On the Computational Complexity of Spatio-Temporal Logics

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Abstract

Recently, a hierarchy of spatio-temporal languages based on the propositional temporal logic $\mathbf{PTL}$ and the spatial languages $\mathcal{RCC}$-8, $\mathcal{BRCC}$-8 and $\mathbf{S4}_u$ has been introduced. Although a number of results on their computational properties were obtained, the most important questions were left open. In this paper, we solve these problems and provide a clear picture of the balance between expressiveness and 'computational realisability' within the hierarchy.

Introduction

In principle, it is not too difficult to devise a formalism suitable for spatio-temporal representation and reasoning—provided, of course, that spatial and temporal components are given. A real problem arises if one wants this formalism to be *effective* (or *practical*) from the computational point of view. For as is well known in logic and KR&R, the factor of *multi-dimensionality* can easily undermine the good algorithmic behaviour of those components when they are combined in a system with interacting spatial and temporal constructors.

Some aspects of qualitative spatio-temporal reasoning in this computational perspective were analysed in (Wolter and Zakharyaschev 2002; 2000b), where, in particular, the following hierarchy of spatio-temporal logics was suggested. The spatial components of the logics are mereotopological and can vary from $\mathcal{RCC}$-8 (Egenhofer and Franzosa 1991; Randell, Cui, and Cohn 1992) and $\mathcal{BRCC}$-8 (Wolter and Zakharyaschev 2000a) (i.e., $\mathcal{RCC}$-8 with Boolean region terms) to modal logic $\mathbf{S4}_u$ interpreted over topological spaces and containing both $\mathcal{RCC}$-8 and $\mathcal{BRCC}$-8 as fragments (Bennett 1996; Wolter and Zakharyaschev 2000a). The underlying temporal logic is the standard propositional $\mathbf{PTL}$ (known also as $\mathbf{LTL}$) with the operators $\mathcal{U}$ ('until'), $\mathcal{S}$ ('since') and their derivatives ($\bigcirc$ ('tomorrow'), $\Box_F$ ('always in the future'), $\Diamond_F$ ('eventually'), etc.) over various flows of time, say, $\langle \mathbb{N}, < \rangle$. The hierarchy contains the logics:

- $\mathcal{ST}_0$, which allows applications of temporal operators only to (combinations of) $\mathcal{BRCC}$-8 formulas, but not to region terms;
- $\mathcal{ST}_1$, which extends $\mathcal{ST}_0$ by allowing applications of the next-time operator $\bigcirc$ to region terms;
- $\mathcal{ST}_2$, where the temporal operators can be applied to both formulas and region terms;
- $\mathcal{ST}_i^-$— the variants of the above based on $\mathcal{RCC}$-8.

Here are some simple examples illustrating what can be said in these languages:

$$\mathsf{DC}(\mathit{Kaliningrad}, \mathit{EU})\, \mathcal{U}\, \mathsf{TPP}(\mathit{Poland}, \mathit{EU}) \qquad (1)$$

('Kaliningrad is disconnected from the EU until Poland becomes a tangential proper part of the EU'),

$$\mathsf{EQ}(\mathit{Russia}, T) \wedge \Diamond_F \mathsf{P}(T, \mathit{EU}) \wedge \Box_F \mathsf{EQ}(T, \bigcirc T) \qquad (2)$$

('some day in the future the present territory of Russia will be part of the EU'),

$$\mathsf{P}(\mathit{Russia}, \Diamond_F \mathit{EU}) \qquad (3)$$

('all points of Russia as it is today will belong to the EU in the future (but perhaps at different moments of time)')

$$\mathsf{EQ}(\bigcirc\bigcirc \mathit{EU}, \mathit{EU} \sqcup \mathit{Poland} \sqcup \mathit{Hungary} \sqcup \dots) \qquad (4)$$

('in two years the EU will be extended with Poland, Hungary, etc.')

$$\mathsf{EQ}(\mathit{Greece}, \Box_F \mathit{Euro\text{-}zone}) \qquad (5)$$

('only Greece in its present territory will always be in the euro-zone').

The most expressive formalism of the hierarchy, called $\mathcal{PST}$ (propositional spatio-temporal logic), combines (in fact, is the product of) $\mathbf{S4}_u$ and $\mathbf{PTL}$. The propositional variables of $\mathcal{PST}$ play the role of *region variables*—at each moment of time they are interpreted as subsets of a topological space. The Boolean connectives, the interior $\mathbf{I}$ and closure $\mathbf{C}$ operators, as well as the universal modalities $\boxdot$ and $\diamondplus$ ('for all points' and 'for some point in the space,' respectively) of $\mathbf{S4}_u$ allow us to express more complex relations among regions than those available in $\mathcal{RCC}$-8. For example, we can define a ternary relation

$$\mathsf{EC3}(X, Y, Z) = \boxdot\neg(\mathbf{I}X \wedge \mathbf{I}Y \wedge \mathbf{I}Z) \wedge \diamondplus(X \wedge Y \wedge Z)$$

and write $\mathsf{EC3}(\mathit{Russia}, \mathit{Poland}, \mathit{Lithuania})$ to say that Russia, Poland and Lithuania have a common border but no common interior point. Unlike $\mathcal{RCC}$-8, where regions are usually assumed to be regular closed (i.e, satisfying the condition $\boxdot(X \leftrightarrow \mathbf{CI}X)$), $\mathbf{S4}_u$ allows for more flexibility. In the extreme, we can express such 'pathological' properties of sets as 'X is dense in Y, but has no interior:'

$$\boxdot\neg\mathbf{I}X \wedge \boxdot(\mathbf{C}X \leftrightarrow Y).$$

Consider, for instance, the following $\mathcal{PST}$-formulas

$$\Box_F \boxdot(\neg\mathbf{I}\textit{cockroach} \wedge (\mathbf{C}\textit{cockroach} \leftrightarrow \textit{habitat})),$$
$$\Box_F \boxdot(\textit{habitat} \rightarrow \bigcirc\textit{habitat}),$$
$$\boxdot\mathbf{C}\Diamond_F\textit{cockroach},$$

saying that (a) the cockroaches form a dense set in their habitat (but for humans they are invisible), (b) the cockroach habitat will never contract, and (c) sooner or later, cockroaches will appear in the neighbourhood of every place on Earth.

The results obtained in (Wolter and Zakharyaschev 2000b; 2002) show that the satisfiability problem for the languages $\mathcal{ST}_i$ in topological temporal models over various flows of time is decidable (for $\mathcal{ST}_2$ under the so called *finite state assumption* **FSA**, see below). Moreover, if we consider the flow of time $\langle\mathbb{N}, <\rangle$, then $\mathcal{ST}_2$ (under **FSA**) and $\mathcal{ST}_1$ are decidable in EXPSPACE, while $\mathcal{ST}_0$ is PSPACE-complete. The problem of finding the lower bounds for the $\mathcal{ST}_i$, the complexity of $\mathcal{ST}_1^-$, as well as the decision problem for $\mathcal{PST}$ have remained open.

The main aim of this paper is to provide solutions to these problems. The obtained results are summarised in Table 1.

logic	**FSA**	no **FSA**
$\mathcal{PST}$	undecidable	undecidable
$\mathcal{ST}_2$	EXPSPACE-complete	?
$\mathcal{ST}_2^-$	in EXPSPACE	?
$\mathcal{ST}_1$	EXPSPACE-complete	EXPSPACE-complete
$\mathcal{ST}_1^-$	PSPACE-complete	PSPACE-complete
$\mathcal{ST}_0$	PSPACE-complete	PSPACE-complete

Table 1: Complexity of spatio-temporal logics over $\langle\mathbb{N}, <\rangle$.

To investigate the computational behaviour of the $\mathcal{ST}_i$, we (polynomially) embed these logics into the one-variable fragment of quantified temporal logic $\mathcal{QTL}$ based on the appropriate flow of time. Recent results of (Hodkinson, Wolter, and Zakharyaschev 2000) on the decidability of monodic fragments of first-order temporal logic show immediately that all the $\mathcal{ST}_i$ are decidable and provide upper bounds for their computational complexity over the flow of time $\langle\mathbb{N}, <\rangle$. To establish the matching lower bounds for $\mathcal{ST}_2$ (under **FSA**) and $\mathcal{ST}_1$, we identify the fragment of $\mathcal{QTL}$, which can be polynomially embedded into $\mathcal{ST}_1$, and show its EXPSPACE-hardness. It then comes as a surprise that $\mathcal{ST}_1^-$ turns out to be PSPACE-complete, i.e., as complex as pure $\mathbf{PTL}$. Another surprising result is the undecidability of full $\mathcal{PST}$, which is established by a reduction of Post's correspondence problem.

It is worth noting that the embedding of the $\mathcal{ST}_i$ into the one-variable fragment of $\mathcal{QTL}$ makes it possible to use the tableau- and resolution-based reasoning procedures recently developed in (Kontchakov *et al.* 2002; Degtyarev and Fisher 2001).

A full paper containing all proofs will be available on the web.

Propositional Spatio-Temporal Logic $\mathcal{PST}$

The language $\mathcal{PST}$, or *propositional spatio-temporal language*, is based on the following alphabet

- propositional (or rather spatial) variables $p_0, p_1, \ldots$,
- the Booleans (say, $\neg$ and $\wedge$),
- the binary temporal operators $\mathcal{U}$ (until) and $\mathcal{S}$ (since),
- the interior $\mathbf{I}$ and closure $\mathbf{C}$ operators, and
- the 'universal modalities' $\boxdot$ and $\diamondplus$.

We will freely use other Booleans definable via $\neg$ and $\wedge$, as well as the temporal operators $\bigcirc$ ('next-time'), $\Diamond_F$ ('sometime in the future') and $\Box_F$ ('always in the future') definable via $\mathcal{U}$

$$\bigcirc\psi = \bot\,\mathcal{U}\,\psi, \quad \Diamond_F\psi = \top\,\mathcal{U}\,\psi, \quad \Box_F\psi = \neg\Diamond_F\neg\psi.$$

The intended models of $\mathcal{PST}$, called *topological $\mathcal{PST}$-models*, are triples of the form $\mathfrak{M} = \langle\mathfrak{F}, \mathfrak{T}, \mathfrak{U}\rangle$, where $\mathfrak{F} = \langle W, <\rangle$ is a strict linear order representing the *flow of time*, $\mathfrak{T} = \langle U, \mathbb{I}\rangle$ a topological space[1] and $\mathfrak{U}$, a *valuation*, is a map associating with every variable p_i and every $w \in W$ a set $\mathfrak{U}(p_i, w) \subseteq U$—the 'region' occupied by p_i at moment w. The valuation $\mathfrak{U}$ is inductively extended to arbitrary $\mathcal{PST}$-formulas in the following way, where by (u, v) we denote the open interval $\{w \in W \mid u < w < v\}$:

- $\mathfrak{U}(\neg\psi, w) = U - \mathfrak{U}(\psi, w)$;
- $\mathfrak{U}(\psi_1 \wedge \psi_2, w) = \mathfrak{U}(\psi_1, w) \cap \mathfrak{U}(\psi_2, w)$;
- $\mathfrak{U}(\mathbf{I}\psi, w) = \mathbb{I}\mathfrak{U}(\psi, w)$;
- $\mathfrak{U}(\boxdot\psi, w) = \begin{cases} U, & \text{if } \mathfrak{U}(\psi, w) = U, \\ \emptyset, & \text{otherwise;} \end{cases}$
- $\mathfrak{U}(\psi_1\,\mathcal{U}\,\psi_2, w) = \{x \in U \mid \exists v > w\,(x \in \mathfrak{U}(\psi_2, v) \wedge \forall u\,(u \in (w, v) \rightarrow x \in \mathfrak{U}(\psi_1, u)))\}$;
- $\mathfrak{U}(\psi_1\,\mathcal{S}\,\psi_2, w) = \{x \in U \mid \exists v < w\,(x \in \mathfrak{U}(\psi_2, v) \wedge \forall u\,(u \in (v, w) \rightarrow x \in \mathfrak{U}(\psi_1, u)))\}$.

In particular,

$$\mathfrak{U}(\Diamond_F\psi, w) = \bigcup_{v>w} \mathfrak{U}(\psi, v), \quad \mathfrak{U}(\Box_F\psi, w) = \bigcap_{v>w} \mathfrak{U}(\psi, v),$$

and if $\mathfrak{F}$ is discrete then

$$\mathfrak{U}(\bigcirc\psi, w) = \mathfrak{U}(\psi, w+1),$$

where $w + 1$ is the immediate successor of w in $\mathfrak{F}$.

A $\mathcal{PST}$-formula φ is *satisfied* in $\mathfrak{M}$ if $\mathfrak{U}(\varphi, w) \neq \emptyset$ for some $w \in W$.

[1] Here U is a non-empty set, the *universe* of the space, and $\mathbb{I}$ is the *interior operator* on U satisfying the standard *Kuratowski axioms*: for all $X, Y \subseteq U$, $\mathbb{I}(X \cap Y) = \mathbb{I}X \cap \mathbb{I}Y$, $\mathbb{I}X \subseteq \mathbb{I}\mathbb{I}X$, $\mathbb{I}X \subseteq X$, $\mathbb{I}U = U$. The operator dual to $\mathbb{I}$ is called the *closure operator* and denoted by $\mathbb{C}$. Thus $\mathbb{C}X = U - \mathbb{I}(U - X)$.

Recall now (cf. e.g., (Chagrov and Zakharyaschev 1997)) that every Kripke frame $\mathfrak{G} = \langle U, R\rangle$ for Lewis's modal logic $\mathbf{S4}_u$ (R is a transitive and reflexive binary relation on U) gives rise to the topological space $\mathfrak{T}_{\mathfrak{G}} = \langle U, \mathbb{I}_{\mathfrak{G}}\rangle$, where for every $V \subseteq U$,

$$\mathbb{I}_{\mathfrak{G}} V = \{v \in V \mid \forall u \in U\, (vRu \to u \in V)\}.$$

(The main difference between arbitrary topological spaces and those generated by quasi-orders and known as *Aleksandrov spaces* is that all intersections of open sets are open in the latter, but this is not the case in the former.) This observation motivates the following definition.

A *Kripke $\mathcal{PST}$-model* is a topological $\mathcal{PST}$-model in which the topological space is of the form $\langle U, \mathbb{I}_{\mathfrak{G}}\rangle$ for some transitive and reflexive Kripke frame $\mathfrak{G}$.

Note that the set of $\mathcal{PST}$-formulas satisfiable in Kripke models is a proper subset of those satisfiable in topological models. Consider, e.g., the formula $\Box_F \mathbf{I}p \leftrightarrow \mathbf{I}\Box_F p$. It is clearly true in every Kripke $\mathcal{PST}$-model. On the other hand, we can satisfy the negation of this formula in a topological $\mathcal{PST}$-model: it suffices to take the flow of time $\mathfrak{F} = \langle \mathbb{N}, <\rangle$ and the topology $\mathfrak{T} = \langle \mathbb{R}, \mathbb{I}\rangle$ with the standard interior operator $\mathbb{I}$ on the real line, select a sequence X_n of open sets such that $\bigcap_{n\in\mathbb{N}} X_n$ is not open, for example $X_n = (-1/n, 1/n)$, and put $\mathfrak{U}(p, n) = X_n$.

A fundamental result of (Wolter and Zakharyaschev 2000b; 2002) is that, nevertheless, under certain natural semantical and syntactical conditions it is sufficient to work with Kripke $\mathcal{PST}$-models. First, this is the case for $\mathcal{PST}$-models satisfying the *finite state assumption* (**FSA**) which is formulated as follows. Say that a topological $\mathcal{PST}$-model $\langle \mathfrak{F}, \mathfrak{T}, \mathfrak{U}\rangle$ (or a Kripke $\mathcal{PST}$-model $\langle \mathfrak{F}, \mathfrak{G}, \mathfrak{U}\rangle$) satisfies **FSA** if for every variable p there exist finitely many sets $A_1, \dots, A_k$ such that

$$\{\mathfrak{U}(p, w) \mid w \in W\} = \{A_1, \dots, A_k\}.$$

Second, this also is the case for so-called u-formulas. A $\mathcal{PST}$-formula of the from $\boxdot\psi$ or $\diamondsuit\psi$, where ψ is built from propositional variables using the Booleans, the temporal operator $\bigcirc$ and the $\mathbf{S4}$-operators $\mathbf{C}$ and $\mathbf{I}$, is called a *basic u-formula*. A *u-formula* is a $\mathcal{PST}$-formula constructed from basic u-formulas using arbitrary connectives of $\mathcal{PST}$. (Clearly, $\Box_F \mathbf{I}p \leftrightarrow \mathbf{I}\Box_F p$ above is not a u-formula.)

Lemma 1. [W&Z 2000b] (i) *If a $\mathcal{PST}$-formula φ is satisfied in a topological $\mathcal{PST}$-model satisfying* **FSA** *and based on a flow of time $\mathfrak{F}$, then φ is satisfied in a Kripke $\mathcal{PST}$-model satisfying* **FSA** *and based on $\mathfrak{F}$.*

(ii) *If a u-formula φ is satisfied in a topological $\mathcal{PST}$-model based on a flow of time $\mathfrak{F}$, then φ is satisfied in a Kripke $\mathcal{PST}$-model based on $\mathfrak{F}$ as well.*

Lemma 1 serves as a basis for both 'positive' and 'negative' complexity results below. The reason is that relational structures are much easier to manipulate with than topological spaces. First, we use the lemma to show that the full language $\mathcal{PST}$ is 'too expressive,' at least when interpreted in models over discrete flows of time.

Theorem 2. *Let $\mathcal{C}$ be one of the following classes of flows of time: $\{\langle \mathbb{N}, <\rangle\}$, $\{\langle \mathbb{Z}, <\rangle\}$, or the class of all finite strict linear orders. Then the satisfiability problem for $\mathcal{PST}$-formulas in Kripke* (*and topological*) *$\mathcal{PST}$-models* (*with or without* **FSA**) *over the flows of time in $\mathcal{C}$ is undecidable.*

The proof is by reduction of the undecidable Post's correspondence problem (PCP) (Post 1946). The encoding of the problem relies heavily upon the fact that words over a given alphabet can be regarded as ascending chains of points in a Kripke model, where the propositional variables are regarded as members of the alphabet, and the fact that the flow of time is discrete. Theorem 2 holds even for the fragment of $\mathcal{PST}$ with the sole temporal operator $\Box_F$ and without $\boxdot$.

Spatio-Temporal Logics Based on $\mathcal{BRCC}$-8

We remind the reader that the language of $\mathcal{RCC}$-8 consists of a countably infinite set of *region variables* $X_0, X_1, \dots$, the eight binary predicate symbols DC, EQ, PO, EC, TPP, NTTP, TPPi, NTTPi and the Booleans out of which we construct *spatial formulas*. Spatial formulas are interpreted in topological spaces, with the region variables ranging over regular closed sets of $\mathfrak{T}$ and the intended meaning of the eight predicates being 'disconnected,' 'equal,' 'partial overlap,' 'externally connected,' 'tangential proper part,' 'non-tangential proper part' and the inverses of the last two, respectively.

Given a topological space $\mathfrak{T} = \langle U, \mathbb{I}\rangle$ and an assignment $\mathfrak{a}$ associating with every region variable X_i a regular closed set $\mathfrak{a}(X_i) \subseteq U$, the *truth-relation* $\mathfrak{T} \models^{\mathfrak{a}} \varphi$ for atomic formulas of $\mathcal{RCC}$-8 is defined in the natural way:

$\mathfrak{T} \models^{\mathfrak{a}} \mathsf{DC}(X_1, X_2)$ iff $\neg\exists x\; x \in \mathfrak{a}(X_1) \cap \mathfrak{a}(X_2)$;

$\mathfrak{T} \models^{\mathfrak{a}} \mathsf{EQ}(X_1, X_2)$ iff $\forall x\; (x \in \mathfrak{a}(X_1) \leftrightarrow x \in \mathfrak{a}(X_2))$;

$\mathfrak{T} \models^{\mathfrak{a}} \mathsf{PO}(X_1, X_2)$ iff $\exists x, y, z\quad x \in \mathbb{I}\mathfrak{a}(X_1) \cap \mathbb{I}\mathfrak{a}(X_2)$, $y \in \mathfrak{a}(X_1) \cap (U - \mathfrak{a}(X_2))$ and $z \in (U - \mathfrak{a}(X_1)) \cap \mathfrak{a}(X_2)$;

$\mathfrak{T} \models^{\mathfrak{a}} \mathsf{EC}(X_1, X_2)$ iff $\exists x\; x \in \mathfrak{a}(X_1) \cap \mathfrak{a}(X_2)$ and $\neg\exists x\; x \in \mathbb{I}\mathfrak{a}(X_1) \cap \mathbb{I}\mathfrak{a}(X_2)$;

$\mathfrak{T} \models^{\mathfrak{a}} \mathsf{TPP}(X_1, X_2)$ iff $\forall x\; x \in (U - \mathfrak{a}(X_1)) \cup \mathfrak{a}(X_2)$ and $\exists y, z\;\; y \in \mathfrak{a}(X_1) \cap (U - \mathbb{I}\mathfrak{a}(X_2))$, $z \in (U - \mathfrak{a}(X_1)) \cap \mathfrak{a}(X_2)$;

$\mathfrak{T} \models^{\mathfrak{a}} \mathsf{NTPP}(X_1, X_2)$ iff $\forall x\; x \in (U - \mathfrak{a}(X_1)) \cup \mathbb{I}\mathfrak{a}(X_2)$ and $\exists x\; x \in (U - \mathfrak{a}(X_1)) \cap \mathfrak{a}(X_2)$.

An $\mathcal{RCC}$-8-formula is said to be *satisfiable* in a topological space $\mathfrak{T}$ if there is an assignment $\mathfrak{a}$ in $\mathfrak{T}$ such that $\mathfrak{T} \models^{\mathfrak{a}} \varphi$.

Remark 3. It is often required that region variables are only interpreted by *non-empty* regular closed sets. We do not impose this restriction for two reasons. First, we can always express that region X is non-empty by constraints $\mathsf{EC}(X, Y)$ or $\mathsf{TPP}(X, Y)$, etc., for some dummy variable Y. And second we will need empty regions to interpret complex region terms of $\mathcal{BRCC}$-8 below.

The satisfiability problem for $\mathcal{RCC}$-8 formulas in topological spaces is NP-complete (Renz and Nebel 1999), while that for $\mathbf{S4}_u$ formulas is PSPACE-complete. So by replacing $\mathbf{S4}_u$ with $\mathcal{RCC}$-8 we may hope to get a spatio-temporal formalism with better computational properties.

One apparent 'deficit' of $\mathcal{RCC}$-8 is that it operates only with *atomic* regions. Denote by $\mathcal{BRCC}$-8 the extension of $\mathcal{RCC}$-8 which allows for *Boolean region terms*, i.e., combinations of region variables using the Booleans $\sqcup$, $\sqcap$ and

$\neg$, as arguments of $\mathcal{RCC}$-8 predicates. The *value* $\mathfrak{a}(t)$ of a Boolean region term t in a topological space $\mathfrak{T} = \langle U, \mathbb{I}\rangle$ under an assignment $\mathfrak{a}$ is defined inductively as follows:

$$\begin{aligned}\mathfrak{a}(\neg t) &= \mathbb{CI}\,(U - \mathfrak{a}(t)),\\ \mathfrak{a}(t_1 \sqcup t_2) &= \mathbb{CI}\,(\mathfrak{a}(t_1) \cup \mathfrak{a}(t_2)),\\ \mathfrak{a}(t_1 \sqcap t_2) &= \mathbb{CI}\,(\mathfrak{a}(t_1) \cap \mathfrak{a}(t_2)).\end{aligned}$$

As the Boolean operators do not in general preserve the property of being regular closed, we need the prefix $\mathbb{CI}$ in the right-hand parts of these definitions.

The computational behaviour of $\mathcal{BRCC}$-8 in arbitrary topological spaces is precisely the same as that of $\mathcal{RCC}$-8 (Wolter and Zakharyaschev 2000a).

There are different ways of introducing a temporal dimension into the syntax of $\mathcal{BRCC}$-8, which give rise to a hierarchy of possible spatio-temporal logics.

$\mathcal{ST}_0$. The most obvious one allows applications of the temporal operators $\mathcal{U}$ and $\mathcal{S}$ only to spatial $\mathcal{BRCC}$-8 formulas. More precisely, every formula of $\mathcal{BRCC}$-8 is an $\mathcal{ST}_0$-formula, and if ψ_1 and ψ_2 are $\mathcal{ST}_0$-formulas, then so are $\neg\psi_1$, $\psi_1 \wedge \psi_2$, $\psi_1 \mathcal{U} \psi_2$ and $\psi_1 \mathcal{S} \psi_2$. However, the expressive power of $\mathcal{ST}_0$ is rather limited. In particular, we can compare regions only at one moment of time, but are not able to connect a region as it is 'today' with its state 'tomorrow' to say, e.g., that it is expanding or remains the same.

$\mathcal{ST}_1$. To capture this dynamics, we extend $\mathcal{ST}_0$ by allowing applications of the next-time operator $\bigcirc$ not only to formulas but also to Boolean region terms. Thus, arguments of the predicate symbols of $\mathcal{RCC}$-8 can be now arbitrary region $\bigcirc$-*terms* which are constructed from region variables using the Booleans and $\bigcirc$. For instance, $\bigcirc\bigcirc X$ represents region X as it will be 'the day after tomorrow.' Denote the resulting language by $\mathcal{ST}_1$. Obviously, $\mathcal{ST}_1$ is more expressive than $\mathcal{ST}_0$ only for *discrete* flows of time; in dense flows of time like $\langle\mathbb{Q}, <\rangle$ or $\langle\mathbb{R}, <\rangle$ the 'next-time' operator makes no sense.

$\mathcal{ST}_2$. We can also extend $\mathcal{ST}_0$ by allowing the use of *temporal region terms*, constructed from region variables, the Booleans and the temporal operators $\mathcal{U}$ and $\mathcal{S}$ as arguments of the $\mathcal{RCC}$-8 predicates. In other words, every region variable is a temporal region term, and if t_1 and t_2 are temporal region terms then so are $\neg t_1$, $t_1 \sqcap t_2$, $t_1 \sqcup t_2$, $t_1 \mathcal{U} t_2$, $t_1 \mathcal{S} t_2$. The resulting language is denoted by $\mathcal{ST}_2$.

$\mathcal{ST}_i^-$. The languages $\mathcal{ST}_i^-$, $i = 0, 1, 2$, do not allow the use of the Booleans for constructing region terms—we can use only region variables and the permitted temporal operators.

The spatio-temporal languages $\mathcal{ST}_0$, $\mathcal{ST}_1$ and $\mathcal{ST}_2$ are interpreted in *topological temporal models* (or *tt-models*, for short) $\mathfrak{M} = \langle\mathfrak{F}, \mathfrak{T}, \mathfrak{a}\rangle$ which differ from topological $\mathcal{PST}$-models only in that $\mathfrak{a}$ assigns to every region variable X_i and every moment of time w in $\mathfrak{F}$ a regular closed set $\mathfrak{a}(X, w)$ in $\mathfrak{T}$, the *state* of X at w. The assignment $\mathfrak{a}$ is extended to temporal region terms as follows:

$$\begin{aligned}\mathfrak{a}(\neg t, w) &= \mathbb{CI}\,(U - \mathfrak{a}(t, w)),\\ \mathfrak{a}(t_1 \sqcup t_2, w) &= \mathbb{CI}\,(\mathfrak{a}(t_1, w) \cup \mathfrak{a}(t_2, w)),\\ \mathfrak{a}(t_1 \sqcap t_2, w) &= \mathbb{CI}\,(\mathfrak{a}(t_1, w) \cap \mathfrak{a}(t_2, w)),\\ \mathfrak{a}(t_1 \mathcal{U}\, t_2, w) &= \mathbb{CI}\,\{x \mid \exists v > w\,(x \in \mathfrak{a}(t_2, v) \wedge \\ &\qquad \forall u\,(u \in (w, v) \to x \in \mathfrak{a}(t_1, u)))\},\\ \mathfrak{a}(t_1 \mathcal{S}\, t_2, w) &= \mathbb{CI}\,\{x \mid \exists v < w\,(x \in \mathfrak{a}(t_2, v) \wedge \\ &\qquad \forall u\,(u \in (v, w) \to x \in \mathfrak{a}(t_1, u)))\}.\end{aligned}$$

In particular, $\mathfrak{a}(\bigcirc t, w) = \mathfrak{a}(t, w + 1)$ and

$$\mathfrak{a}(\Box_F t, w) = \mathbb{CI} \bigcap_{v > w} \mathfrak{a}(t, v),\ \mathfrak{a}(\Diamond_F t, w) = \mathbb{CI} \bigcup_{v > w} \mathfrak{a}(t, v).$$

For a tt-model $\mathfrak{M} = \langle\mathfrak{F}, \mathfrak{T}, \mathfrak{a}\rangle$, an $\mathcal{ST}_i$-formula φ and a w in W, we define the *truth-relation* $(\mathfrak{M}, w) \models \varphi$ ('φ holds in $\mathfrak{M}$ at moment w') by induction on the construction of φ:

- $(\mathfrak{M}, w) \models \mathsf{R}(t_1, t_2)$ iff $\mathfrak{T} \models^{\mathfrak{b}} \mathsf{R}(X_1, X_2)$, where R is one of the $\mathcal{RCC}$-8-relations, t_1 and t_2 are temporal region terms, and $\mathfrak{b}(X_j) = \mathfrak{a}(t_j, w)$, for $j = 1, 2$;
- $(\mathfrak{M}, w) \models \neg\psi$ iff $(\mathfrak{M}, w) \not\models \psi$;
- $(\mathfrak{M}, w) \models \psi_1 \wedge \psi_2$ iff $(\mathfrak{M}, w) \models \psi_1$ and $(\mathfrak{M}, w) \models \psi_2$;
- $(\mathfrak{M}, w) \models \psi_1\ \mathcal{U}\ \psi_2$ iff there exists $v > w$ such that $(\mathfrak{M}, v) \models \psi_2$ and $(\mathfrak{M}, u) \models \psi_1$ for every $u \in (w, v)$,
- $(\mathfrak{M}, w) \models \psi_1\ \mathcal{S}\ \psi_2$ iff there exists $v < w$ such that $(\mathfrak{M}, v) \models \psi_2$ and $(\mathfrak{M}, u) \models \psi_1$ for every $u \in (v, w)$.

As before, we say that a tt-model $\mathfrak{M} = \langle\mathfrak{F}, \mathfrak{T}, \mathfrak{a}\rangle$ satisfies **FSA** if for every region variable X there are only finitely many regular closed sets $A_1, \dots, A_k \subseteq U$ such that

$$\{\mathfrak{a}(X, w) \mid w \in W\} = \{A_1, \dots, A_k\}.$$

As was shown in (Wolter and Zakharyaschev 2000b; 2002), the languages $\mathcal{ST}_i$ can be embedded in $\mathcal{PST}$. More precisely, one can construct a polynomial translation $\cdot^\dagger$ which associates with every $\mathcal{ST}_i$-formula φ a $\mathcal{PST}$-formula $\varphi^\dagger$ in such a way that, for any tt-model $\mathfrak{M} = \langle\mathfrak{F}, \mathfrak{T}, \mathfrak{a}\rangle$, if we take the $\mathcal{PST}$-model $\mathfrak{M}' = \langle\mathfrak{F}, \mathfrak{T}, \mathfrak{U}\rangle$, where $\mathfrak{U}(p_i, w) = \mathfrak{a}(X_i, w)$, then

$$(\mathfrak{M}, w) \models \varphi \quad \text{iff} \quad \mathfrak{U}(\varphi, w) \neq \emptyset.$$

The languages $\mathcal{ST}_i$ are computationally simpler than $\mathcal{PST}$, because the modal translations of $\mathcal{ST}_2$-formulas form a rather special fragment of the modal language $\mathcal{PST}$. Renz (1998) showed that an $\mathcal{RCC}$-8-formula φ is satisfiable iff its translation $\varphi^\dagger$ is satisfiable in a Kripke model based on *saws*—disjoint unions of *forks*—with a fork being a three-point frame $\langle\{b, r, l\}, R\rangle$ such that bRr and bRl. Actually, it turns out that this result can be generalised to $\mathcal{ST}_2$- and $\mathcal{ST}_1$-formulas (in the former case we assume **FSA**).

Theorem 4. [W&Z 2000b] (i) *An $\mathcal{ST}_2$-formula φ is satisfied in a tt-model with* **FSA** *and based on a flow of time $\mathfrak{F}$ iff $\varphi^\dagger$ is satisfied in a Kripke $\mathcal{PST}$-model $\langle\mathfrak{F}, \mathfrak{G}, \mathfrak{U}\rangle$ with* **FSA** *whose underlying* **S4**-*frame $\mathfrak{G}$ is a saw.*

(ii) *An $\mathcal{ST}_1$-formula φ is satisfied in a tt-model based on a flow of time $\mathfrak{F}$ iff $\varphi^\dagger$ is satisfied in a Kripke $\mathcal{PST}$-model $\langle\mathfrak{F}, \mathfrak{G}, \mathfrak{U}\rangle$ whose underlying* **S4**-*frame $\mathfrak{G}$ is a saw.*

Starting from this result, (Wolter and Zakharyaschev 2000b; 2002) obtain EXPSPACE-upper bounds for the computational complexity of the $\mathcal{ST}_i$ using the observation that disjoint unions of forks (i.e., saws) can be encoded into first-order models in which the forks are regarded as primitive objects whose properties are encoded by unary predicates. One individual variable turned out to be sufficient to express the required properties. Thus, the $\mathcal{PST}$-translations of $\mathcal{ST}_2$-formulas were shown to be polynomially embeddable into the EXPSPACE-complete one-variable fragment of first-order temporal logic. However, it remained open whether the resulting EXPSPACE upper bound is optimal.

The following result provides the missing matching lower bounds:

Theorem 5. (i) *The satisfiability problem for $\mathcal{ST}_0$ in tt-models over the flow of time $\langle \mathbb{N}, < \rangle$ is* PSPACE-*complete.*

(ii) *The satisfiability problem for $\mathcal{ST}_1$ in tt-models over the flow of time $\langle \mathbb{N}, < \rangle$ is* EXPSPACE-*complete.*

(iii) *The satisfiability problem for $\mathcal{ST}_2$ in tt-models over the flow of time $\langle \mathbb{N}, < \rangle$ with* **FSA** *is* EXPSPACE-*complete.*

The proof is based on a rather sophisticated reduction of the EXPSPACE-complete 2^n-corridor tiling problem to the satisfiability problem for the fragment of first-order temporal logic (both with and without **FSA**) which corresponds to $\mathcal{ST}_1$, i.e., the fragment consisting of those one-variable first-order temporal formulas in which $\mathcal{U}$ and $\mathcal{S}$ can only be applied to sentences, while $\bigcirc$ is applicable to open formulas as well.

Spatio-Temporal Logics Based on $\mathcal{RCC}$-8

As we saw above, $\mathcal{ST}_1$ is EXPSPACE-complete. EXPSPACE is also the upper bound determined in (Wolter and Zakharyaschev 2000b; 2002) for $\mathcal{ST}_1^-$. We show here that, in contrast to $\mathcal{ST}_1$, this complexity bound is far from optimal: $\mathcal{ST}_1^-$-satisfiability is PSPACE-complete.

To this end, we first prove that $\mathcal{RCC}$-8 has a 'completion property' of (Balbiani and Condotta 2002) in the class of topological spaces generated by maximal saws with countably many forks of each type (in contrast to arbitrary topological spaces). Using this property, we show then that one can check satisfiability of $\mathcal{ST}_1^-$-formulas by a simple, almost modular, combination of the satisfiability checking algorithm for $\mathbf{PTL}$ of (Sistla and Clarke 1985) and any algorithm checking satisfiability of $\mathcal{RCC}$-8-formulas. This approach to determining the computational complexity of combinations of $\mathbf{PTL}$ with constraint systems like Allen's interval algebra and the orientation logic of (Ligozat 1998) has been introduced by Balbiani and Condotta (2002) and further developed for constraint systems without the completion property by Demri and D'Souza (2002).

Theorem 6. *The satisfiability problem for $\mathcal{ST}_1^-$ in tt-models over the flow of time $\langle \mathbb{N}, < \rangle$ is* PSPACE-*complete.*

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