Interpreting Topological Logics over Euclidean Spaces

Roman Kontchakov and **Michael Zakharyaschev**
Department of Computer Science and Information Systems
Birkbeck College London, UK
{roman,michael}@dcs.bbk.ac.uk

Ian Pratt-Hartmann
School of Computer Science
University of Manchester, UK
ipratt@cs.man.ac.uk

Introduction

Topological logics are a family of languages for representing and reasoning about topological data. The non-logical primitives of these languages stand for various topological relations and operations, and their valid formulas encode our knowledge about those relations and operations. Consider, for example, the six relations illustrated in Fig. 1. By em-

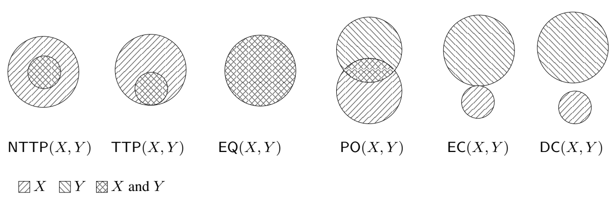


Figure 1: $\mathcal{RCC}8$-relations over disc-homeomorphs in $\mathbb{R}^2$.

ploying the binary predicates DC (disconnection), EC (external contact), PO (partial overlap), EQ (equality), TPP (tangential proper parthood) and NTPP (non-tangential proper parthood) to stand for these relations, the formula

$$\mathsf{TPP}(r_1, r_2) \wedge \mathsf{NTPP}(r_1, r_3) \rightarrow \mathsf{PO}(r_2, r_3) \vee \mathsf{TPP}(r_2, r_3) \vee \mathsf{NTPP}(r_2, r_3) \quad (1)$$

makes the intuitively reasonable assertion that, if region r_1 externally contacts region r_2 and is a non-tangential proper part of region r_3, then r_2 either partially overlaps, or else is a proper part (tangential or non-tangential) of r_3. This particular topological logic, known as $\mathcal{RCC}8$, has been intensively analysed in the literature on qualitative spatial reasoning; see e.g., (Egenhofer and Franzosa 1991; Randell, Cui, and Cohn 1992; Renz and Nebel 1999).

We referred to r_1, r_2 and r_3 above as 'regions,' and depicted them as discs in the plane. But a moment's thought shows that the set of valid formulas of any topological logic depends on the precise collection of regions we have in mind. For instance, there are $\mathcal{RCC}8$-formulas that are valid as long as regions are taken to be discs in the plane, but invalid when regions are allowed to be *disconnected* (i.e. to consist of more than one 'piece'). Thus, an important semantic issue for a topological logic like $\mathcal{RCC}8$ is to identify the intended models. In this paper, we show how even relatively inexpressive topological logics are sensitive both to the spaces they are interpreted over and—more particularly—to the subsets of those spaces over which their variables are allowed to range. We identify the crucial notion of *tameness*, and chart the surprising patterns of sensitivity to the presence of non-tame regions exhibited by a range of topological logics in low-dimensional Euclidean spaces.

Historically, $\mathcal{RCC}8$ has typically been interpreted over the *regular closed* sets of *arbitrary* topological spaces. (A set is *regular closed* if it is the topological closure of an open set.) This very general interpretation is *prima facie* surprising: after all, in qualitative spatial reasoning, it is the low-dimensional Euclidean spaces $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$ that interest us—not the arbitrary topological spaces found in mathematics textbooks! The answer to this objection is that if an $\mathcal{RCC}8$-formula is valid over regular closed subsets of $\mathbb{R}^n$ for some $n \geq 1$, then it is valid in all topological spaces whatsoever (Renz 1998). Put another way: the language $\mathcal{RCC}8$ is almost totally *in*sensitive to the underlying space.

However, this insensitivity disappears when we increase the expressive resources at our command. To illustrate, consider the effect of adding a unary predicate c, where $c(r)$ means 'r is connected.' We call the resulting language $\mathcal{RCC}8c$. Thus, the $\mathcal{RCC}8c$-formula

$$\bigwedge_{1 \leq i \leq 3} c(r_i) \quad \rightarrow \quad \bigvee_{1 \leq i < j \leq 3} \neg\mathsf{EC}(r_i, r_j) \qquad (2)$$

states that no three connected regions r_1, r_2 and r_3 can externally contact each other. In $\mathbb{R}^2$ (or in $\mathbb{R}^3$), this formula is clearly *in*valid. However, (2) is *valid* when interpreted over $\mathbb{R}$, since the non-empty connected regular closed subsets of $\mathbb{R}$ are simply the (non-empty) closed intervals.

Actually, the standard notion of connectedness may be inappropriate for many applications of qualitative spatial reasoning, particularly in the context of geographical information systems (GIS). Consider, for example, the (closed) region formed by two triangles touching externally at a common vertex. Mathematically speaking, this set is connected; yet we are loath to take it to represent, say, a connected plot of land on a map. Accordingly, we introduce the unary predicate c°, where $c^\circ(r)$ means 'the topological interior of r is connected,' and denote by $\mathcal{RCC}8c^\circ$ the result of adding c° to $\mathcal{RCC}8$. Again, one can show that $\mathcal{RCC}8c^\circ$, like $\mathcal{RCC}8c$, is sensitive to the topological space in which it is interpreted.

lang.	$\mathbb{R}$			$\mathbb{R}^2$			$\mathbb{R}^3$				RC
	RCP		RC	RCP		RC	RCP		RC		
$\mathcal{RCC}8c^\circ$	NP	$\neq$	NP	NP Th. 6, 9			NP				
$\mathcal{RCC}8c$		Th. 2		NP Th. 6, 9			Th. 12				
$\mathcal{B}c^\circ$	NP			$\geq$ ExpTime Th. 11	$\neq$ Th. 7	?	ExpTime Th. 15	$\neq$ Th. 14	?	?	NP Th. 16
$\mathcal{B}c$	Th. 3			$\geq$ ExpTime Th. 10	$\neq$ Th. 8	$\geq$ ExpTime Th. 10	$\geq$ ExpTime	?	$\geq$ ExpTime	?	ExpTime
$\mathcal{C}c^\circ$	PSpace	$\neq$	PSpace	$\geq$ ExpTime Th. 10	$\neq$ Th. 7	$\geq$ ExpTime Th. 10	$\geq$ ExpTime	$\neq$ Th. 14	$\geq$ ExpTime	$\neq$ Th. 13	ExpTime
$\mathcal{C}c$		Th. 2		$\geq$ ExpTime Th. 10	$\neq$ Th. 8	$\geq$ ExpTime Th. 10	$\geq$ ExpTime	?	$\geq$ ExpTime	?	ExpTime

Figure 2: Summary of the expressiveness and complexity results.

Another way to increase the expressive power of $\mathcal{RCC}8$ is to provide the means to talk about *combinations* of regions. Thus, in the language known as *Boolean* $\mathcal{RCC}8$ (Wolter and Zakharyaschev 2000), we use $r_1 + r_2$, $r_1 \cdot r_2$ and $-r$ for the regular closures of $r_1 \cup r_2$, $r_1 \cap r_2$ and the complement of r, respectively. We denote this extension of $\mathcal{RCC}8$ by $\mathcal{C}$ (this nomenclature will be justified in the sequel). By extending $\mathcal{C}$ with either of the predicates c or c°, we obtain the languages $\mathcal{C}c$ and $\mathcal{C}c^\circ$. For example, the $\mathcal{C}c^\circ$-formula

$$c^\circ(-r_1) \wedge c^\circ(-r_2) \wedge \neg c^\circ(-(r_1+r_2)) \rightarrow \neg \mathsf{DC}(r_1, r_2) \quad (3)$$

can be shown to be valid for regular closed sets in Euclidean space of any dimension (Theorem 13); yet it is invalid in other topological spaces—for example, the torus (Fig. 3).

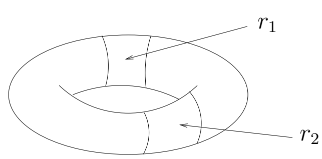


Figure 3: Invalidating (3) in the torus.

Once we have connectedness predicates at our disposal, two further topological languages suggest themselves. We denote by $\mathcal{B}c$ the language featuring the Boolean function symbols $+$, $\cdot$ and $-$, together with the equality predicate $=$ and the connectedness predicate c; the language $\mathcal{B}c^\circ$ is defined similarly, but with c replaced by c°.

The language $\mathcal{B}c^\circ$ nicely illustrates a subtle but important semantic issue which is often neglected in discussions of topological logics. Consider the $\mathcal{B}c^\circ$-formulas (for $m \geq 3$):

$$\bigwedge_{1 \leq i \leq m} c^\circ(r_i) \;\wedge\; c^\circ\big(\sum_{1 \leq i \leq m} r_i\big) \;\rightarrow\; \bigvee_{2 \leq i \leq m} c^\circ(r_1 + r_i). \quad (4)$$

Interpreted over the regular closed subsets of $\mathbb{R}^2$, these formulas are *in*valid: Fig. 4 shows a counterexample (with $m = 3$) in which the boundary between r_2 and r_3 is formed by the curve $\sin(1/x)$ over the interval $(0, 1]$. Yet r_2 and r_3 are hardly plausible models of, for instance, regions occupied by physical objects resting on a surface, or plots of land in a cadastre. Crucially, it can be shown (Lemma 1) that (4) becomes valid as soon as we restrict attention to 'well-behaved' (as we shall say: *tame*) subsets of $\mathbb{R}^n$—in

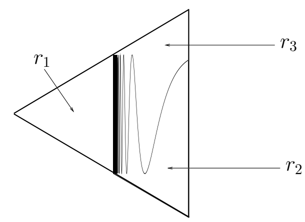


Figure 4: Three regular closed sets in $\mathbb{R}^2$ satisfying (4).

particular, to polyhedra (or polygons). Therefore, in deciding how to interpret topological logics for spatial representation and reasoning, it is not sufficient merely to fix the topological space in question: we must also specify which subsets of that space we wish to count as *bona fide* regions.

This paper undertakes the kind of semantic analysis of topological logics we have just argued for. We consider the six languages introduced above, and interpret them in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$. We show that the dimensionality of the space is important for all of our languages, but that, in addition, these languages exhibit varying patterns of sensitivity to tameness in different dimensions. It turns out that only $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$ are insensitive to tameness in $\mathbb{R}^2$ and $\mathbb{R}^3$, while—surprisingly—$\mathcal{B}c$ and $\mathcal{B}c^\circ$ do not feel it in $\mathbb{R}$; in all these cases reasoning (satisfiability) proves to be NP-complete. Reasoning with $\mathcal{B}c$, $\mathcal{B}c^\circ$, $\mathcal{C}c$ and $\mathcal{C}c^\circ$ in $\mathbb{R}^n$, for $n \geq 2$, is shown to be generally ExpTime-hard (apart from two cases that are still open). A matching upper bound is obtained for $\mathcal{B}c^\circ$ over polyhedra in $\mathbb{R}^n$, $n \geq 3$. The obtained results are collected in Fig. 2. The proofs of these results are occasionally intricate, and can only be sketched here. Full details can be found at `http://www.dcs.bbk.ac.uk/~roman`.

Preliminaries

Let T be a topological space. We denote the closure of any $X \subseteq T$ by X^-, its interior by X° and its boundary by $\delta X = X^- \setminus X^\circ$. We call X *regular closed* if $X = X^{\circ -}$, and denote by $\mathsf{RC}(T)$ the set of all regular closed subsets of T. It is known that $\mathsf{RC}(T)$ forms a Boolean algebra under the operations $r_1 + r_2 = r_1 \cup r_2$, $r_1 \cdot r_2 = (r_1 \cup r_2)^{\circ -}$ and $-r_1 = (T \setminus r_1)^-$. A subset $X \subseteq T$ is *connected* if it cannot be covered by the union of two non-empty and disjoint subsets which are open in the subspace topology on X. We say that X is *interior-connected* if X° is connected.

By a *topological language* we mean a language featuring an infinite set of variables, a fixed non-logical signature of function symbols and predicates (with standard meanings as topological operations and relations), and the usual connectives of propositional logic. If $\mathcal{L}$ is a topological language, a *frame* for $\mathcal{L}$ is a collection $\mathfrak{F}$ of subsets of some topological space T; and a *model* for $\mathcal{L}$ over $\mathfrak{F}$ is a pair $(\mathfrak{F}, \sigma)$, where σ is a function from variables to elements of $\mathfrak{F}$. Thus, for any topological space T, $\mathsf{RC}(T)$ is a frame for any of our topological languages; we denote by RC the class of all such frames. In this paper, we shall be concerned exclusively with frames which form subsets of $\mathsf{RC}(T)$ for some T. Restricting attention to regular closed sets is regarded as a convenient means of ignoring the boundaries of spatial regions.

Since the meanings of the non-logical primitives of $\mathcal{L}$ are fixed, any model defines a notion of truth for $\mathcal{L}$-formulas in the obvious way. If $\mathfrak{F}$ is a frame and φ an $\mathcal{L}$-formula, we say that φ is *satisfiable over* $\mathfrak{F}$ if φ is true in some model over $\mathfrak{F}$. If $\mathcal{K}$ is a class of frames, we say that φ is *satisfiable over* $\mathcal{K}$ if φ is satisfiable over some frame in $\mathcal{K}$; and we say that φ is *valid over* $\mathcal{K}$ if $\neg\varphi$ is not satisfiable over $\mathcal{K}$. Thus, satisfiability and validity are dual notions in the usual sense. A *topological logic* is a pair $(\mathcal{L}, \mathcal{K})$ where $\mathcal{L}$ is a topological language and $\mathcal{K}$ a class of frames for $\mathcal{L}$. The *satisfiability problem* for $(\mathcal{L}, \mathcal{K})$ is denoted by *Sat*$(\mathcal{L}, \mathcal{K})$.

For regular closed sets, the $\mathcal{RCC}8$-predicates are standardly interpreted as follows:

$$
\begin{array}{rll}
\mathsf{DC}(r_1, r_2) & \text{iff} & r_1 \cap r_2 = \emptyset, \\
\mathsf{EC}(r_1, r_2) & \text{iff} & r_1 \cap r_2 \neq \emptyset \text{ but } r_1^\circ \cap r_2^\circ = \emptyset, \\
\mathsf{PO}(r_1, r_2) & \text{iff} & r_1^\circ \cap r_2^\circ,\ r_1^\circ \setminus r_2,\ r_2^\circ \setminus r_1 \neq \emptyset \\
\mathsf{EQ}(r_1, r_2) & \text{iff} & r_1 = r_2, \\
\mathsf{TPP}(r_1, r_2) & \text{iff} & r_1 \subseteq r_2 \text{ but } r_1 \not\subseteq r_2^\circ \text{ and } r_2 \not\subseteq r_1, \\
\mathsf{NTPP}(r_1, r_2) & \text{iff} & r_1 \subseteq r_2^\circ \text{ but } r_2 \not\subseteq r_1.
\end{array}
$$

The unary predicates c and c° are interpreted as the properties of connectedness and interior-connectedness, respectively. The function symbols $+, \cdot, -$ and constants 0 and 1 are interpreted as the corresponding operations and elements in $\mathsf{RC}(T)$. The *contact* predicate C holds between regions r_1 and r_2 if and only if $r_1 \cap r_2 \neq \emptyset$. Thus, $C(r_1, r_2)$ is equivalent to $\neg\mathsf{DC}(r_1, r_2)$. In the presence of the Boolean functions, all the $\mathcal{RCC}8$-predicates can be expressed in terms of C and $=$ (i.e., EQ), and *vice versa* (Kontchakov et al. 2009); hence the name $\mathcal{C}$ for the Boolean extension of $\mathcal{RCC}8$.

Fixing $n \geq 1$, any $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$ bounds two regions in $\mathsf{RC}(\mathbb{R}^n)$; let us call these regions *half-spaces*. We denote by $\mathsf{RCP}(\mathbb{R}^n)$ the Boolean subalgebra of $\mathsf{RC}(\mathbb{R}^n)$ generated by the half-spaces. We call the elements of $\mathsf{RCP}(\mathbb{R}^n)$ *polyhedra* in $\mathbb{R}^n$, and the elements of $\mathsf{RCP}(\mathbb{R}^2)$ *polygons*. We have: (*i*) every polyhedron is the union of finitely many connected polyhedra; and (*ii*) every polyhedron satisfies the *curve-selection lemma* (Bochnak et al. 1998, p. 38). (The regions r_2 and r_3 of Fig. 4 lack curve-selection.) We call any collection of (regular closed) sets satisfying these two properties *tame*. Tame regions are regarded as well-behaved.

More generally, a subset of $\mathbb{R}^n$ is *semi-algebraic* if it is definable by a formula with n free variables in the first-order language of fields; we denote the collection of regular closed semi-algebraic subsets of $\mathbb{R}^n$ by $\mathsf{RCS}(\mathbb{R}^n)$. Semi-algebraic sets are certainly representationally adequate for all practical purposes, yet they are tame in the above sense. Since, however, even very expressive topological languages standardly cannot distinguish between semi-algebraic sets and polyhedra, we may as well, for the purpose of restricting attention to tame regions, focus on polyhedra; and that is what we do in the sequel. Note also that, in GISs, regions are usually represented as polygons.

One-dimensional Euclidean space

First we consider the logics $(\mathcal{L}, \mathsf{RC}(\mathbb{R}))$ and $(\mathcal{L}, \mathsf{RCP}(\mathbb{R}))$, where $\mathcal{L}$ is any of topological languages introduced above. The one-dimensional case is simple to analyse, yet illustrates well the kinds of phenomena that will occupy us at greater length when we come to the 2D and 3D cases.

Over $\mathbb{R}$, the notions of connectedness and interior-connectedness coincide; hence, we have only the languages $\mathcal{RCC}8c$, $\mathcal{B}c$ and $\mathcal{C}c$ to consider. We begin by observing that the dimensionality of the space is significant.

Theorem 1. *For any* $\mathcal{L} \in \{\mathcal{RCC}8c, \mathcal{B}c, \mathcal{C}c\}$ *and* $n \geq 2$, *Sat*$(\mathcal{L}, \mathsf{RC}(\mathbb{R})) \subsetneqq$ *Sat*$(\mathcal{L}, \mathsf{RC}(\mathbb{R}^n))$.

Proof. The inclusion holds because any tuple in $\mathsf{RC}(\mathbb{R})$ can easily be cylindrified to form a tuple in $\mathsf{RC}(\mathbb{R}^n)$, $n \geq 2$, satisfying the same $\mathcal{L}$-formulas. To see that it is proper, we observed above that (2) is invalid over $\mathsf{RC}(\mathbb{R}^n)$ for $n \geq 2$; but it is valid over $\mathsf{RC}(\mathbb{R})$. For $\mathcal{B}c$, consider $\bigwedge_{1 \leq i \leq 3}\big(c(r_i) \wedge (r_i \neq 0)\big) \to \bigvee_{1 \leq i < j \leq 3} \neg\big((r_i \cdot r_j = 0) \wedge c(r_i + r_j)\big)$. □

Next, we consider the issue of tameness. Over $\mathbb{R}$, the languages $\mathcal{RCC}8c$ and $\mathcal{C}c$ are sensitive to the presence of non-tame regions:

Theorem 2. *Sat*$(\mathcal{RCC}8c, \mathsf{RCP}(\mathbb{R})) \subsetneqq$ *Sat*$(\mathcal{RCC}8c, \mathsf{RC}(\mathbb{R}))$ *and Sat*$(\mathcal{C}c, \mathsf{RCP}(\mathbb{R})) \subsetneqq$ *Sat*$(\mathcal{C}c, \mathsf{RC}(\mathbb{R}))$.

Proof. The inclusions are trivial. To show that they are proper, the $\mathcal{RCC}8c$-formula $c(r_1) \wedge \bigwedge_{1 \leq i < j \leq 4} \mathsf{EC}(r_i, r_j)$ is satisfiable over $\mathsf{RC}(\mathbb{R})$, but not over $\mathsf{RCP}(\mathbb{R})$; see Fig. 5. □

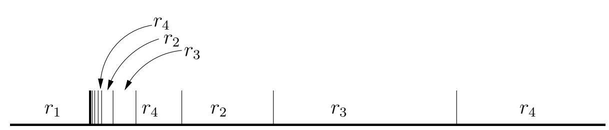


Figure 5: Subsets of $\mathbb{R}$ used in the proof of Theorem 2.

The language $\mathcal{B}c$, by contrast, is not sensitive to tameness:

Theorem 3. *Sat*$(\mathcal{B}c, \mathsf{RC}(\mathbb{R})) =$ *Sat*$(\mathcal{B}c, \mathsf{RCP}(\mathbb{R}))$.

Proof. See `http://www.dcs.bbk.ac.uk/~roman`. □

Turning now to complexity, it is already known that *Sat*$(\mathcal{B}c, \mathsf{RC}(\mathbb{R}))$ is NP-complete (Kontchakov et al. 2009), and *Sat*$(\mathcal{C}c, \mathsf{RC}(\mathbb{R}))$ is PSPACE-complete (Kontchakov et al. 2008). The picture is completed by the following results:

Theorem 4. *The problems Sat*$(\mathcal{RCC}8c, \mathsf{RC}(\mathbb{R}))$ *and Sat*$(\mathcal{RCC}8c, \mathsf{RCP}(\mathbb{R}))$ *are both* NP*-complete*; *the problem Sat*$(\mathcal{C}c, \mathsf{RCP}(\mathbb{R}))$ *is* PSPACE*-complete.*

Proof. See `http://www.dcs.bbk.ac.uk/~roman`. □

Two-dimensional Euclidean space

We first observe that, for the languages we consider, confining attention to the space $\mathbb{R}^2$ is significant for satisfiability.

Theorem 5. *For any of the languages $\mathcal{L}$ considered in this paper, and any $n \geq 3$, $Sat(\mathcal{L}, \mathsf{RC}(\mathbb{R}^2)) \subsetneqq Sat(\mathcal{L}, \mathsf{RC}(\mathbb{R}^n))$.*

Proof. Again, inclusion follows by cylindrification.

To show that it is proper, let r_i $(1 \leq i \leq 5)$ and $r_{i,j}$ $(1 \leq i < j \leq 5)$ be variables, let φ be the $\mathcal{RCC}8c$-formula

$$\bigwedge_{1 \leq i < j \leq 5} c(r_{i,j}) \;\wedge \bigwedge_{\{i,j\} \cap \{k,\ell\} = \emptyset} \mathsf{DC}(r_{i,j}, r_{k,\ell}) \;\wedge \bigwedge_{i \in \{j,k\}} \mathsf{TPP}(r_i, r_{j,k}),$$

and let φ° be the $\mathcal{RCC}8c^\circ$-formula obtained by replacing all occurrences of c by c°. Thus, φ° entails φ. A simple argument based on the non-planarity of the graph K_5 shows that φ (and hence φ°) is not satisfiable over $\mathsf{RC}(\mathbb{R}^2)$. On the other hand, φ° (and hence φ) is satisfiable over $\mathsf{RC}(\mathbb{R}^n)$ for $n \geq 3$. This deals with $\mathcal{RCC}8c$, $\mathcal{RCC}8c^\circ$, $\mathcal{C}$ and $\mathcal{C}c^\circ$. For $\mathcal{B}c$, replace $\mathsf{DC}(r_{i,j}, r_{k,\ell})$ by $\neg c(r_{i,j} + r_{k,\ell})$ and $\mathsf{TPP}(r_i, r_{j,k})$ by $(r_i \cdot r_{j,k} = r_i) \wedge (r_i \neq 0)$. For $\mathcal{B}c^\circ$, use the formula $\bigwedge_{1 \leq i \leq 5}\big(c^\circ(r_i) \wedge (r_i \neq 0)\big) \wedge \bigwedge_{1 \leq i < j \leq 5}\big(c^\circ(r_j + r_j) \wedge (r_i \cdot r_j = 0)\big)$. □

We now proceed to show (Theorems 6–8) that, over $\mathbb{R}^2$, our topological languages exhibit a different pattern of sensitivity to tameness to that which we observed over $\mathbb{R}$.

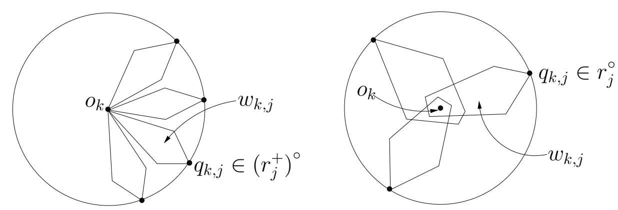
Theorem 6. *If an $\mathcal{RCC}8c$- or $\mathcal{RCC}8c^\circ$-formula is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$, then it can be satisfied over the frame of bounded regular closed polygons. In consequence:*

$$Sat(\mathcal{RCC}8c, \mathsf{RC}(\mathbb{R}^2)) = Sat(\mathcal{RCC}8c, \mathsf{RCP}(\mathbb{R}^2)),$$
$$Sat(\mathcal{RCC}8c^\circ, \mathsf{RC}(\mathbb{R}^2)) = Sat(\mathcal{RCC}8c^\circ, \mathsf{RCP}(\mathbb{R}^2)).$$

Proof. For the first statement, it suffices to construct, for any tuple $r_1, \ldots, r_n$ in $\mathsf{RC}(\mathbb{R}^2)$, a corresponding tuple $p_1, \ldots, p_n$ in $\mathsf{RCP}(\mathbb{R}^2)$ satisfying exactly the same atomic $\mathcal{RCC}8c$-formulas. We may assume that the r_i are distinct and non-empty. By reordering the variables if necessary, we can ensure that $r_i \subseteq r_j$ implies $i \leq j$. For all i, j $(1 \leq i < j \leq n)$, let $R_{ij} \in \{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{TPP}, \mathsf{NTPP}\}$ be the unique relation such that $R_{ij}(r_i, r_j)$.

First, we construct regular closed sets $r_1^+, \ldots, r_n^+$ such that $r_j \subseteq (r_j^+)^\circ$, $(r_j^+)^\circ$ is connected whenever r_j is connected, and $r_j^+ \cap r_{j'}^+ = \emptyset$ whenever $r_j \cap r_{j'} = \emptyset$, for all j, j'. This is possible because $\mathbb{R}^2$ is a normal, locally connected topological space and the r_i are regular closed subsets of $\mathbb{R}^2$. For all i, j $(1 \leq i < j \leq n)$, pick points o, o', o'' satisfying the conditions: (*i*) if $R_{ij} = \mathsf{EC}$, then $o \in \delta r_i \cap \delta r_j$; (*ii*) if $R_{ij} = \mathsf{PO}$, then $o \in r_i^\circ \cap r_j^\circ$, $o' \in r_i^\circ \setminus r_j$, and $o'' \in r_j^\circ \setminus r_i$; (*iii*) if $R_{ij} = \mathsf{TPP}$, then $o \in \delta r_i \cap \delta r_j$ and $o' \in r_j^\circ \setminus r_i$; (*iv*) if $R_{ij} = \mathsf{NTPP}$, then $o \in r_j^\circ \setminus r_i$. And for all i $(1 \leq i \leq n)$, pick points o, o' satisfying the condition that, if r_i is not connected, then o and o' lie in different components of r_i. Enumerate the chosen (distinct) points as $o_1, \ldots, o_m$: we call them *witness points*. We can draw disjoint closed disks $d_1, \ldots, d_m$, centred on the respective witness points o_k, such that, for all $j \leq n$ and $k \leq m$: $o_k \in r_j^\circ$ implies $d_k \subseteq r_j^\circ$; and $o_k \in (r_j^+)^\circ$ implies $d_k \subseteq (r_j^+)^\circ$. Indeed, we can ensure that none of the sets $(r_j^+)^\circ$ is disconnected by (simultaneous) removal of $d_1, \ldots, d_m$.

We now begin the construction of the $p_1, \ldots, p_n$. First, for each set r_j and each witness point o_k, we select a polygon $w_{k,j}$ inside d_k. We refer to the $w_{k,j}$ as *wedges*: for each $j \leq n$, and each $k \leq m$ we will ensure below that $w_{k,j} \subseteq p_j$. Wedges are selected as follows. (*i*) If $o_k \in \delta r_j$, pick a point $q_{k,j} \in \delta d_k \subseteq (r_j^+)^\circ$, and let $w_{k,j}$ be a lozenge with $w_{k,j} \subseteq d_k$ and $o_k, q_{k,j} \in \delta w_{k,j}$; see Fig. 6a. We may pick the $q_{k,j}$ to be distinct, and construct the $w_{k,j}$ so that no two such $w_{k,j}$ have intersecting interiors. (*ii*) If $o_k \in r_j^\circ$, pick a point $q_{k,j} \in \delta d_k \subseteq r_j^\circ$, and let $w_{k,j}$ be a lozenge such that $w_{k,j} \subseteq d_k$, $o_k \in (w_{k,j})^\circ$ and $q_{k,j} \in \delta w_{k,j}$. Again, we may pick the $q_{k,j}$ to be distinct from each other and from the $q_{k,j}$ selected in (*i*); see Fig. 6b. (*iii*) Otherwise, i.e., if $o_k \notin r_j$, let $w_{k,j} = \emptyset$. The wedges $w_{k,j}$ will ensure that $p_1, \ldots, p_n$ contain certain witness points required for the satisfaction of the relevant atomic $\mathcal{RCC}8c$-formulas. For example, if $\mathsf{PO}(r_i, r_j)$, there will exist a witness point $o_k \in r_i^\circ \cap r_j^\circ$; but then $o_k \in w_{k,i}^\circ \cap w_{k,j}^\circ$, whence $o_k \in p_i^\circ \cap p_j^\circ$, which is required to ensure that $\mathsf{PO}(p_i, p_j)$.



a) $o_k \in \delta r_j$ b) $o_k \in r_j^\circ$

Figure 6: Wedges involving the witness point o_k.

Fix any $j \leq n$. If r_j is connected, we need to connect up the various wedges $w_{k,j}$ so as to ensure that p_j is connected. Specifically, we construct a connected, regular closed polygon $a_j \subseteq (r_j^+)^\circ$ such that a_j is externally connected to all the non-empty $w_{k,j}$ (with k varying). The construction is quite elaborate, but the basic technique is illustrated in Fig. 7. The crucial point is that the a_j need only be connected—they need not have connected interiors; hence they may be crossed by regions $a_{j'}$ $(j' \neq j)$ as long as we do not have $\mathsf{DC}(r_j, r_{j'})$. The polygon a_j illustrated in Fig. 7 is crossed twice in this way. If r_j is not connected, set $a_j = 0$. For all $j \leq n$, define $b_j = a_j + \sum_{1 \leq k \leq m} w_{k,j}$. Then b_j will be connected if and only if r_j is.

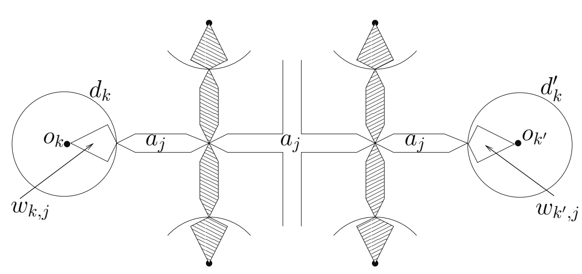


Figure 7: Connecting together wedges $w_{k,j}$ and $w_{k',j}$.

Using the polygons $b_1, \ldots, b_n$, we construct the desired polygons $p_1, \ldots, p_n$, relying on the fact that we earlier ensured that $r_i \subseteq r_j$ implies $i \leq j$. Start by setting $p_1 = b_1$, and let p_1^+ be a polygon containing p_1 in its interior, but which is 'close' to p_1 (specifically: p_1^+ does not intersect any polygon involved in this construction that p_1 does not intersect). For $2 \leq j \leq n$, set

$$p_j \;=\; b_j \;+ \sum_{\mathsf{TPP}(r_i,r_j)} p_i \;+ \sum_{\mathsf{NTPP}(r_i,r_j)} p_i^+$$

and again let p_j^+ be a polygon containing p_j in its interior, and 'close' to p_j. This ensures that $r_i \subseteq r_j$ implies $p_i \subseteq p_j$, and $r_i \subseteq r_j^\circ$ implies $p_i \subseteq p_j^\circ$. We can then show that $p_1, \ldots, p_n$ satisfy exactly the same atomic $\mathcal{RCC}8c$-formulas as the $r_1, \ldots, r_n$.

A similar (but not identical) construction can be carried out for the case of $\mathcal{RCC}8c^\circ$. □

An analogous result to Theorem 6 for a more expressive spatial logic can be found in (Davis et al. 1991, Sec. 8.1).

For the language $\mathcal{B}c$, however, tameness does make a difference in two dimensions, both for connectedness and for interior-connectedness. The latter is easily dealt with:

Lemma 1. *The $\mathcal{B}c^\circ$-formula* (4) *is valid in* $\mathsf{RCP}(\mathbb{R}^n)$ *for all* $m > n \geq 1$.

Proof. The case $n = 1$ is trivial. See (Pratt-Hartmann 2007, p. 40) for the case $n = 2$; the proof applies almost unaltered to higher dimensions. □

Theorem 7. *Sat*$(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^2)) \subsetneqq$ *Sat*$(\mathcal{B}c^\circ, \mathsf{RC}(\mathbb{R}^2))$ *and* *Sat*$(\mathcal{C}c^\circ, \mathsf{RCP}(\mathbb{R}^2)) \subsetneqq$ *Sat*$(\mathcal{C}c^\circ, \mathsf{RC}(\mathbb{R}^2))$.

Proof. We need only show that the inclusions are proper. As observed above, (4) is invalid over $\mathsf{RC}(\mathbb{R}^2)$; but it is valid over $\mathsf{RC}(\mathbb{R}^2)$ by Lemma 1. □

For ordinary connectedness, much more work is required.

Theorem 8. *Sat*$(\mathcal{C}c, \mathsf{RCP}(\mathbb{R}^2)) \subsetneqq$ *Sat*$(\mathcal{C}c, \mathsf{RC}(\mathbb{R}^2))$. *In fact, Sat*$(\mathcal{B}c, \mathsf{RCP}(\mathbb{R}^2)) \subsetneqq$ *Sat*$(\mathcal{B}c, \mathsf{RC}(\mathbb{R}^2))$.

Proof. Again, we need only show that the inclusions are proper. We begin with the language $\mathcal{C}c$; the second statement of the theorem will follow by an easy adaptation. Let V be the set of variables $\{v, h, s, t, t_0, r_0, r_1, r_2, r_3, r_4, r_5\}$ and, for any $x \in V$, let $\hat{x}$ be a fresh variable. Consider the assignment of elements of $\mathsf{RC}(\mathbb{R}^2)$ to these variables shown in Fig. 8. Here, the regions v and h are unbounded, connected polygons, the regions s, t and t_0 are bounded, connected polygons, and the regions r_i $(0 \leq i < 6)$ are all unbounded and have infinitely many components (and hence are not polygons). Also, for all $x \in V$, the region $\hat{x}$ is a slightly 'enlarged' version of x, with x lying in the interior of $\hat{x}$. (In Fig. 8, we have drawn $\hat{v}$, $\hat{t}_0$ and $\hat{h}$ with dotted lines; the other regions $\hat{x}$ are suppressed for clarity.)

Let φ_0 be the conjunction of $x \neq 0$, $x \cdot y = 0$, $\neg C(x, -\hat{x})$, for distinct $x, y \in V$. The last of these ensures that $\hat{x}$ represents a region whose interior contains x.

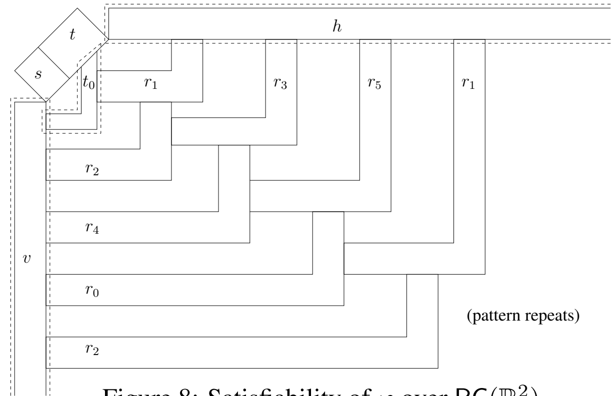


Figure 8: Satisfiability of φ over $\mathsf{RC}(\mathbb{R}^2)$.

Let φ_1 be the conjunction of the following formulas:

$$c(h) \qquad c(v) \qquad \neg C(v,h) \qquad c(h+t+s+v)$$
$$\neg C(s,\hat{h}) \qquad c((t+t_0)\cdot(-\hat{h})+v)$$
$$\neg C(\hat{v},\hat{t}) \qquad c((t_0+r_1)\cdot(-\hat{t})\cdot(-\hat{v})+h)$$
$$\neg C(\hat{t}_0,\hat{h}) \quad \neg C(t_0,s) \quad c((r_1+r_2)\cdot(-\hat{t}_0)\cdot(-\hat{h})+v).$$

Let φ_2 be the conjunction of the following formulas, for $i = 0, 2, 4$, and with arithmetic in the subscripts modulo 6:

$$\neg C(r_i,s) \qquad \neg C(r_i,t) \qquad \neg C(\hat{r}_i,\hat{h})$$
$$c((r_i+r_{i+1})\cdot(-\hat{r}_{i-1})\cdot(-\hat{v})+h).$$

Let φ_3 be the conjunction of the following formulas, for $i = 1, 3, 5$, with arithmetic in the subscripts modulo 6:

$$\neg C(r_i,s) \qquad \neg C(r_i,t) \qquad \neg C(\hat{r}_i,\hat{v})$$
$$c((r_i+r_{i+1})\cdot(-\hat{r}_{i-1})\cdot(-\hat{h})+v).$$

Let φ_4 be the conjunction of the following formulas:

$$\neg C(r_i,r_j), \qquad (0 \leq i < j < 6,\ |i-j| > 1),$$
$$\neg C(t_0,r_j), \qquad (2 \leq j < 6).$$

Let $\varphi = \bigwedge_{i=0}^4 \varphi_i$. We claim that the model in Fig. 8 satisfies φ. The truth of φ_0 and φ_1 is easily checked. For φ_2, note that, for i even, each component of r_i contacts v, but not h; on the other hand, the components of r_i together connect all the components of r_{i-1} to v (arithmetic modulo 6). The conjunct φ_3 is handled analogously. For φ_4, observe that the components of $r_0, \ldots r_5$ form a repeating pattern, with $r_i \cap r_j = \emptyset$ whenever i and j differ by more than 1. Thus, φ is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$.

We outline the proof that φ is not satisfiable over $\mathsf{RCP}(\mathbb{R}^2)$. Refer to Fig. 8, and pick any i $(0 \leq i < 6)$ and any component of r_i' of r_i. Let r_{i+1}' be the component of r_{i+1} which contacts r_i'. Draw a Jordan curve in $r_i' + r_{i+1}' + v + s + t + h$ enclosing t_0. By doing this for all r_i' and r_{i+1}', we obtain an infinite sequence $\{\gamma_{i,j}\}$ of nested Jordan curves $(0 \leq i < 6, 1 \leq j)$, with each $\gamma_{i,j}$ drawn in $r_i + r_{i+1} + v + s + t + h$. Suppose, then that φ is satisfied by any tuple of regions $x, \hat{x}$, for $x \in V$. On the assumption that these regions are in $\mathsf{RCP}(\mathbb{R}^2)$, it can be shown that just such a sequence of Jordan curves $\{\gamma_{i,j}\}$ must exist. But then each set $\mathbb{R}^2 \setminus (r_i + r_{i+1} + v + s + t + h)$ has infinitely many components, contradicting the supposition that the satisfying tuple is in $\mathsf{RCP}(\mathbb{R}^2)$. □

We turn now to complexity-theoretic issues, employing a surprising theorem on graph-drawing. Let $\mathbf{D}$ be the frame consisting of all regular closed subsets of $\mathbb{R}^2$ homeomorphic to closed discs. (It does not matter that $\mathbf{D}$ is not a Boolean algebra.) Then $Sat(\mathcal{RCC}8, \mathbf{D})$ is in NP (Schaefer, Sedgwick, and Štefankovič 2003). Using Theorem 6, we can show that $Sat(\mathcal{RCC}8c, \mathsf{RC}(\mathbb{R}^2))$ is also in NP. The following lemma enables us to reduce $Sat(\mathcal{RCC}8c, \mathsf{RCP}(\mathbb{R}^2))$ non-deterministically to $Sat(\mathcal{RCC}8, \mathbf{D})$.

Lemma 2. *Let φ be an $\mathcal{RCC}8c$-formula, and suppose φ is satisfied by bounded polygons $r_1, \ldots, r_n$. Then φ is satisfied by bounded polygons $r'_1, \ldots, r'_n$ such that: for all $i \leq n$, (a) if u is a connected component of r_i°, then $u^- \in \mathbf{D}$; and (b) $(r'_i)^\circ$ has at most $O(n^3)$ components.*

Theorem 9. *The problems $Sat(\mathcal{RCC}8c, \mathsf{RCP}(\mathbb{R}^2))$ and $Sat(\mathcal{RCC}8c^\circ, \mathsf{RCP}(\mathbb{R}^2))$ are both NP-complete.*

Proof. Suppose φ is an $\mathcal{RCC}8c$-formula with n variables. We describe an NP procedure for determining whether φ is satisfiable over $\mathsf{RCP}(\mathbb{R}^2)$. For each i ($1 \leq i \leq n$), take up to $O(n^3)$ fresh variables $t_{i,1}, \ldots, t_{i,m_i}$, and list all these variables as $t_1, \ldots, t_m$. For all i, j ($1 \leq i < j \leq m$), guess an $\mathcal{RCC}8$-relation R_{ij}, and let ψ be the conjunction of all the formulas $R_{ij}(t_i, t_j)$. By the result of (Schaefer, Sedgwick, and Štefankovič 2003), we check, in NP, that ψ is satisfiable over $\mathbf{D}$. Finally, we check, in deterministic polynomial time, that, if ψ is satisfied by the $t_1, \ldots, t_m$, then φ is satisfied by the regions $r_1, \ldots, r_n$, where $r_i = t_{i,1} + \cdots + t_{i,m_i}$. If both of these tests succeed, we report that ψ is satisfiable. It follows from Theorem 8 and Lemma 2 this procedure has a successful run if and only if φ is satisfied over $\mathsf{RCP}(\mathbb{R}^2)$ (and $\mathsf{RC}(\mathbb{R}^2)$). The case of $\mathcal{RCC}8c^\circ$ is handled similarly. □

The precise computational complexity of the languages $\mathcal{B}c$, $\mathcal{C}c$ and $\mathcal{C}c^\circ$ is not known; we have only the following:

Theorem 10. *$Sat(\mathcal{L}, \mathsf{RC}(\mathbb{R}^2))$ and $Sat(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^2))$ are all* EXPTIME*-hard, for $\mathcal{L} \in \{\mathcal{B}c, \mathcal{C}c, \mathcal{C}c^\circ\}$.*

Proof. The proof employs a technique developed in (Kontchakov et al. 2009, Theorem 5.9) For details, see `http://www.dcs.bbk.ac.uk/~roman`. □

Theorem 11. *$Sat(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^2))$ is* EXPTIME*-hard.*

Proof. See the proof of Theorem 15. □

At the time of writing, no non-trivial lower complexity bound is known for $Sat(\mathcal{B}c^\circ, \mathsf{RC}(\mathbb{R}^2))$. Moreover, no upper bound at all is known for these problems. Thus, we do not know whether $Sat(\mathcal{B}c, \mathsf{RC}(\mathbb{R}^2))$, $Sat(\mathcal{B}c, \mathsf{RCP}(\mathbb{R}^2))$, etc. are even decidable.

Three-dimensional Euclidean space

Languages based on $\mathcal{RCC}8$ cannot distinguish between Euclidean spaces of more than 3 dimensions. Indeed, they are even insensitive to the tameness of sets, and to the distinction between connectedness and interior-connectedness.

Theorem 12. *The problems $Sat(\mathcal{RCC}8c, \mathcal{K})$ are identical, where $\mathcal{K}$ is any of RC, $\mathsf{RC}(\mathbb{R}^n)$ or $\mathsf{RCP}(\mathbb{R}^n)$ for any $n \geq 3$. Further, for an $\mathcal{RCC}8c$-formula φ, let φ° be the result of replacing all occurrences of c with c°. Then φ is satisfiable over $\mathcal{K}$ if and only if φ° is.*

Proof. Follows from the observation (Renz 1998) that any satisfiable $\mathcal{RCC}8$-formula is satisfied by (interior-) connected polyhedra in $\mathbb{R}^n$, for $n \geq 3$. □

It is open whether $Sat(\mathcal{L}, \mathsf{RC}(\mathbb{R}^n)) = Sat(\mathcal{L}, \mathsf{RC}(\mathbb{R}^3))$, for $n > 3$, where $\mathcal{L}$ is any of $\mathcal{B}c$, $\mathcal{C}c$ or $\mathcal{C}c^\circ$. The best result we have is:

Theorem 13. *$Sat(\mathcal{C}c^\circ, \mathsf{RC}(\mathbb{R}^n)) \subsetneq Sat(\mathcal{C}c^\circ, \mathsf{RC})$ for $n \geq 1$.*

Proof. Recall that (3) can be invalidated in the torus (Fig. 3). To show it is valid over Euclidean spaces, we use the following fact (Newman 1951, p. 137): if $r_1, r_2 \in \mathsf{RC}(\mathbb{R}^n)$ are non-intersecting, and points p_1 and p_2 lie in the same component of $\mathbb{R}^n \setminus r_i = (-r_i)^\circ$ for $i = 1, 2$, then p_1 and p_2 lie in the same component of $\mathbb{R}^n \setminus (r_1 \cup r_2) = (-(r_1 + r_2))^\circ$. Thus, (3) is valid over $\mathsf{RC}(\mathbb{R}^n)$. □

In the case $\mathcal{L} = \mathcal{B}c^\circ$, however, we can give an answer. A *connected partition in* $\mathsf{RCP}(\mathbb{R}^n)$ is a tuple of non-empty, pairwise disjoint elements of $\mathsf{RCP}(\mathbb{R}^n)$, having connected interiors, which sum to the entire space. If $r_1, \ldots, r_n$ is a connected partition, its *neighbourhood graph* is the graph (V, E) with vertices $V = \{r_1, \ldots, r_n\}$ and edges $E = \{(r_i, r_j) \mid i \neq j \text{ and } (r_i + r_j)^\circ \text{ is connected}\}$.

Lemma 3. *Let G be a connected graph. Then G is (isomorphic to) to the neighbourhood graph of some connected partition in $\mathbb{R}^n$, $n \geq 3$. If G is also planar, it is the neighbourhood graph of some connected partition in $\mathbb{R}^2$.*

Proof. To prove the second statement, take a plane embedding H of G, and let H^* be its geometric dual (which we may draw with piecewise linear edges). The faces of H^* then form a connected partition in $\mathsf{RCP}(\mathbb{R}^2)$, and the geometric dual H^{**} of H^* is a drawing of the neighbourhood graph of this connected partition. Since H is connected, H^{**} is isomorphic to H, and hence to G. For the first statement, we proceed by induction on the number k of vertices of G. The case $k = 1$ is trivial. If $k > 1$, let $G' = G/e$ be the minor of G formed by collapsing some edge e of G into a single node. By inductive hypothesis, let $r_1, \ldots, r_{k-2}, r'_{k-1}$ be a connected partition in $\mathbb{R}^n$ whose neighbourhood graph is G', with the interior-connected polyhedron r'_{k-1} corresponding to the node e. It is routine to decompose r'_{k-1} into two interior-connected polyhedra r_{k-1} and r_k so that the neighbourhood graph of $r_1, \ldots, r_{k-1}, r_k$ is G. □

A *graph model* is a pair $\mathfrak{G} = (G, \sigma)$, where $G = (V, E)$ is a graph and σ is a function mapping any variable of $\mathcal{B}c^\circ$ to a subset of V. The function symbols $+$, $\cdot$ and $-$ are interpreted, respectively, as union, intersection and complement in the power-set algebra on V, and c° is interpreted as the property of graph-theoretic connectedness.

Lemma 4. *(i) A $\mathcal{B}c^\circ$-formula φ is satisfiable over $\mathsf{RCP}(\mathbb{R}^2)$ if and only if it is true in a connected planar graph model.*
(ii) A $\mathcal{B}c^\circ$-formula φ is satisfiable over $\mathsf{RCP}(\mathbb{R}^n)$, $n \geq 3$, if and only if it is true in a connected graph model.

Proof. We prove (*ii*); the proof of (*i*) is similar. Suppose φ is satisfiable over $\mathsf{RCP}(\mathbb{R}^n)$, $n \geq 3$. Let $\bar{s} = s_1, \ldots, s_k$ be a tuple of polyhedra satisfying φ, and $\bar{t}$ a connected partition in $\mathsf{RCP}(\mathbb{R}^2)$ such that, for all $i \leq k$, there exists a subset $R_i \subseteq \bar{t}$ of these elements such that $s_i = \sum R_i$. Let G be the neighbourhood graph of $\bar{t}$; and make G into a graph model $\mathfrak{G}$ by assigning to each variable x_i the set of nodes R_i. (Such a $\bar{t}$ always exists as long as the $\bar{s}$ are polyhedra.) Using Lemma 1, we can show that φ is true in $\mathfrak{G}$. Conversely, suppose φ is true in a connected graph model $\mathfrak{G} = (G, \sigma)$. By Lemma 3, let $\bar{r}$ be a connected partition in $\mathsf{RCP}(\mathbb{R}^n)$ whose neighbourhood graph is isomorphic to G. Taking the nodes of G to be the elements $\bar{r}$, we define a model over $\mathsf{RCP}(\mathbb{R}^n)$ as follows: if $\mathfrak{G}$ maps a variable x to the set of elements $R \subseteq \bar{r}$, interpret x as $\sum R$. One can check that φ is true in this model. ☐

Turning next to tameness, Theorem 12 has already shown that $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$ are not sensitive to the difference between $\mathsf{RC}(\mathbb{R}^n)$ and $\mathsf{RCP}(\mathbb{R}^n)$ for $n \geq 3$. By contrast:

Theorem 14. *$Sat(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^n)) \subsetneq Sat(\mathcal{B}c^\circ, \mathsf{RC}(\mathbb{R}^n))$ and $Sat(\mathcal{C}c^\circ, \mathsf{RCP}(\mathbb{R}^n)) \subsetneq Sat(\mathcal{C}c^\circ, \mathsf{RC}(\mathbb{R}^n))$, for all $n \geq 3$.*

Proof. By cylindrification of Fig. 4 and Lemma 1. ☐

For $n \geq 3$, the question of whether $Sat(\mathcal{L}, \mathsf{RC}(\mathbb{R}^n)) = Sat(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^n))$, where $\mathcal{L}$ is either $\mathcal{B}c$ or $\mathcal{C}c$, is open.

Finally, we address the complexity of satisfiability. As well as settling the insensitivity of $\mathcal{B}c^\circ$ to dimension ≥ 3 in Euclidean spaces, Lemma 4 gives us some complexity-theoretic information. Using (Kontchakov et al. 2009, Theorems 5.3 and 5.19), we can show

Lemma 5. *The problem of determining whether a $\mathcal{B}c^\circ$-formula has a graph-model is* EXPTIME*-complete. It is* EXPTIME*-hard to decide whether a $\mathcal{B}c^\circ$-formula has a planar graph-model.*

Theorem 15. *$Sat(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^3)) = Sat(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^n))$, for $n > 3$ and the problem is* EXPTIME*-complete.*
$Sat(\mathcal{B}c^\circ, \mathsf{RCP}(\mathbb{R}^2))$ is EXPTIME*-hard.*

Proof. Follows from Lemmas 4 and 5. ☐

This result is, however, in stark contrast to the following:

Theorem 16. *$Sat(\mathcal{B}c^\circ, \mathsf{RC})$ is* NP*-complete.*

Proof. See `http://www.dcs.bbk.ac.uk/~roman`. ☐

As shown in (Kontchakov et al. 2009), $Sat(\mathcal{L}, \mathsf{RC}(\mathbb{R}^n))$ and $Sat(\mathcal{L}, \mathsf{RCP}(\mathbb{R}^n))$ are EXPTIME-hard, for $\mathcal{L}$ any of $\mathcal{B}c$, $\mathcal{C}c$ or $\mathcal{C}c^\circ$ and $n \geq 3$. The upper complexity bounds for these problems are open.

Conclusions

We investigated the six languages $\mathcal{RCC}8c$, $\mathcal{B}c$, $\mathcal{C}c$, $\mathcal{RCC}8c^\circ$, $\mathcal{B}c^\circ$ and $\mathcal{C}c^\circ$ obtained by extending $\mathcal{RCC}8$ with the connectedness and interior connectedness predicates c and c°, as well as the Boolean function-symbols $+$, $\cdot$ and $-$, paying particular regard to issues that arise when interpreting these languages over low-dimensional Euclidean spaces. We showed that—in contrast to the less expressive $\mathcal{RCC}8$—the dimensionality of the space is important for all of our languages, and that, in addition, these languages exhibit varying patterns of sensitivity to tameness in different dimensions. Thus, $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$ both distinguish between $\mathsf{RC}(\mathbb{R}^n)$ and $\mathsf{RCP}(\mathbb{R}^n)$ for $n = 1$, but do not for $n \geq 2$; $\mathcal{B}c$ does for $n = 2$, but not for $n = 1$; $\mathcal{B}c^\circ$ does for $n = 2$, but not for $n = 1$ or $n \geq 3$; $\mathcal{C}c$ does for $n = 1$ and 2; $\mathcal{C}c^\circ$ does in all dimensions. We also obtained results on the complexity of reasoning in these logics. For example, the satisfiability problems for $\mathcal{RCC}8c$ and $\mathcal{RCC}8c^\circ$, under all the interpretations considered here, are NP-complete, whereas the corresponding problems for $\mathcal{B}c$, $\mathcal{B}c^\circ$, $\mathcal{C}c$ and $\mathcal{C}c^\circ$ in $\mathbb{R}^n$, for $n \geq 2$, are generally EXPTIME-hard. (Two cases are still open). A matching EXPTIME upper bound was proved for $\mathcal{B}c^\circ$ over polyhedra in $\mathbb{R}^n$, $n \geq 3$. The expressiveness and complexity problems that still remain open are indicated in Fig. 2.

Acknowledgments

The work on this paper was partially supported by the U.K. EPSRC research grants EP/E034942/1 and EP/E035248/1.

References

Bochnak, J.; Coste, M.; and Roy, M.-F. 1998. *Real Algebraic Geometry*. Springer.

Davis, E.; Gotts, N.; and Cohn A. 1999. Constraint networks of topological relations and convexity. *Constraints* 4:241–280.

Egenhofer, M., and Franzosa, R. 1991. Point-set topological spatial relations. *International J. of Geographical Information Systems* 5:161–174.

Kontchakov, R.; Pratt-Hartmann, I.; Wolter, F.; and Zakharyaschev, M. 2008. Topology, connectedness and modal logic. In *AiML 7*, 151–176. College Publications.

Kontchakov, R.; Pratt-Hartmann, I.; Wolter, F.; and Zakharyaschev, M. 2009. Spatial logics with connectedness predicate. Submitted; available at `http://www.dcs.bbk.ac.uk/~michael/connect-5.pdf`

Newman, M. 1951. *Elements of the Topology of Plane Sets of Points*. 2nd edition. Cambridge.

Pratt-Hartmann, I. 2007. First-order mereotopology. In Aiello, M.; Pratt-Hartmann, I.; and van Benthem, J., eds., *Handbook of Spatial Logics*. Springer. 13–97.

Randell, D.; Cui, Z.; and Cohn, A. 1992. A spatial logic based on regions and connection. In *Proc. of KR*, 165–176.

Renz, J., and Nebel, B. 1999. On the complexity of qualitative spatial reasoning. *Artificial Intelligence* 108:69–123.

Renz, J. 1998. A canonical model of the region connection calculus. In *Proc. of KR*, 330–341.

Schaefer, M.; Sedgwick, E.; and Štefankovič, D. 2003. Recognizing string graphs in NP. *J. of Computer and System Sciences* 67:365–380.

Wolter, F., and Zakharyaschev, M. 2000. Spatial reasoning in RCC-8 with Boolean region terms. In *Proc. of ECAI*, 244–248.

Proof of Theorem 3

Theorem 3. *Sat*$(\mathcal{B}c, \mathsf{RC}(\mathbb{R})) =$ *Sat*$(\mathcal{B}c, \mathsf{RCP}(\mathbb{R}))$.

Proof. Let a $\mathcal{B}c$-formula φ be given. We may assume without loss of generality that φ is of the form

$$(\rho = 0)\wedge \bigwedge_{1\le j\le m}(\sigma_j \neq 0)\wedge \bigwedge_{1\le i\le n}\big(c(\pi_i)\wedge(\pi_i \neq 0)\big)\wedge \bigwedge_{1\le k\le p}\neg c(\tau_k),$$

since, given any $\mathcal{B}c$-formula ψ, we may easily guess such a φ and show in polynomial time that φ and ψ are satisfiable over the same domains.

We describe a non-deterministic procedure which, given a formula φ of the form above, terminates with either success or failure in time bounded by a polynomial function of $|\varphi|$. We show that if the procedure has a successful run, then φ is satisfiable over $\mathsf{RCP}(\mathbb{R})$ and if φ is satisfiable over $\mathsf{RC}(\mathbb{R})$ then the procedure has a successful run.

Let $E = 2(m + n + 3p)$. Denote by Ξ the set of regular closed intervals $(-\infty, 0], [0, 1], \dots, [E-1, E], [E, +\infty)$ and by Δ the set of integers in the interval $[0, E]$. In what follows we construct a function λ, which maps Ξ to the power set of the set of subterms of φ. An interval $[a, b]$ with $a, b \in \Delta$ is regular closed if $b - a \ge 1$. We start of with $\lambda(I) = \emptyset$ for every $I \in \Xi$.

1. For every j $(1 \le j \le m)$, choose a regular closed interval $[a, b]$, $a, b \in \Delta$, and add σ_j to $\lambda(I)$, for each $I \in \Xi$ with $I \subseteq [a, b]$.
2. For every i $(1 \le i \le n)$, choose a regular closed interval $[a, b]$, $a, b \in \Delta$, and add π_i to $\lambda(I)$, for each $I \in \Xi$ with $I \subseteq [a, b]$ and add $-\pi_i$ to $\lambda(I)$, for each $I \in \Xi$ with $I \subseteq [0, a] \cup [b, E]$; if $a > 0$ add $-\pi_i$ to $\lambda((-\infty, 0])$, otherwise, add either π_i or $-\pi_i$ to $\lambda((-\infty, 0])$; if $b < E$ add $-\pi_i$ to $\lambda([E, +\infty))$, otherwise, add either π_i or $-\pi_i$ to $\lambda([E, +\infty))$;
3. For every k $(1 \le k \le p)$, choose a pair of regular closed intervals $[a_1, b_1]$ and $[a_2, b_2]$ with $a_1, b_1, a_2, b_2 \in \Delta$ such that $a_2 - b_1 \ge 1$ and add τ_k to $\lambda(I)$, for each $I \in \Xi$ with $I \subseteq [a_1, b_1] \cup [a_2, b_2]$, and add $-\tau_k$ to $\lambda(I)$, for each $I \in \Xi$ with $I \subseteq [b_1, a_2]$ (the latter is possible for at least one $I \in \Xi$).
4. For every $I \in \Xi$, guess a $\mathcal{B}c$-term ξ_I of the form $\prod_{1\le i\le \ell} \pm r_i$, where $r_1, \dots, r_\ell$ are all the variables of φ, and fail if $\xi_I \le \rho$ or $\xi_I \le -\prod \lambda(I)$. Succeed otherwise.

Suppose the procedure has a successful termination. Then we define an interpretation over $\mathsf{RCP}(\mathbb{R})$ by setting r to be the union of all the intervals $I \in \Xi$ with $\xi_I \le r$. Step 4 ensures that $\rho = 0$ and that, for every $I \in \Xi$, $I \subseteq \prod \lambda(I)$. Hence: Step 1 ensures that, for every j $(1 \le j \le m)$, σ_j is non-empty; Step 2 ensures that, for every i $(1 \le i \le n)$, π_i is connected and non-empty (note that π_i need not be bounded); Step 3 ensures that, for every k $(1 \le k \le p)$, τ_j is not connected. Thus, the interpretation is as required.

Conversely, if φ is true in a model over $\mathsf{RC}(\mathbb{R})$, it is easy to see how the intervals and the terms ξ_I may be selected so as to ensure successful termination of the procedure.

This shows that the both problems are in NP. □

Proof of Theorem 16

Theorem 16. *Sat*$(\mathcal{B}c^\circ, \mathsf{RC})$ *is* NP*-complete.*

Proof. Let φ be a $\mathcal{B}c^\circ$-formula of the form

$$(\rho = 0)\wedge \bigwedge_{1\le j\le m}(\sigma_j \neq 0)\wedge \bigwedge_{1\le i\le n}\big(c^\circ(\pi_i)\wedge(\pi_i \neq 0)\big)\wedge \bigwedge_{1\le k\le p}^{m}\neg c^\circ(\tau_k).$$

We show that (*i*) if φ is satisfiable over RC then it is satisfiable in a model $\mathfrak{A}$ over an Aleksandrov space with at most $2 \cdot 2^\ell + n$ points, where ℓ is the number of variables in φ; and (*ii*) how one can select a submodel $\mathfrak{B}$ of $\mathfrak{A}$, which contains at most $m + 2n + 2p + n \cdot p$ points and satisfies φ. Thus, we establish a polynomial finite model property for $\mathcal{B}c^\circ$ over RC, which gives us NP membership (NP-hardness is trivial).

(*i*) By a *type* ξ for φ we mean any term of the form $\prod_{1\le i\le \ell} \pm r_i$, where $r_1, \dots, r_\ell$ are all the variables of φ. Let W_0 contain a pair of distinct points x_ξ and x'_ξ, for each type ξ inconsistent with ρ, i.e., $\xi \not\models \rho$ (we need two points to make some regions disconnected). Let W_1 contain a distinct point z_{π_i} for each positive $c^\circ(\pi_i)$ in φ. Let R be the reflexive closure of

$$\{(z_{\pi_i}, x_\xi), (z_{\pi_i}, x'_\xi) \mid \xi \models \pi_i\}.$$

Define a valuation $\cdot^{\mathfrak{A}}$ by taking

$$r_i^{\mathfrak{A}} = \{x_\xi, x'_\xi \mid \xi \models r_i\} \cup \{z_{\pi_i} \mid \pi_i \models r_i\}.$$

It is readily seen that $r_i^{\mathfrak{A}}$ is a regular closed set in the Aleksandrov topology induced by (W, R).

We show now that φ is true in $\mathfrak{A}$. It is trivial for the first three conjuncts of φ. So, it remains to show that $\mathfrak{A} \not\models c^\circ(\tau_k)$, for each k $(1 \le k \le p)$. Suppose to the contrary that τ_k is interior-connected in $\mathfrak{A}$. Then there is a sequence $x_1, z_1, x_2, z_2, \cdots, x_s$ such that $z_i R x_i$, $z_i R x_{i+1}$ and the x_i and the z_i are *all* the points in $\tau_k^{\mathfrak{A}}$. Then, by the definition of $\mathfrak{A}$, there are $\pi_{i_1}, \dots, \pi_{i_{s-1}}$ such that $x_j, z_j, x_{j+1} \in \pi_{i_j}^{\mathfrak{A}} \cap \tau_k^{\mathfrak{A}}$, for all j $(1 \le j < s)$. It follows then that $z_{i_j} \in (\pi_{i_j}^\circ)^{\mathfrak{A}} \cap (\tau_k^\circ)^{\mathfrak{A}}$ and, as z_{i_j} is the R-predecessor of all points in $\pi_{i_j}^{\mathfrak{A}}$, we obtain $\pi_{i_j}^{\mathfrak{A}} \le \tau_k^{\mathfrak{A}}$. So, we have $(\sum_{1\le j<n} \pi_{i_j})^{\mathfrak{A}} \le \tau_k^{\mathfrak{A}}$ and $(\sum_{1\le j<n} \pi_{i_j})^{\mathfrak{A}}$ is interior-connected. On the other hand, as the path contains all points in $\tau_k^{\mathfrak{A}}$, we obtain $\tau_k^{\mathfrak{A}} \le (\sum_{1\le j<n} \pi_{i_j})^{\mathfrak{A}}$. Therefore, $\tau_k^{\mathfrak{A}}$ coincides with the sum of the π_{i_j} and so, is interior-connected contrary to φ being satisfiable.

(*ii*) We select the following points:

- for each j $(1 \le j \le m)$, pick $x \in W_0 \cap \sigma_j^{\mathfrak{A}}$;
- for each i $(1 \le i \le n)$, pick $x \in W_0 \cap \pi_i^{\mathfrak{A}}$ and $z_{\pi_i} \in W_1$;
- for each k $(1 \le k \le p)$, pick 2 points $x_{\tau_k}, x'_{\tau_k} \in W_0 \cap \tau_k^{\mathfrak{A}}$ form two distinct components of $\tau_k^{\mathfrak{A}}$ and up to n points $y_{\overline{\tau_k}, \pi_i} \in W_0 \cap \pi_i^{\mathfrak{A}} \cap (-\tau_k)^{\mathfrak{A}}$, for $1 \le i \le n$ (the point is picked if the set is not empty).

As φ is true in $\mathfrak{A}$, all the points mentioned above do necessarily exist (apart from the $y_{\overline{\tau_k}, \pi_i}$, some of which may not exist). Let V be the set of all these points and $\mathfrak{B}$ the restriction of $\mathfrak{A}$ onto V. We claim that φ is true in $\mathfrak{B}$. Indeed, this is clearly the case for the first three conjuncts of φ. So, it remains to show that $\mathfrak{B} \not\models c^\circ(\tau_k)$, for each k ($1 \leq k \leq p$). This fact follows from the observation that $z_{\pi_i} \in (\tau_k^\circ)^{\mathfrak{A}}$ if and only if $z_{\pi_i} \in (\tau_k^\circ)^{\mathfrak{B}}$, for each i ($1 \leq i \leq n$) and therefore, τ_k is inerior-connected in $\mathfrak{B}$ if and only if it is interior-connected in $\mathfrak{A}$.

This completes the proof. □

Proof of Theorem 8

Theorem 8. *Sat*$(\mathcal{C}c, \mathsf{RCP}(\mathbb{R}^2)) \subsetneq$ *Sat*$(\mathcal{C}c, \mathsf{RC}(\mathbb{R}^2))$. *In fact, Sat*$(\mathcal{B}c, \mathsf{RCP}(\mathbb{R}^2)) \subsetneq$ *Sat*$(\mathcal{B}c, \mathsf{RC}(\mathbb{R}^2))$.

Proof. Let V be the set of variables $\{v, h, s, t, t_0, r_0, r_1, r_2, r_3, r_4, r_5\}$ and, for any $x \in V$, let $\hat{x}$ be a fresh variable. For all $x \in V$, the region $\hat{x}$ is a slightly 'enlarged' version of x, with x lying in the interior of $\hat{x}$.

The formula φ we are about to construct contains conjuncts $\neg C(x, -\hat{x})$, for $x \in V$, which ensure that $\hat{x}$ represents a region whose interior contains x. The other conjuncts of φ will be presented as they are required in the proof. The claim that φ is not satisfiable over $\mathsf{RCP}(\mathbb{R}^2)$ is established by showing that any satisfying assignment in $\mathsf{RC}(\mathbb{R}^2)$ has the property that the interiors of the regions r_i ($0 \leq i < 6$) have infinitely many components. Suppose we have such a satisfying assignment.

$$(v \neq 0) \wedge (h \neq 0) \tag{5}$$
$$c(v) \wedge c(h) \tag{6}$$
$$c(h + t + s + v) \tag{7}$$
$$\neg C(v, h) \tag{8}$$
$$\neg C(t, v) \tag{9}$$
$$\neg C(s, \hat{h}) \tag{10}$$
$$c((t + t_0) \cdot (-\hat{h}) + v) \tag{11}$$
$$\neg C(t_0, s) \tag{12}$$

First stage: By (5), choose a point in v and a point in h and, by (7), connect the first to the second by an arc α_0^* in $v + s + t + h$. Let q_0 be the first point of α_0^* in h. By (8), $q_0 \notin v$ and q_0 is not the first point of α_0^*. Let p_0 be the last point of α_0^* in v and strictly before q_0. Let α_0 be the segment of α_0^* from p_0 to q_0. Hence, no interior point of α_0 lies in v or h, and so all points of α_0 lie in $s + t$.

By (9), some initial segment of α_0 lies in s, whence, by (10), there exists a point p_1^* on α_0 such that $p_1^* \notin s$ and $p_1^* \notin \hat{h}$. It follows that $p_1^* \in t \cdot (-\hat{h})$. By (11), draw an arc α_1^* from p_1^* to $p_0 \in v$, and lying in $(t + t_0) \cdot (-\hat{h}) + v$. Since $p_1^* \notin v$, let q_1 be the first point of v on α_1^* after p_1^*.

This arc has a last point of contact with α_0 strictly before q_1. For, all points of α_1^* before q_1 are in $t + t_0$, whence, by (9), some segment of α_1^* leading to $q_1 \in v$ must lie entirely in t_0. But then, by (12), this segment does not touch some initial segment of α_0. Therefore, let p_1 be the last point of α_1^* before q_1 lying on α_0. Let α_1 be the segment of α_1^* between p_1 and q_1. Thus, no point of α_1 lies in h,

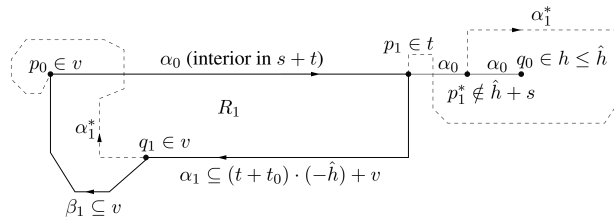


Figure 9: The first stage: the arc α_1 (solid line) is a sub-arc of α_1^*. The Jordan curve (thick lines) is denoted by γ_1; R_1 is the open set it bounds not containing q_0.

and the only point of α_1 lying in v is q_1. By (6), let β_1 be an arc in v from q_1 to p_0. Thus, $\beta_1 \cap \alpha_1 = \{q_1\}$ and $\beta_1 \cap \alpha_0 = \{p_0\}$. The situation is shown in Fig. 9. Let γ_1 be the Jordan curve formed by α_1, β_1 and the segment of α_0 lying between p_0 and p_1, shown in thick lines. We denote the open set bounded by γ_1 and not containing the point q_0 by R_1. By drawing the configuration on the closed plane (with all Jordan arcs avoiding the point at infinity), we may without loss of generality regard R_1 as the 'inside' of γ_1.

$$\neg C(\hat{t}, \hat{v}) \tag{13}$$
$$c((t_0 + r_1) \cdot (-\hat{t}) \cdot (-\hat{v}) + h) \tag{14}$$
$$\neg C(t_0 + r_1, s) \tag{15}$$

Second stage: Since $p_1 \in t$ and $q_1 \in v$, by (13), there exists a point p_2^* on α_1 such that $p_2^* \notin \hat{t}$ and $p_2^* \notin \hat{v}$. It follows that $p_2^* \in t_0 \cdot (-\hat{t}) \cdot (-\hat{v})$. By (14), let α_2^* be an arc from p_2^* to q_0 lying in $(t_0 + r_1) \cdot (-\hat{t}) \cdot (-\hat{v}) + h$. Let p_2 be the last point of α_2^* lying on α_1; hence, $p_2 \in t_0$. Also $p_2 \notin h$, since it lies on α_1. Let q_2 be the first point of α_2^* after p_2 lying in h; and let α_2 be the segment of α_2^* from p_2 to q_2. Thus, no point of α_2 other than q_2 lies in h, and certainly, no point of α_2 lies in v. By (6), let β_2 be an arc from q_2 to q_0 lying entirely in h. Notice that β_2 cannot intersect α_0, α_1 or α_2 except at the endpoints q_2 and q_0, because no other points of these arcs lie in h. We further claim that α_2 cannot enter region R_1, for it is impossible that any point in $(\alpha_2 \cup \beta_2) \setminus \{p_2\}$ lies on γ_1. To see this, note that: (i) by construction, no point of α_2 apart from p_2 lies on α_1, (ii) by (15) and the fact that $\alpha_2 \subseteq (t_0 + r_1) \cdot (-\hat{t})$, no point on α_2 apart from q_2 can lie in $s + t \supseteq \alpha_0$; (iii) no point of α_2 lies in $v \supseteq \beta_1$; (iv) no point of β_2 lies on γ_1, since no point on γ_1 lies in h. It follows that α_2 and β_2 lie on the 'outside' of γ_1 (since q_0 does). The situation is shown in Fig. 10. Thus, α_0, α_2 and β_2 divide the outside of γ_1 into two residual domains; denote that residual domain which does not contain p_0 by R_2. In addition, let γ_2 be the Jordan curve formed by α_2, β_2, α_0, β_1 and α_1 from q_1 to p_2 (shown in thick lines). By drawing the configuration on the closed plane (with all Jordan arcs avoiding the point at infinity), we may without loss of generality regard R_2 as lying 'inside' γ_2. Note that $S_2 = (R_1 \cup R_2)^{-\circ}$ is the bounded open set having γ_2 as its boundary.

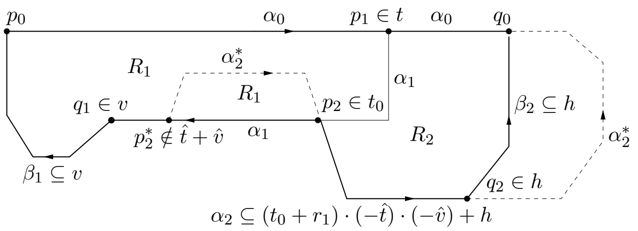


Figure 10: The second stage: the arc α_2 (solid line) is a sub-arc of α_2^*. The Jordan curve (thick lines) is denoted by γ_2. The region $S_2 = (R_1 \cup R_2)^{-\circ}$ is the bounded open set whose boundary is γ_2.

$$\neg C(\hat{t}_0, \hat{h}) \tag{16}$$

$$c((r_1 + r_2) \cdot (-\hat{t_0}) \cdot (-\hat{h}) + v) \tag{17}$$

$$\neg C(r_1 + r_2, t) \tag{18}$$

$$\neg C(r_1 + r_2, s + t) \tag{19}$$

Third stage: Since $p_2 \in t_0$ and $q_2 \in h$, by (16), there exists a point p_3^* on α_2 such that $p_3^* \notin \hat{t}_0$ and $p_3^* \notin \hat{h}$. It follows that $p_3^* \in r_1 \cdot (-\hat{t}_0) \cdot (-\hat{h})$. By (17), let α_3^* be an arc from p_3^* to q_1 lying in $(r_1 + r_2) \cdot (-\hat{t}_0) \cdot (-\hat{h}) + v$. Let p_3 be the last point of α_3^* lying on α_2; hence, p_3 lies in r_1. Let q_3 be the first point of α_3^* after p_3 lying in v; and let α_3 be the segment of α_3^* from p_3 to q_3. Thus, no point of α_3 other than q_3 lies in v, and certainly, no point of α_3 lies in h. By (6), let β_3^* be an arc from q_3 to q_1 lying entirely in v. Let q_3^* be the first point of β_3^* lying on $\gamma_2 \cap v$ (i.e. lying on β_1). Let β_3 be the segment of β_3^* between q_3 and q_3^*. The situation is shown in Fig. 11.

We need to show that the way in which α_3 and β_3 have been drawn is sufficiently general. Recall that that R_1 is bounded by the Jordan curve γ_1, and that $S_2 = (R_1 \cup R_2)^{\circ -}$ is bounded by the Jordan curve γ_2. We first establish that α_3 cannot enter R_1. For: (i) all points of α_1 are in $t_0 + t$, and so by (18) cannot coincide with any point in $(r_1 + r_2) \cdot (-\hat{t}_0)$; (ii) all points of β_3 are in v, and q_1 is the only point of α_1 in v; (iii) $\alpha_3 \setminus \{q_3\}$ has no points in v, and hence none on β_1; (iv) by construction, β_3 stops as soon as it touches γ_2; (v) by (19), no points of $(r_1 + r_2)$ can coincide with any points of $s + t \supseteq \alpha_0$; (vi) no points of $\alpha_0 \setminus \{p_0\}$ lie in v and hence none lie on β_3. We next establish that α_3 cannot enter R_2, either. For, (i) by construction, no point of α_3 apart from the first, can intersect α_2, (ii) by (19) and (18), no points of $(r_1 + r_2)$ can coincide with any points of $s + t \supseteq \alpha_0$; (iii) no point of α_3 lies in h, and hence none lies on β_2. Thus, α_3 cannot enter S_2; and α_3 and β_3 divide the exterior of γ_2 into two regions, forming a larger Jordan curve γ_3, shown by the thick lines in Fig. 11. By inspection, exactly one of these two regions will contain points in v. Drawing the configuration on the closed plane as before, we may without loss of generality regard the region containing points of v as the 'outside' of γ_3. The region $S_3 = (S_2 \cup R_3)^{-\circ}$ is thus the interior of γ_3.

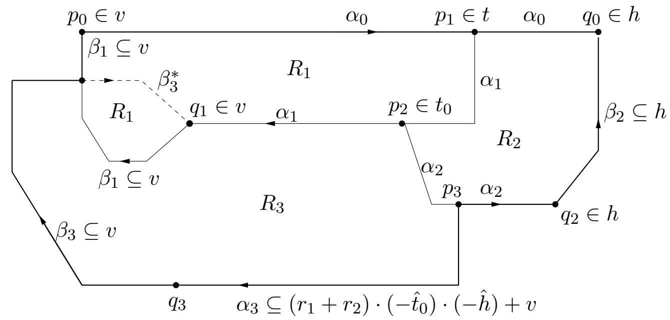


Figure 11: The third stage: the point p_3^* and arc α_3^* are not shown, for clarity; the arc β_3 (solid line) is a sub-arc of β_3^*. The Jordan curve (thick lines) is denoted by γ_3. The region $S_3 = (S_2 \cup R_3)^{-\circ}$ is the bounded open set whose boundary is γ_3.

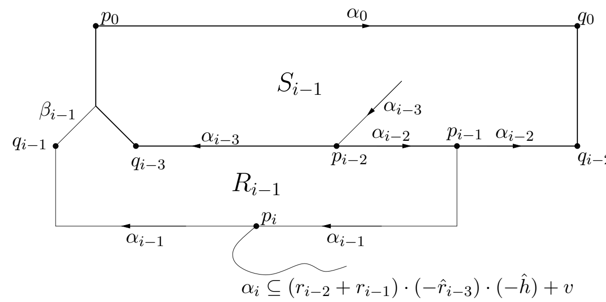


Figure 12: The general case: the arc α_i (i even).

$$\neg C(\hat{r}_{i-3}, \hat{v}) \quad (i \text{ even}) \tag{20}$$

$$\neg C(\hat{r}_{i-3}, \hat{h}) \quad (i \text{ odd}) \tag{21}$$

$$c((r_{i-2} + r_{i-1}) \cdot (-\hat{r}_{i-3}) \cdot (-\hat{v}) + h) \quad (i \text{ even}) \tag{22}$$

$$c((r_{i-2} + r_{i-1}) \cdot (-\hat{r}_{i-3}) \cdot (-\hat{h}) + v) \quad (i \text{ odd}) \tag{23}$$

$$\neg C(r_{i-2} + r_{i-1}, r_{i-4}) \tag{24}$$

$$\neg C(r_{i-2} + r_{i-1}, r_{i-4} + r_{i-5}) \tag{25}$$

$$\neg C(r_{i-2} + r_{i-1}, t + t_0) \quad (i = 4) \tag{26}$$

$$\neg C(r_{i-2} + r_{i-1}, s + t) \tag{27}$$

General stage i: After this point, the process repeats itself through infinitely many stages; at each new stage, the numerical indices in the variables r_i are incremented (modulo 6) and h and v are transposed. The general situation (for i even), is illustrated in Fig. 12. In this stage, α_i and β_i are about to be constructed. The arc $\alpha_i \subseteq (r_{i-1} + r_i) \cdot (-\hat{r}_{i-2}) \cdot (-\hat{v}) + h$ will run from a point p_i on α_{i-1} to a point q_i in h (the starting point p_i^* exists by (20) and the arc by (22)); and the arc $\beta_i \subseteq h$, will run from q_i to some point on $\gamma_{i-1} \cap h = \gamma_{i-2} \cap h$. The Jordan curve γ_{i-2}, enclosing S_{i-2}, is shown in thick lines.

The key observation is that neither α_i nor β_i can enter $S_{i-1} = (S_{i-2} \cup R_{i-1})^{-\circ}$. We first show that α_i cannot enter S_{i-2}. To see this, we note that: (i) since $\alpha_i \subseteq (r_{i-2} + r_{i-1}) \cdot (-\hat{r}_{i-3})$ and $\alpha_{i-2} \subseteq (r_{i-4} + r_{i-3})$, by (24), $\alpha_i \cap \alpha_{i-2} = \emptyset$; (ii) since $\alpha_i \subseteq (r_{i-1} + r_{i-2})$ and $\alpha_{i-3} \subseteq (r_{i-5} +$

r_{i-4}), by (25), $\alpha_i \cap \alpha_{i-3} = \emptyset$; (iii) since $\alpha_i \subseteq -\hat{v}$, α_i does not intersect the segment of γ_{i-2} from q_{i-3} (clockwise) to p_0; (iv) since $\alpha_i \subseteq (r_{i-2} + r_{i-1})$ and $\alpha_0 \subseteq s + t$, by (27), $\alpha_{i+1} \cap \alpha_0 = \emptyset$; (v) only the end-point q_i of α_i is in h and so α_i cannot intersect the segment of γ_{i-2} from q_0 (clockwise) to q_{i-2}. We note here that, for $i = 4$, we use (26) instead of (24) and (25) to show (i) and (ii).

We next show that α_i cannot enter R_{i-1}. To see this, we note that: (i) by construction, $\alpha_i \cap \alpha_{i-1} = \{p_i\}$; (ii) α_i cannot enter S_{i-2}, as we have just argued, and $\beta_i \subseteq h$ certainly cannot cross the boundary between R_{i-1} and S_{i-2}, none of whose points are in h by construction of $\alpha_{i-2}, \alpha_{i-3}$, and by (8); (iii) $\alpha_i \subseteq -\hat{v}$ and so cannot intersect the segment of the boundary of R_{i-1} from q_{i-1} (clockwise) to the point where β_{i-1} reaches γ_{i-2}; β_i cannot intersect this segment either, by (8). But this means that it is impossible for α_i and β_i to connect a point of R_{i-1} to a point of $\gamma_{i-1} \cap h$; hence α_i cannot enter R_{i-1} at all, and so must be drawn (in the closed plane) as shown.

To complete the proof, observe that, for all $k > 0$, $p_{6k+2} \in r_0$; and since r_0 is regular closed, there exist points in the interior of r_0 arbitrarily close to p_{6k+2}. But p_{6k+2} and $p_{6(k+1)+2}$ are separated by (for example) γ_{6k+6}, which lies entirely in the set $s + t + h + v + r_3 + r_4 + r_5$, and hence contains no interior points of r_0. Therefore, r_0° has infinitely many components, as required.

For the second statement of the theorem, we replace every literal $\neg C(x, y)$ in φ with a conjunction

$$(x \leq x^\dagger) \wedge (y \leq y^\dagger) \wedge c(x^\dagger) \wedge c(y^\dagger) \wedge \neg c(x^\dagger + y^\dagger),$$

where the variables $x^\dagger$ and $y^\dagger$ are chosen afresh for each replaced literal. Let the resulting $\mathcal{B}c$-formula be ψ. Trivially, $\mathsf{RCP}(\mathbb{R}^2) \models \psi \to \varphi$, so that ψ is not satisfiable over $\mathsf{RCP}(\mathbb{R}^2)$. That ψ is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$ is almost immediate by inspection of Fig. 8. □

Proof of Theorem 6

Theorem 6. *If an $\mathcal{RCC}8c$- or $\mathcal{RCC}8c^\circ$-formula is satisfiable over $\mathsf{RC}(\mathbb{R}^2)$, then it can be satisfied over the frame of bounded regular closed polygons. In consequence:*

$$Sat(\mathcal{RCC}8c, \mathsf{RC}(\mathbb{R}^2)) = Sat(\mathcal{RCC}8c, \mathsf{RCP}(\mathbb{R}^2)),$$
$$Sat(\mathcal{RCC}8c^\circ, \mathsf{RC}(\mathbb{R}^2)) = Sat(\mathcal{RCC}8c^\circ, \mathsf{RCP}(\mathbb{R}^2)).$$

Proof. For the first statement, it suffices to construct, for any tuple $r_1, \ldots, r_n$ in $\mathsf{RC}(\mathbb{R}^2)$, a corresponding tuple $p_1, \ldots, p_n$ in $\mathsf{RCP}(\mathbb{R}^2)$ satisfying exactly the same atomic $\mathcal{RCC}8c$-formulas. We may assume that the r_i are distinct and non-empty. By reordering the r_i if necessary, we can ensure that $r_i \subseteq r_j$ implies $i \leq j$. For all i, j $(1 \leq i < j \leq n)$, let $R_{ij} \in \{\mathsf{DC}, \mathsf{EC}, \mathsf{PO}, \mathsf{TPP}, \mathsf{NTPP}\}$ be the unique relation such that $R_{ij}(r_i, r_j)$.

Step 1: We construct regular closed sets $r_1^+, \ldots, r_n^+$ such that, for all j $(1 \leq j \leq n)$, $r_j \subseteq (r_j^+)^\circ$,

$$\text{if } r_j \text{ is connected then } (r_j^+)^\circ \text{ is connected,} \tag{28}$$
$$\text{if } r_j \cap r_{j'} = \emptyset \quad \text{then} \quad r_j^+ \cap r_{j'}^+ = \emptyset, \qquad \text{for all } j, j'. \tag{29}$$

Lemma 6. *There are $r_1^+, \ldots, r_n^+$ in $\mathsf{RC}(\mathbb{R}^2)$ with* (28)–(29).

Proof. By the normality of $\mathbb{R}^2$, let $s_1, \ldots, s_n$ be closed sets such that $r_j \subseteq s_j^\circ$, for all $j \leq n$, and if $r_j \cap r_{j'} = \emptyset$ then $s_j \cap s_{j'} = \emptyset$, for all $j, j' \leq n$. Fix any r_j. For all $u \in r_j$, let d_u be a connected and regular closed subset of s_j with $u \in d_u^\circ$ (which is possible because $\mathbb{R}^2$ is locally connected). Let $r_j^+ = \sum_{u \in r_j} d_u$. By construction, $r_j \subseteq (r_j^+)^\circ$ and (29). To show (28), consider $t_j = \bigcup_{u \in r_j} d_u^\circ$. Since $\bigcup_{u \in r_j} (d_u^\circ \cup r_j) = t_j$, the set t_j is connected whenever r_j is. Clearly, $t_j \subseteq (r_j^+)^\circ$. On the other hand, t_j^- is regular closed and $t_j^- \supseteq \sum_{u \in r_j} d_u^{\circ -} = r_j^+$. Thus, $t_j \subseteq (r_j^+)^\circ \subseteq r_j^+ \subseteq t_j^-$, which means that if r_j is connected then $(r_j^+)^\circ$ is sandwiched between a connected set t_j and its closure t_j^-, and hence is itself connected; cf. (28). □

Step 2: For all i, j $(1 \leq i < j \leq n)$, pick points o, o', o'' satisfying the conditions:

- if $R_{ij} = \mathsf{EC}$, then $o \in \delta r_i \cap \delta r_j$;
- if $R_{ij} = \mathsf{PO}$, then $o \in r_i^\circ \cap r_j^\circ$, $o' \in r_i^\circ \setminus r_j$, $o'' \in r_j^\circ \setminus r_i$;
- if $R_{ij} = \mathsf{TPP}$, then $o \in \delta r_i \cap \delta r_j$ and $o' \in r_j^\circ \setminus r_i$;
- if $R_{ij} = \mathsf{NTPP}$, then $o \in r_j^\circ \setminus r_i$.

And for all j $(1 \leq j \leq n)$, pick points o, o' satisfying the condition that, if r_j is not connected, then o and o' lie in different components of r_j. Enumerate the chosen (distinct) points as $o_1, \ldots, o_m$: we call them *witness points*. We can draw disjoint closed disks $d_1, \ldots, d_m$, centred on the respective witness points o_k, such that, for all $j \leq n$ and $k \leq m$:

$$\text{if } o_k \in r_j^\circ \text{ then } d_k \subseteq r_j^\circ, \tag{30}$$
$$\text{if } o_k \in (r_j^+)^\circ \text{ then } d_k \subseteq (r_j^+)^\circ, \tag{31}$$
$$\text{if } (r_j^+)^\circ \text{ is connected then so is } (r_j^+)^\circ \setminus d, \tag{32}$$

where

$$d = \bigcup_{k=1}^{m} d_k.$$

This can be done because, if s is a connected, open subset of $\mathbb{R}^2$ and $u \in s$, then there exists a closed disc d such that $u \in d \subseteq s$ and $s \setminus d$ is connected.

Step 3: We now begin the construction of the $p_1, \ldots, p_n$. First, for each set r_j and each witness point o_k, we select a polygon $w_{k,j}$ such that

$$w_{k,j} \subseteq d_k. \tag{33}$$

We refer to the $w_{k,j}$ as *wedges*: for each $j \leq n$, and each $k \leq m$ we will make $w_{k,j}$ part of p_j. Wedges are selected as follows.

(*i*) If $o_k \in \delta r_j$, pick a point $q_{k,j} \in \delta d_k \subseteq (r_j^+)^\circ$ and let $w_{k,j}$ be a lozenge within d_k such that $o_k, q_{k,j} \in \delta w_{k,j}$; see Fig. 6a. We may pick the $q_{k,j}$ to be distinct, and construct the $w_{k,j}$ so that no two such $w_{k,j}$ have intersecting interiors.

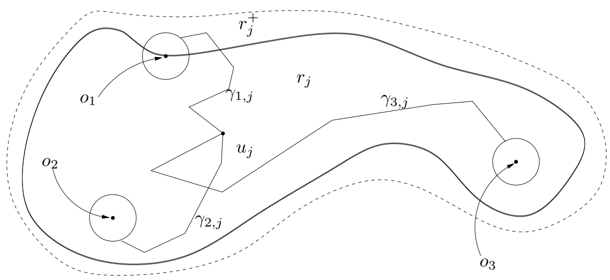


Figure 13: Disposition of the various arcs involving a particular connected element r_j. The outline of r_j is indicated with thickened lines, and its including region r_j^+ with a dotted line. In this example, r_j involves three witness points: o_1 (lying on δr_j) and o_2, o_3 (lying in r_j°). Notice that the arcs $\gamma_{k,j}$ are contained within the larger set r_j^+, and not necessarily within r_j.

(*ii*) If $o_k \in r_j^\circ$, pick a point $q_{k,j} \in \delta d_k \subseteq r_j^\circ$ and let $w_{k,j}$ be a lozenge within d_k such that $o_k \in (w_{k,j})^\circ$, $q_{k,j} \in \delta w_{k,j}$. Again, we may pick the $q_{k,j}$ to be distinct from each other and from the $q_{k,j}$ selected in (*i*); see Fig. 6b.

(*iii*) Otherwise, i.e., if $o_k \notin r_j$, let $w_{k,j} = \emptyset$.

The wedges $w_{k,j}$ will ensure that $p_1, \ldots, p_n$ contain certain witness points required for the satisfaction of the relevant atomic $\mathcal{RCC}8c$-formulas. For example, if $\mathsf{PO}(r_i, r_j)$, there will exist a witness point $o_k \in r_i^\circ \cap r_j^\circ$; but then $o_k \in w_{k,i}^\circ \cap w_{k,j}^\circ$, whence $o_k \in p_i^\circ \cap p_j^\circ$, which is required to ensure that $\mathsf{PO}(p_i, p_j)$.

It is also obvious that the $w_{k,j}$ may be constructed so that distinct $w_{k,j}$ and $w_{k',j'}$ share no bounding line segments. This condition is important in view of the following simple observation about sums of regular closed polygons:

Lemma 7. *Suppose that $s_1, \ldots, s_k$ are elements of $\mathsf{RCP}(\mathbb{R}^2)$ no two of which have any line segment common to their boundaries. Then $(\sum_{1 \le i \le k} s_i)^\circ = \bigcup_{1 \le i \le k} s_i^\circ$.*

Step 4: We must take steps now to connect up the wedges corresponding to the connected regions. For each connected set r_j ($1 \le j \le n$), we do the following:

- Pick a point $u_j \in (r_j^+)^\circ \setminus d$ and, for every $k \le m$, select a piecewise-linear arc $\gamma_{k,j} \subseteq (r_j^+)^\circ \setminus d$ that connects $q_{k,j}$ with u_j.

This is possible because, by (28) and (32), $(r_j^+)^\circ \setminus d$ is connected and $q_{k,j} \in (r_j^+)^\circ \cap \delta d$. Fig. 13 illustrates the disposition of the various arcs $\gamma_{k,j}$ for a particular connected element r_j. We may evidently assume without loss of generality that, if $\gamma_{k,j}$ and $\gamma_{k',j'}$ are defined and distinct, then $\gamma_{k,j}$ and $\gamma_{k',j'}$ intersect at a finite number of points—i.e., do not have any line segments in common.

Step 5: For all j ($1 \le j \le n$), denote

$$s_j \quad = \bigcup_{\substack{j' < j \text{ and} \\ \mathsf{TPP}(r_{j'}, r_j) \text{ or } \mathsf{NTPP}(r_{j'}, r_j)}} \quad \bigcup_{1 \le k \le m} \big(w_{k,j'} \cup \gamma_{k,j'}\big).$$

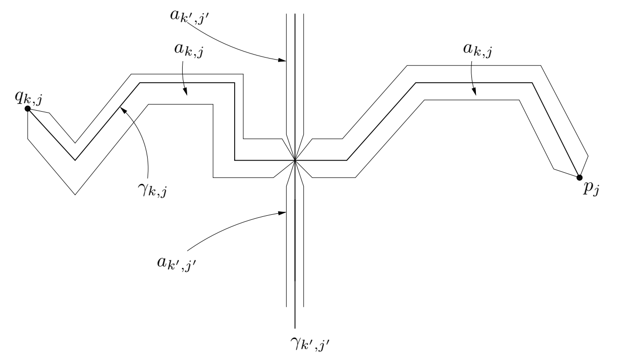


Figure 14: Illustration of the region $a_{k,j}$ surrounding the arc $\gamma_{k,j}$, where $\gamma_{k,j}$ is intersected once by another arc $\gamma_{k',j'}$.

This set is closed and semi-linear. So, therefore, are all of its finitely many components. For each of the connected components s_j^i of s_j, we may construct a connected, regular closed polygon z_j^i such that $s_j^i \subseteq (z_j^i)^\circ$ and

the z_j^i are pairwise disjoint, for fixed j.

Denote $z_j = \sum_i z_j^i$.

Step 6: It is easy to see (Fig. 14) that, for every $\gamma_{k,j}$ defined above, we can construct a regular closed polygon $a_{k,j}$, such that the following conditions are satisfied:

$$\gamma_{k,j} \subseteq a_{k,j} \subseteq \mathbb{R}^2 \setminus d^\circ, \tag{34}$$

$$\text{every } p \in \gamma_{k,j} \cap \delta a_{k,j} \text{ is either an endpoint of } \gamma_{k,j} \text{ or one of the (finitely many) points in } \gamma_{k,j} \cap \gamma_{k',j'}, \tag{35}$$

$$\text{if } \gamma_{k,j} \cap r_i = \emptyset \text{ then } a_{k,j} \cap r_i = \emptyset, \quad \text{for all } i \le n, \tag{36}$$

$$\text{if } \gamma_{k,j} \subseteq (r_i^+)^\circ \text{ then } a_{k,j} \subseteq (r_i^+)^\circ, \quad \text{for all } i \le n, \tag{37}$$

$$\text{if } \gamma_{k,j} \subseteq z_{j'}^\circ \text{ then } a_{k,j} \subseteq z_{j'}^\circ, \quad \text{for all } j' \le j, \tag{38}$$

$$\text{if } \gamma_{k,j} \ne \gamma_{k',j'} \text{ then } (a_{k,j})^\circ \cap (a_{k',j'})^\circ = \emptyset. \tag{39}$$

If the arc $\gamma_{k,j}$ is not defined, set $a_{k,j} = \emptyset$. The $a_{k,j}$ can also be selected so that, $w_{k,j} + a_{k,j}$ and $w_{k',j'} + a_{k',j'}$ share no line segments on their boundaries, for distinct pairs (k, j) and (k', j'). This fact is significant in view of Lemma 7.

Step 7: Now define, inductively, the sequences $p_1, \ldots, p_n$ and $p_1^+, \ldots, p_n^+$ as follows. Suppose that $p_1, \ldots, p_{j-1}$ and $p_1^+, \ldots, p_{j-1}^+$ have been defined. Let

$$p_j \;=\; b_j + \sum_{\substack{j' < j \\ \mathsf{TPP}(r_{j'}, r_j)}} p_{j'} \;+ \sum_{\substack{j' < j \\ \mathsf{NTPP}(r_{j'}, r_j)}} p_{j'}^+, \tag{40}$$

where

$$b_j \;=\; \sum_{1 \le k \le m} (w_{k,j} + a_{k,j}), \tag{41}$$

and let p_j^+ be a regular closed polygon such that $p_j \subseteq (p_j^+)^\circ$

and

$$\text{every component of } p_j^+ \text{ includes a component of } p_j, \quad (42)$$

$$\text{if } p_j \cap p_{j'}^+ = \emptyset \text{ then } p_j^+ \cap p_{j'}^+ = \emptyset, \qquad \text{for } j' < j, \quad (43)$$

$$\text{if } p_j \cap b_i = \emptyset \text{ then } p_j^+ \cap b_i = \emptyset, \qquad \text{for } j < i \leq n, \quad (44)$$

$$\text{if } o_k \notin p_j \text{ then } o_k \notin p_j^+, \qquad \text{for } k \leq m, \quad (45)$$

$$\text{if } p_j \cap \delta z_{j'} = \emptyset \text{ then } p_j^+ \cap \delta z_{j'} = \emptyset, \qquad \text{for } j' \leq j. \quad (46)$$

The polygon p_j^+ can also be chosen so that it shares no line segment with the boundary of any of the sets $p_1, \ldots, p_j$, $p_1^+, \ldots, p_{j-1}^+$. Again, this fact is significant in view of Lemma 7. Since all the regions concerned are bounded polygons, the existence of the p_j^+ is unproblematic.

This concludes the construction of the bounded regular closed polygons $p_1, \ldots, p_n$. We now proceed to show that $p_1, \cdots, p_n$ satisfy the same atomic $\mathcal{RCC}8c$-formulas as the given formula. First, we establish a series of lemmas.

Lemma 8. *For all j and k ($1 \leq j \leq n$, $1 \leq k \leq m$),*
(i) $o_k \in r_j$ if and only if $o_k \in p_j$ and
(ii) $o_k \in r_j^\circ$ if and only if $o_k \in p_j^\circ$.

Proof. (i) If $o_k \in r_j$, then, by construction of $w_{k,j}$, we have $o_k \in w_{k,j} \subseteq b_j \subseteq p_j$. For the converse direction, we first observe that

$$\text{if } o_k \notin r_j \quad \text{then} \quad o_k \notin b_j, \quad \text{for all } j \leq n. \quad (47)$$

Indeed, by (34), $o_k \notin a_{k',j}$, for each k'. By construction, $o_k \notin r_j$ implies $w_{k,j} = \emptyset$ and, for each $k' \neq k$, we have $o_k \notin d_{k'}$, whence, by (33), $o_k \notin w_{k',j}$.

Then we proceed to show (i) by induction on j. The case $j = 1$ is immediate from (47). Let $j > 1$ and $o_k \notin r_j$. By (47), $o_k \notin b_j$ and, for all $j' < j$ with $\mathsf{TPP}(r_{j'}, r_j)$ or $\mathsf{NTPP}(r_{j'}, r_j)$, we have $o_k \notin r_{j'}$, whence, by IH, $o_k \notin p_{j'}$ and, by (45), $o_k \notin p_{j'}^+$. By (40), $o_k \notin p_j$.

(ii) If $o_k \in r_j^\circ$, then, by construction of $w_{k,j}$, we obtain $o_k \in (w_{k,j})^\circ \subseteq b_j^\circ \subseteq p_j^\circ$. For the converse direction, we first show that

$$\text{if } o_k \notin r_j^\circ \quad \text{then} \quad o_k \notin b_j^\circ, \quad \text{for all } j \leq n. \quad (48)$$

In view of Lemma 7, it suffices to prove that $o_k \notin (w_{k',j})^\circ$ and $o_k \notin (a_{k',j})^\circ$, for each k'. The latter is immediate from (34). The former holds because $o_k \notin r_j^\circ$ implies $o_k \notin (w_{k,j})^\circ$ and, for each $k' \neq k$, we have $o_k \notin d_{k'}$, whence, by (33), $o_k \notin (w_{k',j})^\circ$.

Then we proceed to show (ii) by induction on j. The case $j = 1$ is immediate from (48). Let $j > 1$ and $o_k \notin r_j^\circ$. By (48), $o_k \notin b_j^\circ$. For all $j' < j$ with $\mathsf{TPP}(r_{j'}, r_j)$, we have $o_k \notin r_{j'}^\circ$, whence, by IH, $o_k \notin p_{j'}^\circ$. For all $j' < j$ with $\mathsf{NTPP}(r_{j'}, r_j)$, we have $o_k \notin r_{j'}$, whence, by (i), $o_k \notin p_j$ and, by (45), $o_k \notin (p_{j'}^+)^\circ$. It follows from (40) and Lemma 7 that $o_k \notin p_j$. □

Lemma 9. *For all i, j ($1 \leq i, j \leq n$), if $r_i \cap r_j = \emptyset$ then $p_i^+ \cap p_j^+ = \emptyset$.*

Proof. First, we need to prove the following two statements:

$$\text{if } r_i \cap r_j = \emptyset \text{ then } b_i \cap b_j = \emptyset, \quad \text{for } i, j \leq n; \quad (49)$$

$$\text{if } r_j \cap r_i = \emptyset \text{ then } p_j^+ \cap b_i = \emptyset, \quad \text{for } j < i \leq n. \quad (50)$$

Observe that $a_{k,j} + w_{k,j} \subseteq (r_j^+)^\circ$, for all $k \leq m$: indeed, if $a_{k,j} \neq \emptyset$ then $\gamma_{k,j}$ is defined and $\gamma_{k,j} \subseteq (r_j^+)^\circ$, whence, by (37), $a_{k,j} \subseteq (r_j^+)^\circ$; further, if $w_{k,j} \neq \emptyset$ then $o_k \in r_j$ whence, by (31), $w_{k,j} \subseteq d_k \subseteq (r_j^+)^\circ$. Now, if $r_i \cap r_j = \emptyset$ then, by (29), $r_i^+ \cap r_j^+ = \emptyset$ and thus (49).

For each i, we prove (50) by induction on j. The basis of induction, $j = 1$, follows from (49) and (44) as $p_1 = b_1$. For the induction step, let $j > 1$ and $r_j \cap r_i = \emptyset$. By (49), $b_j \cap b_i = \emptyset$. For each $j' < j$ with $\mathsf{TPP}(r_{j'}, r_j)$ or $\mathsf{NTPP}(r_{j'}, r_j)$, we have $j' < j < i$ and thus $r_{j'} \cap r_i = \emptyset$, whence, by IH, $p_{j'}^+ \cap b_i = \emptyset$. Therefore, by (40), $p_j \cap b_i = \emptyset$ and, by (44), $p_j^+ \cap b_i = \emptyset$.

Finally, we prove the statement of the lemma: we show by induction on i that for all $j, j' \leq i$, if $r_{j'} \cap r_j = \emptyset$ then $p_{j'}^+ \cap p_j^+ = \emptyset$. The case $i = 1$ is trivial. Let $i > 1$ and $r_{j'} \cap r_j = \emptyset$. We have to consider only the case $j' < j = i$ (other cases are either immediate from IH or mirror image of this case). By (50), $p_{j'}^+ \cap b_i = \emptyset$. For each $j'' < i$ with $\mathsf{TPP}(r_{j''}, r_i)$ or $\mathsf{NTPP}(r_{j''}, r_i)$, we have $r_{j'} \cap r_{j''} = \emptyset$, whence, by IH, $p_{j'}^+ \cap p_{j''}^+ = \emptyset$. By (40), $p_{j'}^+ \cap p_i = \emptyset$, and thus, by (43), $p_{j'}^+ \cap p_i^+ = \emptyset$. □

Lemma 10. *For all i, j ($1 \leq i, j \leq n$), if $r_i^\circ \cap r_j^\circ = \emptyset$ then $p_i^\circ \cap p_j^\circ = \emptyset$.*

Proof. First, we need to prove the following two statements:

$$\text{if } r_i^\circ \cap r_j^\circ = \emptyset \text{ then } b_i^\circ \cap b_j^\circ = \emptyset, \quad \text{for } i, j \leq n; \quad (51)$$

$$\text{if } r_j^\circ \cap r_i^\circ = \emptyset \text{ then } p_j^\circ \cap b_i^\circ = \emptyset, \quad \text{for } j < i \leq n. \quad (52)$$

To show (51), suppose otherwise, that is, $b_i^\circ \cap b_j^\circ \neq \emptyset$. Then there exists $k \leq m$ such that one of the following holds: $o_k \in w_{k,i} \cap (w_{k,j})^\circ$ or $o_k \in (w_{k,i})^\circ \cap w_{k,j}$ or $o_k \in (w_{k,i})^\circ \cap (w_{k,j})^\circ$. The first case is possible only if $o_k \in r_i \cap r_j^\circ$, the second only if $o_k \in r_i^\circ \cap r_j$, and the third only if $o_k \in r_i^\circ \cap r_j^\circ$. In all cases, $r_i^\circ \cap r_j^\circ \neq \emptyset$.

For each i, we prove (52) by induction on j. The basis of induction, $j = 1$, follows from (51) as $p_1 = b_1$. For the induction step, let $j > 1$ and $r_j^\circ \cap r_i^\circ = \emptyset$. By (51), $b_j^\circ \cap b_i^\circ = \emptyset$. For all $j' < j$ with $\mathsf{TPP}(r_{j'}, r_j)$, we have $r_{j'}^\circ \cap r_i^\circ = \emptyset$, whence, by IH, $p_{j'}^\circ \cap b_i^\circ = \emptyset$; for all $j' < j$ with $\mathsf{NTPP}(r_{j'}, r_j)$, we have $r_{j'} \cap r_i = \emptyset$, whence, by Lemma 9, $p_{j'}^+ \cap b_i = \emptyset$ and $(p_{j'}^+)^\circ \cap b_i^\circ = \emptyset$. By (40) and Lemma 7, $p_j^\circ \cap b_i^\circ = \emptyset$.

Finally, we prove by induction on i that for all $j, j' \leq i$, if $r_{j'}^\circ \cap r_j^\circ = \emptyset$ then $p_{j'}^\circ \cap p_j^\circ = \emptyset$. The case $i = 1$ is trivial. Let $i > 1$ and $r_{j'}^\circ \cap r_j^\circ = \emptyset$. We have to consider only the case $j' < j = i$ (other cases are either immediate from IH or mirror image of this case). By (52), $p_{j'}^\circ \cap b_i^\circ = \emptyset$. For all $j'' < i$ with $\mathsf{TPP}(r_{j''}, r_i)$, we have $r_{j'}^\circ \cap r_{j''}^\circ = \emptyset$, whence by IH, $p_{j'}^\circ \cap p_{j''}^\circ = \emptyset$; for all $j'' < i$ with $\mathsf{NTPP}(r_{j''}, r_i)$, we have $r_{j'} \cap r_{j''} = \emptyset$, whence, by Lemma 9, $p_{j'}^+ \cap p_{j''}^+ = \emptyset$ and so $p_{j'}^\circ \cap (p_{j''}^+)^\circ = \emptyset$. By (40) and Lemma 7, $p_{j'}^\circ \cap p_i^\circ = \emptyset$. □

Lemma 11. *For all j $(1 \leq j \leq n)$, r_j is connected if and only if p_j is connected.*

Proof. ($\Rightarrow$) By construction of the sets $b_1, \dots, b_n$, if r_j is connected then b_j is connected. We show by induction on j that every component of p_j includes a component of b_j. For $j = 1$, we have $p_1 = b_1$. Suppose, $j > 1$ and p_j has a component not including (and hence not intersecting) any component of b_j. By construction, if $r_{j'} \subseteq r_j$ then every component of $b_{j'}$ intersects some component of b_j, since $w_{k,j'} \neq \emptyset$ implies $o_k \in r_{j'} \subseteq r_j$ and thus $w_{k,j} \neq \emptyset$. Then, by (40), there exists $j' < j$ such that either $\mathsf{TPP}(r_{j'}, r_j)$ and some component e of $p_{j'}$ does not intersect any component of b_j or $\mathsf{NTPP}(r_{j'}, r_j)$ and some component of $p^+_{j'}$ does not intersect any component of b_j. In the former case, by IH, e includes some component of $b_{j'}$ but since $r_{j'} \subseteq r_j$, $b_{j'}$ and hence e intersects some component of b_j, a contradiction. The latter case follows similarly, making use of (42).

($\Leftarrow$) Now we show by induction on j that

$$\text{if } r_{j'} \subseteq r_j \text{ then } p_{j'} \cap \delta z_j = \emptyset, \text{ for all } j' \leq j. \qquad (53)$$

As δz_j separates the components of $s_{j'}$, it follows that $(w_{k,j'} \cup \gamma_{k,j'}) \cap \delta z_j = \emptyset$, for all $k \leq m$, and so, by (38), $b_{j'} \cap \delta z_j = \emptyset$. Now, the basis of induction, $j = 1$, is trivial, since $p_1 = b_1$, and $r_{j'} \subseteq r_1$ only if $j' = 1$. Let $j > 1$. We have $b_{j'} \cap \delta z_j = \emptyset$, and, for all $j'' < j'$ with $\mathsf{TPP}(r_{j''}, r_{j'})$ or $\mathsf{NTPP}(r_{j''}, r_{j'})$, we have $r_{j''} \subseteq r_j$, whence, by IH, $p_{j''} \cap \delta z_j = \emptyset$ and, by (46), $p^+_{j''} \cap \delta z_j = \emptyset$. By (40), $p_{j'} \cap \delta z_j = \emptyset$.

If r_j is not connected then there exist o_k and $o_{k'}$ lying in separate components of r_j. We claim that o_k and $o_{k'}$ lie in distinct components of z_j, which, by (53), implies that p_j is not connected. Suppose, to the contrary, that o_k and $o_{k'}$ lie in the same component of z_j; then they lie in the same component of s_j. But then, by the construction of the $\gamma_{k,j}$ and $w_{k,j}$, there exists a sequence of arcs $\gamma_{k_1,j_1}, \dots, \gamma_{k_\ell,j_\ell}$ along which its is possible to pass (possibly via wedges $w_{k'',j'}$ with $r_{j'} \subseteq r_j$) from o_k to $o_{k'}$. But in that case, we have, for all $l \leq \ell$: (i) r_{j_l} is connected; (ii) $r_{j_l} \subseteq r_j$; and (iii) $r_{j_l} \cap r_{j_{l+1}} \neq \emptyset$ if $l < \ell$. To see (iii), suppose $r_{j_l} \cap r_{j_{l+1}} = \emptyset$. By (29), $r^+_{j_l} \cap r^+_{j_{l+1}} = \emptyset$; but $\gamma_{k_l,j_l} \subseteq r^+_{j_l}$ and $\gamma_{k_{l+1},j_{l+1}} \subseteq r^+_{i_{j+1}}$. Thus, o_k and $o_{k'}$ lie in the same component of r_j contrary to our assumption. □

Consider all the atomic formulas: (*i*) if $\mathsf{DC}(r_i, r_j)$ then, by Lemma 9, $p_i \cap p_j = \emptyset$ and so $\mathsf{DC}(r_i, p_j)$; (*ii*) if $\mathsf{EC}(r_i, r_j)$ then, by Lemma 10, $p_i^\circ \cap p_j^\circ = \emptyset$ and, by Lemma 8, $\delta p_i \cap \delta p_j \neq \emptyset$ and so, $\mathsf{EC}(p_i, p_j)$; (*iii*) if $\mathsf{PO}(r_i, r_j)$ then, by Lemma 8, $p_i^\circ \cap p_j^\circ \neq \emptyset$, $p_i^\circ \setminus p_j \neq \emptyset$ and $p_j^\circ \setminus p_i \neq \emptyset$ and so, $\mathsf{PO}(p_i, p_j)$; (*iv*) if $\mathsf{TPP}(r_i, r_j)$ then $i < j$ and, by (40), $p_i \subseteq p_j$; also, by Lemma 8, $\delta p_i \cap \delta p_j \neq \emptyset$ and $p_j^\circ \setminus p_i \neq \emptyset$ and so, $\mathsf{TPP}(p_i, p_j)$; (*v*) if $\mathsf{NTPP}(r_i, r_j)$ then $i < j$, whence, by (40), $p_i \subseteq p_j^\circ$, and so, $\mathsf{NTPP}(p_i, p_j)$; (*vi*) finally, by Lemma 11, r_i is connected if and only if p_i is connected. □

Proof of Theorem 4

Theorem 4. *The problems Sat*$(\mathcal{RCC}8c, \mathsf{RC}(\mathbb{R}))$ *and Sat*$(\mathcal{RCC}8c, \mathsf{RCP}(\mathbb{R}))$ *are both* NP*-complete; the problem Sat*$(\mathcal{C}c, \mathsf{RCP}(\mathbb{R}))$ *is* PSPACE*-complete.*

Proof. (*i*) We show that an $\mathcal{RCC}8c$-formula φ is satisfiable over $\mathsf{RCP}(\mathbb{R})$ if and only if it is satisfiable over the Aleksandrov space induced by a quasi-order of the form $(\{x_0, \dots, x_n, z_0, \dots, z_{n-1}\}, R)$, where R is the reflexive closure of $\{(z_i, x_i), (z_i, x_{i+1}) \mid 1 \leq i < n\}$ and $n \leq |\varphi|^2$.

Without loss of generality we may assume that φ is a conjunction of atoms of the form:

- $(r \cdot r' \neq 0)$, $(r \cdot (-r') \neq 0)$,
- $C(r, r')$, $C(r, -r')$,
- $(r \cdot r' = 0)$, $(r \leq r')$,
- $\neg C(r, r')$, $\neg C(r, -r')$,
- $c(r)$

(negative occurrences of $c(r)$ can be equi-satisfiably replaced by $(r' \leq r) \wedge (r'' \leq r) \wedge \neg C(r', r'')$, where r' and r'' are fresh variables).

Suppose φ is satisfiable in a model $\mathfrak{M}$ over $\mathsf{RCP}(\mathbb{R})$. We construct a model $\mathfrak{B}$ over the Alkesandrov space induced by (W, R) in a number of steps.

Step 1. First we find points for W in the following way:

(sing) if $\mathfrak{M} \models (\tau \neq 0)$, we pick a point $x \in \tau^{\mathfrak{M}}$; and

(fork) if $\mathfrak{M} \models C(\tau, \tau')$, we pick a pair of points $x \in (\tau)^{\mathfrak{M}}$ and $x' \in (\tau')^{\mathfrak{M}}$.

Without loss of generality, we may assume that between no pair of points picked in **(fork)** lies another of the picked points (this can be done by selecting that pair close enough to the point of contact of τ and τ'). We also assume that the same point may be picked twice. Denote by W_0 the set of all the points picked above and let $\prec$ be the strict linear order on W_0 induced by their natural order in $\mathfrak{M}$. Let $W = W_0$ and $R = \emptyset$. Next, for each pair x, x' of points picked in **(fork)**, take a fresh point z, add it to W and (z, x), (z, x') to R. Note that (W, R) is a subgraph of the required quasi-order (i.e., each z has at most two successors and each x has at most two predecessors). Finally, we construct the model $\mathfrak{B}$ based on the Aleksandrov space induced by (W, R) by setting $x \in r^{\mathfrak{B}}$ iff $x \in r^{\mathfrak{M}}$, for each $w \in W_0$ (the valuation in $z \in W \setminus W_0$ needs no definition as if $r^{\mathfrak{B}}$ are to be regular closed we must have $z \in \mathfrak{B}$ iff $x \in r^{\mathfrak{B}}$, for some $x \in W_0$ with zRx). Note that $\mathfrak{B} \models (\tau = 0)$ whenever $\mathfrak{M} \models (\tau = 0)$ and $\mathfrak{B} \models \neg C(\tau, \tau')$ whenever $\mathfrak{M} \models \neg C(\tau, \tau')$. If the space is connected and $\mathfrak{B} \models c(r)$ whenever $\mathfrak{M} \models c(r)$, for all $c(r)$, we are done.

Step 2. So, suppose $c(\tau_0)$, where τ_0 is either r or 1, is false in the model. Pick any two neighbouring (with respect to $\prec$) points $x, x' \in \tau_0^{\mathfrak{B}}$ without a common predecessor and take a fresh point y, add y to W with $y \in r^{\mathfrak{B}}$ iff $x, x' \in r^{\mathfrak{B}}$. Clearly, adding this point to the model does not change the truth value of any subformula of the form $(\tau \neq 0)$, $C(\tau, \tau')$, $(\tau = 0)$ or $\neg C(\tau, \tau')$. So, it remains to connect y to both x and x'. We cannot, however, directly connect y to, say, x by creating a common R-predecessor, i.e., a point of depth 1, because that might make one of the $\neg C(\tau, \tau)$ subformulas false. Let $r_{j_1}, \ldots, r_{j_k}$ be a linear order on the regions containing x such that it is compatible with the subformulas of the form $(r_i \leq r_j)$, i.e., such that $\mathfrak{M} \models (r_{j_i} \leq r_{j_{i'}})$ whenever $j_i \leq j_{i'}$. We proceed our construction in a step-by-step way. For step 0, let $x_0 = y$. For step i, $1 \leq i \leq k$, take fresh points x_i and z_i, add them to W, add (z_i, x_{i-1}) and (z_i, x_i) to R, let

$$x_i \in r^{\mathfrak{B}} \qquad \text{iff} \qquad x_{i-1} \in r^{\mathfrak{B}} \text{ or } \mathfrak{M} \models r_{j_i} \leq r$$

and $z_i \in r^{\mathfrak{B}}$ iff $\{x_{i-1}, x_i\} \cap r^{\mathfrak{B}} \neq \emptyset$. Clearly, all subformulas of the form $(\tau \neq 0)$, $C(\tau, \tau')$ and $(\tau = 0)$ that are true in $\mathfrak{M}$ are also true in $\mathfrak{B}$. It is also clear that the same holds for subformulas of the form $\neg C(r, r')$. So, it remains to show that the same holds for subformulas of the form $\neg C(r, -r')$. To this end we observe that, each triple x_{i-1}, z_i, x_i is either entirely in $r^{\mathfrak{B}}$ or z_i is on the boundary of $r^{\mathfrak{B}}$, in which case $x_{i-1} \notin r^{\mathfrak{B}}$ and $x_i \in r^{\mathfrak{B}}$. It also follows that in the latter case $\mathfrak{M} \models r_{j_k} \leq r$ iff $k \geq i$, and so z_i cannot be a point of contact of r and any $-r'$. Finally, points x_k and x belong to precisely the same regions and can be identified. In the same way we connect y to x'.

Repeating **Step 2** for each pair of neighbouring (with respect to $\prec$) points of depth 0 without a common predecessor, we construct the model as required.

*(*ii*)* For *Sat*$(\mathcal{C}c, \mathsf{RCP}(\mathbb{R}))$, the model can be of exponential size (due to exponential paths required in **Step 2** to connect up disconnected regions) but a non-deterministic algorithm can guess such a model using only a polynoimial number of cells on the tape of a Turing machine. $\square$