On the computational complexity of spatial logics with connectedness constraints

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joint work with

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Motivation

Connectedness

- is one of the most **fundamental** concepts of topology

  (any textbook in the field contains a substantial chapter on connectedness)

- in spatial KR&R, the distinction between

  connected and *disconnected* regions

  is recognized as **indispensable** for various modelling and representation tasks

So far only sporadic attempts have been made to investigate the computational complexity of spatial logics with connectedness constraints

$\mathcal{S}4_u$: syntax and semantics

terms: **subsets of T**

$$\tau \quad ::= \quad v_i \quad | \quad \overline{\tau} \quad | \quad \tau_1 \cap \tau_2 \quad | \quad \tau^\circ \quad | \quad \tau^-$$

complement — interior — closure

formulas: **true** or **false**

$$\varphi \quad ::= \quad \tau_1 = \tau_2 \quad | \quad \neg\varphi \quad | \quad \varphi_1 \wedge \varphi_2$$

e.g., $\mathfrak{M} \models \tau_1 = \tau_2$ iff $\tau_1^{\mathfrak{M}} = \tau_2^{\mathfrak{M}}$

topological model $\mathfrak{M} = (T, \cdot^{\mathfrak{M}})$
- T a topological space
- $\cdot^{\mathfrak{M}}$ a valuation

NB. This definition is as expressive as the 'standard' one

A space is called **Aleksandrov** if *arbitrary* intersections of open sets are open

Aleksandrov spaces = Kripke frames $F = (W, R)$, R is a quasi-order on W

(Shehtman 99, Areces *et. al* 00): **Sat**$(\mathcal{S}4_u, \textsc{All})$ = **Sat**$(\mathcal{S}4_u, \textsc{Alek})$,
and this set is **PSPACE**-complete

NB. **Sat**$(\mathcal{S}4_u, \textsc{All}) \neq$ **Sat**$(\mathcal{S}4_u, \textsc{Con})$ (in contrast with $\mathcal{S}4$)

Connectedness

A topological space is **connected** iff
it is not the union of two non-empty, disjoint, open sets

Example:

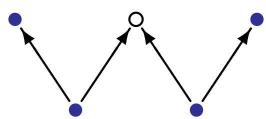
$$(v_1 \neq \mathbf{0}) \wedge (v_2 \neq \mathbf{0}) \wedge (v_1 \cup v_2 = \mathbf{1}) \wedge (v_1^- \cap v_2 = \mathbf{0}) \wedge (v_1 \cap v_2^- = \mathbf{0})$$

is satisfiable in a topological space T iff T is not connected

$X \subseteq T$ is **connected in T** just in case either it is empty,
or the topological space X (with the subspace topology) is connected

A maximal connected subset of X is called a **component** of X

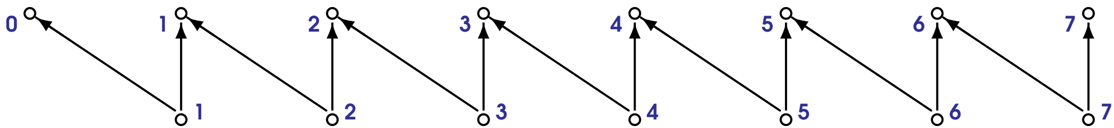
An **Aleksandrov space** induced by $F = (W, R)$ is **connected** iff F is connected
(i.e., between any two points $x, y \in W$ there is a path along the relation $R \cup R^{-1}$)



$\mathcal{S}4_u$ over connected topological spaces

(Shehtman 99): $\textbf{Sat}(\mathcal{S}4_u, \text{CON}) = \textbf{Sat}(\mathcal{S}4_u, \text{CONALEK}) = \textbf{Sat}(\mathcal{S}4_u, \mathbb{R}^n)$, $n \geq 1$,
and this set is **PSPACE**-complete

Example: generating all numbers from $\mathbf{0}$ to $\mathbf{2}^n - \mathbf{1}$:



$\mathbf{0}$ and $\mathbf{2}^n - \mathbf{1}$ are non-empty:

$\overline{v_n} \cap \dots \cap \overline{v_1} \neq \mathbf{0}, \qquad v_n \cap \dots \cap v_1 \neq \mathbf{0}$

the closure of m can share points only with $m + 1$, for $\mathbf{0} \leq m < \mathbf{2}^n - \mathbf{1}$:

$(v_j \cap \overline{v_k})^- \subseteq v_j, \qquad (\overline{v_j} \cap \overline{v_k})^- \subseteq \overline{v_j}, \qquad$ for $n \geq j > k \geq 1$

$(\overline{v_k} \cap v_{k-1} \cap \dots \cap v_1)^- \subseteq (v_k \cap \overline{v_i}) \cup (\overline{v_k} \cap v_i), \qquad$ for $n \geq k > i \geq 1$

$\mathbf{2}^n - \mathbf{1}$ is a closed set:

$(v_n \cap \dots \cap v_1)^- \subseteq v_n \cap \dots \cap v_1$

$\mathcal{S}4_u c$ = $\mathcal{S}4_u$ + connectedness predicate (1)

$\mathcal{S}4_u c$-formulas: $\varphi \quad ::= \quad \tau_1 = \tau_2 \quad | \quad c(\tau) \quad | \quad \neg\varphi \quad | \quad \varphi_1 \wedge \varphi_2$

$\mathfrak{M} \models c(\tau)$ iff $\tau^{\mathfrak{M}}$ is connected in T

$\downarrow$ one occurrence of c

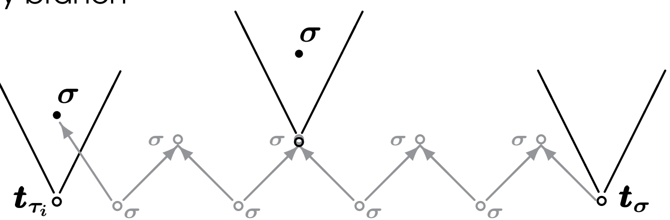
Theorem. **Sat**$(\mathcal{S}4_u c^1, \text{ALL})$ is **PSPACE**-complete

Proof. Let $\psi = (\tau_0 = \mathbf{0}) \wedge \bigwedge_{i=1}^{m} (\tau_i \neq \mathbf{0}) \wedge \big(c(\sigma) \wedge (\sigma \neq \mathbf{0})\big)$ (conjunct of a full DNF)

1. guess a type (Hintikka set) t_σ containing σ and $\overline{\tau_0}^{\circ}$
   and expand the tableau branch by branch (all points with σ are to be connected to t_σ)

2. for each i, guess a type t_{τ_i} containing τ_i and $\overline{\tau_0}^{\circ}$
   and expand the tableau branch by branch

– if σ appears in the tableau
  then we construct a path to t_σ
  (by "divide and conquer")



path of length $2^{|\psi|}$

$\mathcal{S}4_u c$ = $\mathcal{S}4_u$ + connectedness predicate (2)

Theorem. **Sat**$(\mathcal{S}4_u c$, ALL) is in EXPTIME

Proof. Let $\psi = (\tau_0 = \mathbf{0}) \wedge \bigwedge_{i=1}^{m} (\tau_i \neq \mathbf{0}) \wedge \bigwedge_{i=1}^{k} \big(c(\sigma_i) \wedge (\sigma_i \neq \mathbf{0})\big)$ (conjunct of a full DNF)

The proof is by reduction to $\mathcal{PDL}$ with converse and nominals [De Giacomo 95]

Let α and β be atomic programs and ℓ_i a nominal, for each σ_i

- the $\mathcal{S}4$-box is simulated by $[\alpha^*]$:
  $\tau^\dagger$ is the result of replacing in τ each sub-term ϑ° with $[\alpha^*]\vartheta$
- the universal box is simulated by $[\gamma]$, where $\gamma = (\beta \cup \beta^- \cup \alpha \cup \alpha^-)^*$

$$\psi' = [\gamma]\neg\tau_0^\dagger \ \wedge \ \bigwedge_{i=1}^{m} \langle\gamma\rangle\tau_i^\dagger \ \wedge \ \bigwedge_{i=1}^{k} \Big(\langle\gamma\rangle(\ell_i \wedge \sigma_i^\dagger) \ \wedge \ [\gamma](\sigma_i^\dagger \rightarrow \langle(\alpha \cup \alpha^-; \sigma_i^\dagger?)^*\rangle\ell_i)\Big)$$

ψ' is satisfiable iff ψ is satisfiable

NB. Matching lower bound to follow...

$\mathcal{S}4_u cc$ = $\mathcal{S}4_u$ + component counting predicates

$\mathcal{S}4_u cc$-formulas: $\quad \varphi \quad ::= \quad \tau_1 = \tau_2 \quad | \quad c^{\leq k}(\tau) \quad | \quad \neg\varphi \quad | \quad \varphi_1 \wedge \varphi_2$

$\mathfrak{M} \models c^{\leq k}(\tau)$ iff $\tau^{\mathfrak{M}}$ has at most k components in T

reduction to $\mathcal{S}4_u c$: (the v_i are fresh variables) **exponential** if k coded in binary!

- $c^{\leq k}(\tau) \rightarrow \quad (\tau = \bigcup_{1 \leq i \leq k} v_i) \wedge \bigwedge_{1 \leq i \leq k} c(v_i)$
- $\neg c^{\leq k}(\tau) \rightarrow (\tau = \bigcup_{1 \leq i \leq k+1} v_i) \wedge \bigwedge_{1 \leq i \leq k+1} (v_i \neq \mathbf{0}) \wedge \bigwedge_{1 \leq i < j \leq k+1} (\tau \cap v_i^- \cap v_j^- = \mathbf{0})$

(Pratt-Hartmann 02): **Sat**$(\mathcal{S}4_u cc$, All$)$ = **Sat**$(\mathcal{S}4_u cc$, Alek$)$; this set is in **NExpTime**

Proof. 1. Full $\mathcal{S}4_u cc$ is logspace-reducible to its fragment with no negative occurrences of $c^{\leq k}(\tau)$

2. This fragment of $\mathcal{S}4_u cc$ has exponential fmp (by continuous topological filtration)

$\mathcal{S}4_u c$ in Euclidean spaces

- satisfiable in $\mathbb{R}^2$ but not in $\mathbb{R}$:

$$\bigwedge_{1\leq i\leq 3} c(v_i) \quad \wedge \bigwedge_{1\leq i<j\leq 3} (v_i \cap v_j \neq \mathbf{0}) \quad \wedge \quad (v_1 \cap v_2 \cap v_3 = \mathbf{0})$$

- satisfiable in $\mathbb{R}^3$ but not in $\mathbb{R}^2$:

$$\bigwedge_{i\in\{j,k\}} (v_i \subseteq e^{\circ}_{j,k}) \;\wedge \bigwedge_{1\leq i\leq 5} (v_i \neq \mathbf{0}) \;\wedge \bigwedge_{\{i,j\}\cap\{k,l\}=\emptyset} (e_{i,j} \cap e_{k,l} = \mathbf{0}) \;\wedge \bigwedge_{1\leq i<j\leq 5} c(e^{\circ}_{i,j})$$

- satisfiable in connected spaces but not in $\mathbb{R}^n$, for any $n \geq 1$:

$$(v_1 \cap v_2 = \mathbf{0}) \;\wedge \bigwedge_{i=1,2} \big((v_i^- \subseteq v_i) \;\wedge\; c(\overline{v_i})\big) \;\wedge\; \neg c(\overline{v_1} \cap \overline{v_2})$$

Theorem. $\textsf{Sat}(\mathcal{S}4_u cc, \mathbb{R})$ is PSPACE-complete

Proof. Encoding in temporal logic with $\mathcal{S}$ and $\mathcal{U}$ over $(\mathbb{R}, <)$

Regular closed sets and $\mathcal{B}$

$X \subseteq T$ is **regular closed** if $X = X^{\circ -}$

RC(T) regular closed subsets of T

RC(T) = sets of the form $X^{\circ -}$, for $X \subseteq T$

RC(T) is a Boolean algebra $(\mathbf{RC}(T), +, -, \emptyset, T)$

where $X + Y = X \cup Y$ and $-X = (\overline{X})^{-}$

$\mathcal{B}$-terms: $\tau \quad ::= \quad r_i \quad | \quad -\tau \quad | \quad \tau_1 \cdot \tau_2$ **regular closed sets!**

$\mathcal{B}$-formulas: $\varphi \quad ::= \quad \tau_1 = \tau_2 \quad | \quad \neg\varphi \quad | \quad \varphi_1 \wedge \varphi_2$

$\mathcal{B}$ is a **fragment** of $\mathcal{S}4_u$: $\mathcal{B}$-terms $\xrightarrow{h}$ $\mathcal{S}4$-terms

$$h(r_i) = v_i^{\circ -}, \quad h(\tau_1 \cdot \tau_2) = (h(\tau_1) \cap h(\tau_2))^{\circ -}, \quad h(-\tau_1) = \left(\overline{h(\tau_1)}\right)^{-}$$

Theorem. $\mathbf{Sat}(\mathcal{B}, \textsc{Reg}) = \mathbf{Sat}(\mathcal{B}, \textsc{ConReg}) = \mathbf{Sat}(\mathcal{B}, \mathbf{RC}(\mathbb{R}^n))$, $n \geq 1$,
and this set is **NP**-complete

Proof. Every satisfiable $\mathcal{B}$-formula φ is satisfied
in a discrete topological space with $\leq |\varphi|$ points

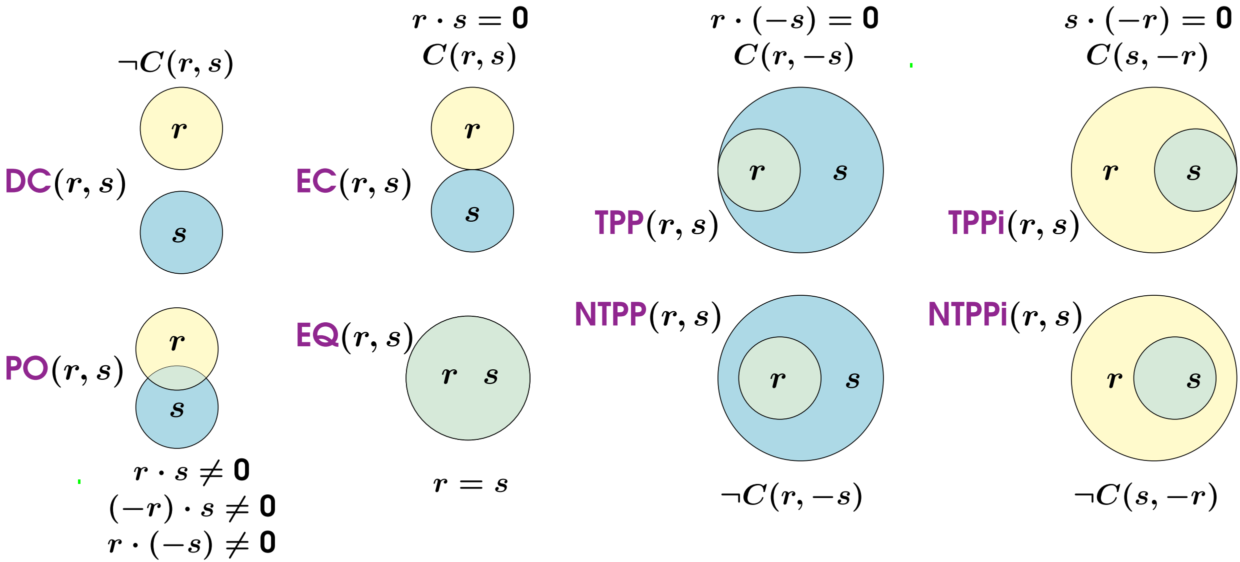
$\mathcal{C}$ = $\mathcal{B}$ + contact predicate

$\downarrow$Whitehead's 'connection' relation

$\mathcal{C}$-formulas: $\varphi \quad ::= \quad \tau_1 = \tau_2 \quad | \quad C(\tau_1, \tau_2) \quad | \quad \neg\varphi \quad | \quad \varphi_1 \wedge \varphi_2$

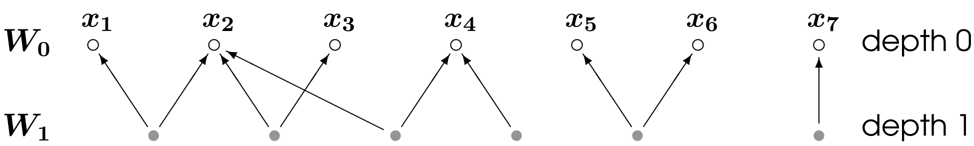
$\mathfrak{M} \models C(\tau_1, \tau_2)$ iff $\tau_1^{\mathfrak{M}} \cap \tau_2^{\mathfrak{M}} \neq \emptyset$

a.k.a. $\mathcal{BRCC}$-8



Quasi-saw models for $\mathcal{C}cc$

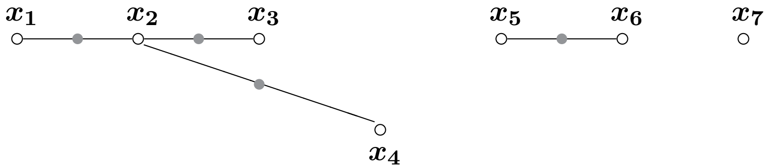
Lemma. Every satisfiable $\mathcal{C}cc$-formula is satisfied in a quasi-saw model



A valuation may be defined only on points of depth 0

and 'computed' on points of depth 1

$$z \in \tau^{\mathfrak{M}} \cap W_1 \text{ iff there is } x \in \tau^{\mathfrak{M}} \cap W_0 \text{ with } zRx$$



$\mathcal{C}c$ is ExpTime-hard

Theorem. **Sat**$(\mathcal{C}c, \text{Reg})$ is ExpTime-hard

Proof. Let $\mathcal{D}_2^f$ be the bimodal logic of the full infinite binary tree $\mathfrak{G} = (V, R_1, R_2)$ with **functional** R_1 and R_2

Reduction of the **global consequence relation** $\psi \models_2^f \chi$:

1. $(a \neq \mathbf{0}) \ \wedge \ c(f_0 + a) \ \wedge \ c(f_1 + a)$
2. every component of f_j contains a **sequence** of points in $s_j^0, s_j^1, \dots, s_j^5$ (provided it contains a point in s_j^0)
3. d marks points representing nodes of the binary tree, $d = s_0^0 + s_1^0$
   for each φ, q_φ means **'φ holds at the point'**
4. $q_{\neg\psi} \cdot s_0^0 \neq \mathbf{0}$ and $d \subseteq q_\chi$
5. $d \cdot q_{\neg\varphi} = d \cdot (-q_\varphi)$ and $d \cdot q_{\varphi_1 \wedge \varphi_2} = d \cdot (q_{\varphi_1} \cdot q_{\varphi_2})$
6. s_j^2 is the R_1-successor, s_j^4 is the R_2-successor: $s_j^2 \subseteq s_{j\oplus 1}^0$, $s_1^4 \subseteq s_{j\oplus 1}^0$, $j = 0, 1$
7. for each $\Box_i\varphi$, $m_\varphi^{i,j}$ means **'φ holds at the R_i-successor'**
   $\neg C(f_j \cdot m_\varphi^{i,j}, \ f_j \cdot m_{\neg\varphi}^{i,j})$
   $(s_j^0 \cdot q_{\Box_i\varphi} \subseteq m_\varphi^{i,j})$ and $(m_\varphi^{i,j} \cdot s_j^{2i} \subseteq q_\varphi)$ (similarly for $m_{\neg\varphi}^{i,j}$)

$\mathcal{C}cc$ is NExpTime-hard

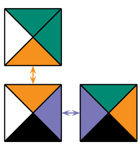
Theorem. **Sat**$(\mathcal{C}c, \text{Reg})$ is NExpTime-hard

Proof. By reduction of the $2^n \times 2^n$ origin constrained tiling

Given $n \in \mathbb{N}$, a finite set T of tile types $t = (\mathit{left}(t), \mathit{right}(t), \mathit{up}(t), \mathit{down}(t))$ and $t_0 \in T$

decide whether there exists $\tau : [0, 2^n] \times [0, 2^n] \longrightarrow T$ such that

(i) for all i, j,

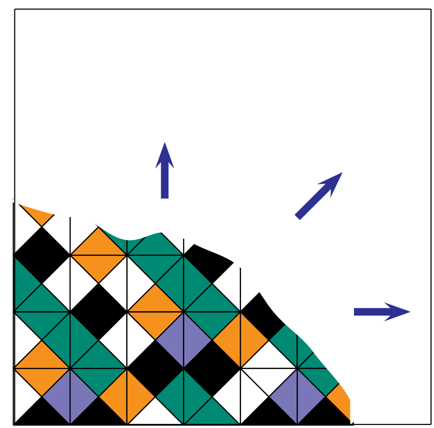


$\mathit{up}(\tau(i,j)) = \mathit{down}(\tau(i,j+1))$

and

$\mathit{left}(\tau(i,j)) = \mathit{right}(\tau(i+1,j))$

(ii) $\tau(0,0) = t_0$.



The $2^n \times 2^n$ origin constrained tiling is NExpTime-complete

$\mathcal{C}cc$ is NExpTime-hard

Theorem. **Sat**$(\mathcal{C}c, \text{Reg})$ is NExpTime-hard

Proof. By reduction of the $2^n \times 2^n$ origin constrained tiling

1. 2^n-counter formulas $X_n, \dots, X_1$ and 2^n-counter formulas for $Y_n, \dots, Y_1$

2. 4-neighbours: $\neg C(X_j \cdot Y_k,\ (-X_j) \cdot (-Y_k))$ and $\neg C((-X_j) \cdot Y_k,\ X_j \cdot (-Y_k))$

3. perimeter: $0_X \cdot 0_Y \neq \mathbf{0}$, $(2^d - 1)_X \cdot (2^d - 1)_Y \neq \mathbf{0}$,
$c(0_X + (2^d - 1)_Y)$, $c((2^d - 1)_X + 0_Y)$

4. interior: $c((-X_1) + 0_Y)$, $c(X_1 + 0_Y)$, $c(0_X + (-Y_1))$, $c(0_X + Y_1)$

5. chessboard: $\mathbf{b} = \big(X_1 \cdot (-Y_1)\big) + \big((-X_1) \cdot Y_1\big)$ $c^{\leq 2^{n-1}}(\mathbf{b})$
$\mathbf{w} = \big((-X_1) \cdot (-Y_1)\big) + \big(X_1 \cdot Y_1\big)$ $c^{\leq 2^{n-1}}(\mathbf{w})$

Note that (1)–(4) imply that each **b** and **w** contains **at least** 2^{n-1} components

6. $\neg C(\mathbf{b} \cdot T,\ \mathbf{b} \cdot T')$ and $\neg C(\mathbf{w} \cdot T,\ \mathbf{w} \cdot T')$, for $T \neq T'$

7. standard tiling formulas

Reduction from $\mathcal{C}c$ to $\mathcal{B}c$

$\mathcal{B}c$ is a fragment of $\mathcal{C}c$ and the following formula is a $\mathcal{C}c$-validity:

$$c(\tau_1) \wedge c(\tau_2) \longrightarrow \big(c(\tau_1 + \tau_2) \leftrightarrow C(\tau_1, \tau_2)\big)$$

Let φ be a $\mathcal{C}c$-formula

- positive occurrence of $C(\tau_1, \tau_2)$:

$$\varphi^* \quad = \quad \varphi[t = \mathbf{0}]^+ \quad \wedge \quad \big((t = \mathbf{0}) \;\rightarrow\; c(t_1 + t_2) \;\wedge \bigwedge_{i=1,2} (t_i \leq \tau_i) \wedge c(t_i)\big)$$

- negative occurrence of $C(\tau_1, \tau_2)$:

$$\varphi^* \quad = \quad \big(\varphi[t = \mathbf{0}]^-\big)_{|s} \quad \wedge \quad \big(\neg(t = \mathbf{0}) \;\rightarrow\; \neg c(t_1 + t_2) \wedge \bigwedge_{i=1,2} c(t_i) \wedge (\tau_i \cdot s \leq t_i)\big)$$

Then φ is satisfiable in an Aleksandrov space iff

φ^* is satisfiable in an Aleksandrov space

Summary of the results

	REG	CONREG	RC($\mathbb{R}^n$) $n > 2$	RC($\mathbb{R}^2$)	RC($\mathbb{R}$)
$\mathcal{RCC}$-8	NP				
$\mathcal{RCC}$-8c	NP			?	≤PSPACE,≥NP
$\mathcal{RCC}$-8cc	NP			?	≤PSPACE,≥NP
$\mathcal{B}$	NP				
$\mathcal{B}c$	EXPTIME	EXPTIME	?	?	≤PSPACE,≥NP
$\mathcal{B}cc$	NEXPTIME	NEXPTIME	?	?	≤PSPACE,≥NP
$\mathcal{C}$	NP	PSPACE			
$\mathcal{C}c$	EXPTIME	EXPTIME	≥EXPTIME	≥EXPTIME	PSPACE
$\mathcal{C}cc$	NEXPTIME	NEXPTIME	≥NEXPTIME	≥NEXPTIME	PSPACE
$\mathcal{C}^m$	NP	PSPACE		PSPACE	PSPACE
$\mathcal{C}^m c$	EXPTIME	EXPTIME	≥EXPTIME	≥EXPTIME	PSPACE
$\mathcal{C}^m cc$	NEXPTIME	NEXPTIME	≥NEXPTIME	≥NEXPTIME	PSPACE

	ALL	CON	$\mathbb{R}^n$, $n > 2$	$\mathbb{R}^2$	$\mathbb{R}$
$\mathcal{S}4_u$	PSPACE	PSPACE			
$\mathcal{S}4_u c$	EXPTIME	EXPTIME	≥EXPTIME	≥EXPTIME	PSPACE
$\mathcal{S}4_u cc$	NEXPTIME	NEXPTIME	≥NEXPTIME	≥NEXPTIME	PSPACE