}}(A) \subseteq M^{\mathcal{A}(A,M)}$$

Functoriality: follow your nose!

We get a functor from $\mathcal{A}$ to $\mathcal{A}_{\mathfrak{T}} := \mathbb{IS}_{\mathrm{c}}\mathbb{P}(M)$, the class of isomorphic copies of topologically closed substructures of powers of M (empty structure included).

For $u \in \mathcal{A}(A, B)$, define $n_{\mathcal{A}}(u)\colon n_{\mathcal{A}}(A) \to n_{\mathcal{A}}(B)$ by

$$n_{\mathcal{A}}(u) := \widehat{u}{\restriction}n_{\mathcal{A}}(A) \quad \text{where } (\widehat{u}(\varphi))(y) := \varphi(y \circ u),$$

$$\text{for } \varphi \in M^{\mathcal{A}}(A, M) \text{ and } y \in \mathcal{A}(B, M).$$

Summarising

Propaganda for the natural extension:

- In the lattice-based case, $n_{\mathcal{A}}(A)$ is not just a dense and compact lattice completion, it is an algebra in $\mathcal{A}$.
- All operations in $e_{\mathcal{A}}(A) \cong A$ and $n_{\mathcal{A}}(A)$ are obtained simply by pointwise lifting from those of $\mathcal{M}$.
- The natural extension construction is functorial in a natural way.
- $n_{\mathcal{A}}(A)$ is a **topological algebra**—more on this shortly.

Structure of the natural extension $C := n_{\mathcal{A}}(A)$

In the **lattice-based case**

- C comes equipped with the subspace topology from the product it sits in. But, **better**, the topology on $C = n_{\mathcal{A}}(A)$ is in fact its **interval topology**, ι_C, having closed subbase the sets $\uparrow x$ and $\downarrow x$ ($x \in C$).
- C is bi-algebraic (that is, algebraic and dually algebraic). Hence $M^\infty(C)$ is meet-dense on C and $J^\infty(C)$ is join-dense in C—as expected of a canonical extension.
- C is a Priestley space.

Contextualising, II

Topological algebra meets domain theory:— Let C be a bi-algebraic lattice. Then TFAE

(1) C is a Priestley space wrt interval topology ι_C;

(2) C is Hausdorff wrt ι_C;

(3) Lawson, dual Lawson and interval topologies on C are all equal (C is **linked bi-algebraic**);

Let C be a complete lattice. Then TFAE:

(a) C is bi-algebraic and satisfies (1)–(3);

(b) C is a Boolean topological lattice wrt ι_C;

(c) C is a Boolean topological lattice wrt some topology.

An observation Convergence wrt Lawson topology is convergence wrt liminf topology, and dually. This can be used to show that topological density on a linked bi-algebraic lattice equals Δ_1-density.

What happened to f^σ and f^π?

For a lattice-based algebra A, denote by L_A the underlying lattice.

The traditional way to lift a basic operation f on A from L_A to L_A^δ is to consider

$$
\begin{aligned}
f^\sigma(x) := \bigvee\Big\{ \bigwedge\{\, f(a) \mid a \in A \text{ and } p \leqslant a \leqslant q \,\} \mid \\
p \in K(C), q \in O(C) \text{ and } p \leqslant x \leqslant q \Big\}, \\
f^\pi(x) := \bigwedge\Big\{ \bigvee\{\, f(a) \mid a \in A \text{ and } p \leqslant a \leqslant q \,\} \mid \\
p \in K(C), q \in O(C) \text{ and } p \leqslant x \leqslant q \Big\},
\end{aligned}
$$

$p \in K(C)$ iff p is a meet of elements from A (closed (filter) elements),

$q \in O(C)$ iff q is a join of elements from A (open (ideal) elements).

Our strategy:—

$n_{\mathcal{A}}(A)$ comes equipped with an interval-continuous extension $\widehat{f}$ of f on $\mathcal{A}$. We can prove $\widehat{f}$ coincides with f^σ and with f^π, so

- f is **smooth**;
- $\mathcal{A}$ is **canonical** using the standard extensions of operations; and
- $f^\sigma = f^\pi$ is **interval-continuous**.

For this:

- We use ONLY interval-continuity of $\widehat{f}$, simple properties of $n_{\mathcal{A}}(A)$, of canonical extensions and of f^σ and f^π.
- We do NOT need the δ-topology introduced by Gehrke/Jónsson (distributive case) and extended to lattice case by Vosmaer.
- We do NOT need to establish directly that f^σ (or f^π) is interval-continuous.

Structure of canonical extensions in the finitely generated case

Take $C = n_{\mathcal{A}}(A)$ as before. Let $x \nleqslant y$ in C. Then there exist $j \in J^{\infty}(C)$ and a **finite** set $\mathcal{M}_j \subseteq M^{\infty}(C)$ such that

- $j \leqslant x$ and $j \nleqslant y$,
- $C \backslash {\uparrow} j = {\downarrow} \mathcal{M}_j$.

and dually.

Contextualising, III

- **The distributive case**: $|\mathcal{M}_j| = 1$. Completely join-irreducible elements = completely join-prime elements and the canonical extension is completely distributive and linked bi-algebraic.
- **The non-finitely generated case**: we still have a set $\mathcal{M}_j$ of elements of $M^{\infty}(C)$ such that $C \backslash {\uparrow} j = {\downarrow} \mathcal{M}_j$ but $\mathcal{M}_j$ is not generally finite. Canonical extension need not be bi-algebraic (meet-continuity can fail—example due to Mai Gehrke).

Contextualising, IV

Profinite completions

Let $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ where now $\mathcal{M}$ is **any** set of finite algebras (of same type) —an **internally residually finite prevariety**.

Consider the family of congruences

$$\mathcal{S}_A := \{\, \alpha \in \mathrm{Con}(A) \mid A/\alpha \in \mathcal{A} \text{ and } A/\alpha \text{ is finite}\,\}.$$

Then $\mathcal{S}_A$ is directed wrt $\supseteq$. **Define**

$$\mathrm{pro}_{\mathcal{A}}(A) := \left\{ c \in \prod_{\alpha \in \mathcal{S}_A} A/\alpha \;\middle|\; (\forall \alpha, \beta \in \mathcal{S}_A)\ \alpha \subseteq \beta \implies \varphi_{\alpha\beta}(c(\alpha)) = c(\beta) \right\}.$$

Then A embeds in $\mathrm{pro}_{\mathcal{A}}(A)$ via μ_A, where $\mu_A(a)(\alpha) := a/\alpha$ $(a \in A,\ \alpha \in \mathcal{S}_A)$.

Profinite completion: reconciliations

A finitely generated lattice-based variety

$$
\begin{tikzcd}
 & \text{canonical extension} \arrow[dl, <->] \arrow[dr, <->] & \\
\text{profinite completion} & & \text{natural extension}
\end{tikzcd}
$$

Any IRF prevariety

$$
\begin{tikzcd}
\text{profinite completion} \arrow[r, <->] & \text{natural extension}
\end{tikzcd}
$$

Back to duality theory

For distributive lattices we can use Priestley duality to access the canonical extension as a lattice of order-preserving maps into $\mathbf{2} = (\{0,1\}; \leqslant)$.

For simplicity, take $\mathcal{A} = \mathbb{ISP}(M)$ where M is a finite lattice-based algebra. An **algebraic relation** is a subalgebra of some M^n.

Theorem: *Let $A \in \mathcal{A}$ and let $b\colon \mathcal{A}(A, M) \to M$. Then TFAE:*

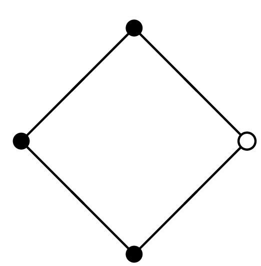
(1) *b belongs to $n_{\mathcal{A}}(A)$;*

(2) *b preserves every algebraic relation on M.*

If, further, $\mathcal{R}$ is a set of algebraic relations yielding a duality on $\mathcal{A}$ (in sense of natural duality theory), then (2) is equivalent to

(3) *b preserves every relation in $\mathcal{R}$.*

For $\mathcal{A} = \mathcal{D}$, we take $M = \mathbf{2}$ and $\mathcal{R} = \{\leqslant\}$ (a subalgebra of $\mathbf{2}^2$).



For residuated lattices enthusiasts

Now let $\mathcal{M}$ be a finite set of finite algebras, $\mathcal{T}$ the discrete topology. Let

$$\underset{\sim}{\mathcal{M}}_{\mathrm{Iso}} := \big\langle \dot{\bigcup}\{\, M \mid M \in \mathcal{M}\,\}; \mathrm{Iso}(\mathcal{M}), \mathcal{T} \big\rangle,$$

where $\mathrm{Iso}(\mathcal{M})$ = inverse semigroup (under composition of partial maps) of isomorphisms between subalgebras of algebras in $\mathcal{M}$.

Theorem (finitely generated discriminator varieties): *Let $\mathcal{V} = \mathbb{HSP}(\mathcal{M})$ with $\mathcal{M}$ as above. Then TFAE:*

(1) *$\mathcal{V}$ is quasiprimal (congruence permutable and congruence distributive and every nontrivial subalgebra of each $M \in \mathcal{M}$ is simple);*

(2) *$\exists$ a term yielding the ternary discriminator on each algebra in $\mathcal{M}$;*

(3) *$\underset{\sim}{\mathcal{M}}_{\mathrm{Iso}}$ yields a multisorted duality on $\mathcal{V}$.*

Canonical extensions which are full direct products

Theorem: *Let $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$, where $\mathcal{M}$ is a finite set of finite algebras, let $A \in \mathcal{A}$ and let B be the algebra of all $\mathcal{M}$-sorted maps*

$$b\colon \dot{\bigcup}\{\, \mathcal{A}(A, M) \mid M \in \mathcal{M} \,\} \to \dot{\bigcup}\{\, M \mid M \in \mathcal{M} \,\}$$

that preserve $\mathrm{Iso}(\mathcal{M})$*. Then B is isomorphic to a full direct product of* algebras *from* $\mathbb{S}(\mathcal{M})$*.*

Corollary: *The natural extensionof an algebra in a finitely generated lattice-based discriminator variety is isomorphic to a direct product of quasiprimal algebras.*

And finally:

‘A cardinal principle of modern mathematical research may be stated as a maxim:

“One must always topologize.” ’

Marshall Stone (1938)