Intuitionistic Modal Logic

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0 Intuitionistic modal logics originate from different sources and have different areas of application. They include philosophy (see e.g. Prior [1957], Ewald [1986], Williamson [1992]), the foundations of mathematics (Kuznetsov [1985], Kuznetsov and Muravitskij [1986]), and computer science (Plotkin and Stirling [1986], Stirling [1987], Wijesekera [1990]). Modalities are added to intuitionistic logic in the framework of studying "new intuitionistic connectives" (Bessonov [1977], Gabbay [1977], Yashin [1994]) and to simulate the monadic fragment of intuitionistic first order logic (Bull [1966], Ono [1977], Ono and Suzuki [1988], Bezhanishvili [1997]). The multitude of constructed logics was examined "piecewise", often by means of creating a special semantical and syntactical apparatus. A broader perspective is to try combining the well-developed general theories of classical modal logics and non-modal superintuitionistic (alias intermediate) logics in order to embrace the classes of extensions of some reasonable basic modal systems on the intuitionistic base.

In this paper we give an overview of the model theory that results from such a combination and demonstrate a number of applications.

1 Unlike the classical case, the intuitionistic necessity ($\Box$) and possibility ($\Diamond$) operators are not supposed to be dual, which provides more possibilities for defining intuitionistic modal logics. For a non-empty set M of modal operators, let $\mathcal{L}_\mathsf{M}$ be the standard propositional language augmented by the connectives in M. By an *intuitionistic modal logic* in the language $\mathcal{L}_\mathsf{M}$ we understand in this paper any subset of $\mathcal{L}_\mathsf{M}$ containing intuitionistic logic **Int** and closed under modus ponens, substitution and the regularity rule $\varphi \to \psi / \bigcirc \varphi \to \bigcirc\psi$, for every $\bigcirc \in \mathsf{M}$. Given such a logic L, we denote by $\mathrm{Ext}L$ the lattice of all logics (in $\mathcal{L}_\mathsf{M}$) extending L. The minimal logic in $\mathrm{Ext}L$ containing a set of $\mathcal{L}_\mathsf{M}$-formulas Γ is denoted by $L \oplus \Gamma$.

There are three ways of defining intuitionistic analogues of (classical) normal modal logics. First, one can take the family of logics extending the basic system $\mathbf{IntK}_\Box$ in the language $\mathcal{L}_\Box$ which is axiomatized by adding to **Int** the standard axioms of **K**

$$\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q \text{ and } \Box\top.$$

An example of a logic in this family is Kuznetsov's [1985] intuitionistic provability logic $\mathbf{I}^{\triangle}$ (Kuznetsov used $\triangle$ instead of $\Box$), the intuitionistic analog of the Gödel–Löb classical provability logic **GL**. It can be obtained by adding to $\mathbf{IntK}_\Box$ (and even to **Int**) the axioms

$$p \to \Box p,\ (\Box p \to p) \to p,\ ((p \to q) \to p) \to (\Box q \to p).$$

A model theory for logics in Ext$\mathbf{IntK}_\Box$ was developed by Ono [1977], Božić and Došen [1984], Došen [1985], Sotirov [1984] and Wolter and Zakharyaschev [1997]; we discuss it in the next section. Font [1984, 1986] considered these logics from the algebraic point of view, and Luppi [1996] investigates their interpolation property by proving, in particular, that the superamalgamability of the corresponding varieties of algebras is equivalent to interpolation.

A possibility operator $\Diamond$ in logics of this sort can be defined in the classical way by taking $\Diamond\varphi = \neg\Box\neg\varphi$. Note, however, that in general this $\Diamond$ does not distribute over disjunction and that the connection via negation between $\Box$ and $\Diamond$ is too strong from the intuitionistic standpoint (actually, the situation here is similar to that in intuitionistic predicate logic where $\exists$ and $\forall$ are not dual.)

Another family of "normal" intuitionistic modal logics can be defined in the language $\mathcal{L}_\Diamond$ by taking as the basic system the smallest logic in $\mathcal{L}_\Diamond$ to contain the axioms

$$\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q \text{ and } \neg\Diamond\bot;$$

it will be denoted by $\mathbf{IntK}_\Diamond$. Logics in Ext$\mathbf{IntK}_\Diamond$ were studied by Božić and Došen [1984], Došen [1985], Sotirov [1984] and Wolter [1997b].

Finally, we can define intuitionistic modal logics with independent $\Box$ and $\Diamond$. These are extensions of $\mathbf{IntK}_{\Box\Diamond}$, the smallest logic in the language $\mathcal{L}_{\Box\Diamond}$ containing both $\mathbf{IntK}_\Box$ and $\mathbf{IntK}_\Diamond$. Fischer Servi [1980, 1984] constructed an interesting logic in Ext$\mathbf{IntK}_{\Box\Diamond}$ by imposing a weak connection between the necessity and possibility operators:

$$\mathbf{FS} = \mathbf{IntK}_{\Box\Diamond} \oplus \Diamond(p \to q) \to (\Box p \to \Diamond q) \oplus (\Diamond p \to \Box q) \to \Box(p \to q).$$

She claimed **FS** to be the "true" intuitionistic analog of the minimal classical normal modal system **K** by showing that a natural generalization of the Gödel translation embeds **FS** into an extension of the fusion of **S4** and **K** (see Section 3). Another remarkable feature of **FS**, also supporting Fischer Servi's claim, is that the standard translation ST of modal formulas into first order ones defined by

$ST(p_i) = P_i(x),\ \ ST(\bot) = \bot,$
$ST(\psi \odot \chi) = ST(\psi) \odot ST(\chi)$, for $\odot \in \{\wedge, \vee, \to\}$,
$ST(\Box\psi) = \forall y\ (xRy \to ST(\psi)\{y/x\}),$

$ST(\Diamond\psi) = \exists y \ (xRy \wedge ST(\psi)\{y/x\})$

(where y is an individual variable not occurring in $ST(\psi)$) not only embeds **K** into classical predicate logic but also **FS** into intuitionistic first order logic: φ belongs to the former iff $ST(\varphi)$ is a theorem of the latter. According to Simpson [1994], this result was proved by C. Stirling; see also Grefe [1997].

Various extensions of **FS** were studied by Bull [1966], Ono [1977], Fischer Servi [1977, 1980, 1984], Amati and Pirri [1994], Ewald [1986], Wolter and Zakharyaschev [1996], Wolter [1997b]. The best known one is probably the logic

$$\begin{aligned}\mathbf{MIPC} \quad = \quad & \mathbf{FS} \oplus \Box p \to p \oplus \Box p \to \Box\Box p \oplus \Diamond p \to \Box\Diamond p \oplus \\ & p \to \Diamond p \oplus \Diamond\Diamond p \to \Diamond p \oplus \Diamond\Box p \to \Box p\end{aligned}$$

introduced by Prior [1957]. Bull [1966] noticed that the translation * defined by

$(p_i)^* = P_i(x), \ \ \bot^* = \bot,$
$(\psi \odot \chi)^* = \psi^* \odot \chi^*$, for $\odot \in \{\wedge, \vee, \to\}$,
$(\Box\psi)^* = \forall x \ \psi^*, \ \ (\Diamond\psi)^* = \exists x \ \psi^*$

is an embedding of **MIPC** into the monadic fragment of intuitionistic predicate logic. Ono [1977], Ono and Suzuki [1988], Suzuki [1990], and Bezhanishvili [1997] investigated the relations between logics in Ext**MIPC** and superintuitionistic predicate logics induced by that translation.

In this paper we restrict attention only to the classes of "normal" intuitionistic modal logics introduced above. An interesting example of a non-normal system was constructed by Wijesekera [1990]. A general model theory for such logics is developed by Sotirov [1984] and Wolter and Zakharyaschev [1996].

2 Now let us consider the algebraic and relational semantics for the logics introduced in the preceding section. All the semantical concepts to be defined below turn out to be natural combinations of the corresponding notions developed for classical modal and superintuitionistic logics. For details and proofs we refer the reader to Wolter and Zakharyaschev [1996].

From the algebraic point of view, every logic $L \in \mathrm{Ext}\mathbf{IntK}_{\mathsf{M}}$, $\mathsf{M} \subseteq \{\Box, \Diamond\}$, corresponds to the variety (equationally definable class) of Heyting algebras with one or two operators validating L (for a definition and discussion of Heyting algebras see e.g. Rasiowa and Sikorski [1963]). The variety of algebras for $\mathbf{IntK}_{\mathsf{M}}$ will be called the *variety of* M*-algebras*.

To construct the relational (Stone–Jónsson–Tarski) representations of M-algebras, recall that an intuitionistic (general) frame is a structure $\langle W, R, P\rangle$ such that R is a partial order on W and P a set of R-cones (upward closed sets) in it containing W and closed under intersection, union and the operation

$$X \to Y = \{x \in W : \forall y \in W \ (xRy \wedge y \in X \to y \in Y)\}.$$

Now we define a $\Box$-*frame* to be a structure of the form $\langle W, R, R_\Box, P\rangle$ in which $\langle W, R, P\rangle$ is an intuitionistic frame, $R_\Box$ a binary relation on W such that

$$R \circ R_\Box \circ R \ = R_\Box$$

and P is closed under the operation

$$\Box X = \{x \in W : \forall y \in W \ (x R_\Box y \to y \in X)\}.$$

A $\Diamond$-*frame* has the form $\langle W, R, R_\Diamond, P\rangle$, where $\langle W, R, P\rangle$ is again an intuitionistic frame, $R_\Diamond$ a binary relation on W satisfying the condition

$$R^{-1} \circ R_\Diamond \circ R^{-1} = R_\Diamond$$

and P is closed under

$$\Diamond X = \{x \in W : \exists y \in X \ x R_\Diamond y\}.$$

Finally, a $\Box\Diamond$-*frame* is a structure $\langle W, R, R_\Box, R_\Diamond, P\rangle$ the unimodal reducts $\langle W, R, R_\Box, P\rangle$ and $\langle W, R, R_\Diamond, P\rangle$ of which are $\Box$- and $\Diamond$-frames, respectively. (To see why the intuitionistic and modal accessibility relations are connected by the conditions above the reader can construct in the standard way the canonical models for the logics under consideration. The important point here is that we take the Leibnizean definition of the truth-relation for the modal operators. Other definitions may impose different connecting conditions; see the end of this section.)

Given a $\Box\Diamond$-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$, it is easy to check that its *dual*

$$\mathfrak{F}^+ = \langle P, \cap, \cup, \to, \emptyset, \Box, \Diamond\rangle$$

is a $\Box\Diamond$-algebra. Conversely, for each $\Box\Diamond$-algebra $\mathfrak{A} = \langle A, \wedge, \vee, \to, \bot, \Box, \Diamond\rangle$ we can define the *dual frame*

$$\mathfrak{A}_+ = \langle W, R, R_\Box, R_\Diamond, P\rangle$$

by taking $\langle W, R, P\rangle$ to be the dual of the Heyting algebra $\langle A, \wedge, \vee, \to, \bot\rangle$ (i.e., W is the set of prime filters in $\mathfrak{A}$, R just $\subseteq$ and $P = \{\{\nabla \in W : a \in \nabla\} : a \in A\}$) and putting

$$\nabla_1 R_\Box \nabla_2 \text{ iff } \forall a \in A \ (\Box a \in \nabla_1 \to a \in \nabla_2),$$

$$\nabla_1 R_\Diamond \nabla_2 \text{ iff } \forall a \in A \ (a \in \nabla_2 \to \Diamond a \in \nabla_1).$$

$\mathfrak{A}_+$ is a $\Box\Diamond$-frame and moreover we have $\mathfrak{A} \cong (\mathfrak{A}_+)^+$. Using the standard technique of the model theory for classical modal and superintuitionistic logics (see e.g. Chagrov and Zakharyaschev [1997]), one can show that a $\Box\Diamond$-frame $\mathfrak{F}$ is isomorphic to its bidual $(\mathfrak{F}^+)_+$ iff $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ is *descriptive*, i.e., $\langle W, R, P\rangle$ is a descriptive intuitionistic frame and, for every $x, y \in W$,

$$x R_\Box y \text{ iff } \forall X \in P \ (x \in \Box X \to y \in X),$$

$$x R_\Diamond y \text{ iff } \forall X \in P \ (y \in X \to x \in \Diamond X).$$

Thus we get the following completeness theorem:

Theorem 1 *Every logic $L \in \mathrm{Ext}\mathbf{IntK}_{\Box\Diamond}$ is characterized by a suitable class of (descriptive) $\Box\Diamond$-frames, e.g. by the class $\{\mathfrak{A}_+ : \mathfrak{A} \models L\}$.*

Needless to say that similar results hold for logics in $\mathrm{Ext}\mathbf{IntK}_{\Box}$ and $\mathrm{Ext}\mathbf{IntK}_{\Diamond}$.

As usual, by a *Kripke* ($\Box\Diamond$-) *frame* we understand a frame $\langle W, R, R_\Box, R_\Diamond, P\rangle$ in which P consists of all R-cones; in this case we shall omit P. An intuitionistic modal logic L is said to be $\mathcal{D}$-*persistent* if the underlying Kripke frame of each descriptive frame for L validates L. For example, **FS** as well as the logics

$$\mathbf{L}(k,l,m,n) = \mathbf{IntK}_{\Box\Diamond} \oplus \Diamond^k\Box^l p \to \Box^m\Diamond^n p, \quad \text{for } k,l,m,n \geq 0$$

are $\mathcal{D}$-persistent and so Kripke complete (see Wolter and Zakharyaschev [1996]). Descriptive frames validating **FS** satisfy the conditions

$$xR_\Diamond y \;\to\; \exists z\ (yRz \wedge xR_\Box z \wedge xR_\Diamond z),$$

$$xR_\Box y \;\to\; \exists z\ (xRz \wedge zR_\Box y \wedge zR_\Diamond y),$$

and those for $\mathbf{L}(k,l,m,n)$ satisfy

$$xR_\Diamond^k y \wedge xR_\Box^m y \to \exists u\ (yR_\Box^l u \wedge zR_\Diamond^n u).$$

It follows, in particular, that **MIPC** is $\mathcal{D}$-persistent; its Kripke frames have the properties: $R_\Box$ is a quasi-order, $R_\Diamond = R_\Box^{-1}$ and $R_\Box = R \circ (R_\Box \cap R_\Diamond)$. On the contrary, $\mathbf{I}^\triangle$ is not $\mathcal{D}$-persistent, although it is complete with respect to the class of Kripke frames $\langle W, R, R_\Box\rangle$ such that $\langle W, R_\Box\rangle$ is a frame for **GL** and R the reflexive closure of $R_\Box$. As was noticed by L. Esakia, also not $\mathcal{D}$-persistent is the modal analog of Casari's logic

$$\mathbf{MIPC} \oplus \Box((p \to \Box p) \to \Box p) \to \Box p.$$

The next step in constructing duality theory of M-algebras and M-frames is to find relational counterparts of the algebraic operations of forming homomorphisms, subalgebras and direct products. Let $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ be a $\Box\Diamond$-frame and V a non-empty subset of W such that

$$\forall x \in V \forall y \in W\ (xR_\Box y \vee xRy \to y \in V),$$

$$\forall x \in V \forall y \in W\ (xR_\Diamond y \to \exists z \in V\ (xR_\Diamond z \wedge yRz)).$$

Then $\mathfrak{G} = \langle V, R\!\upharpoonright\! V, R_\Box\!\upharpoonright\! V, R_\Diamond\!\upharpoonright\! V, \{X \cap V : X \in P\}\rangle$ is also a $\Box\Diamond$-frame which is called the *subframe of $\mathfrak{F}$ generated by V*. The former of the two conditions above is standard: it requires V to be upward closed with respect to both R and $R_\Box$. However, the latter one does not imply that V is upward closed with respect to $R_\Diamond$: the frame $\mathfrak{G}$ in Fig. 1 is a generated subframe of $\mathfrak{F}$, although the set $\{x, z\}$ is not an $R_\Diamond$-cone in $\mathfrak{F}$. This is one difference from the standard (classical modal or intuitionistic) case. Another one arises when we define the relational analog of subalgebras.

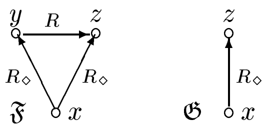

Figure 1:

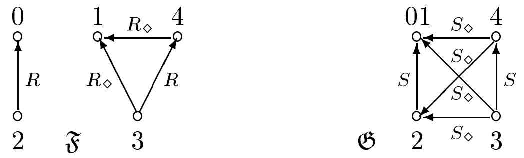

Figure 2:

Given $\Box\Diamond$-frames $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ and $\mathfrak{G} = \langle V, S, S_\Box, S_\Diamond, Q\rangle$, we say a map f from W onto V is a *reduction* (or *p-morphism*) of $\mathfrak{F}$ to $\mathfrak{G}$ if $f^{-1}(X) \in P$ for every $X \in Q$ and, for every $x, y \in W$ and every $u \in V$,

xRy implies $f(x)Sf(y)$,
$xR_\bigcirc y$ implies $f(x)S_\bigcirc f(y)$, for $\bigcirc \in \{\Box, \Diamond\}$,
$f(x)Su$ implies $\exists z \in f^{-1}(u)\ xRz$,
$f(x)S_\Box u$ implies $\exists z \in f^{-1}(u)\ xR_\Box z$,
$f(x)S_\Diamond u$ implies $\exists z \in W\ (xR_\Diamond z \wedge uSf(z))$,

Again, the last condition differs from the standard one: given $f(x)S_\Diamond f(y)$, in general we do not have a point z such that $xR_\Diamond z$ and $f(y) = f(z)$, witness the map gluing the points 0 and 1 in the frame $\mathfrak{F}$ in Fig. 2 and reducing it to $\mathfrak{G}$.

Note that both these concepts coincide with the standard ones in classical modal frames, where R and S are the diagonals. The relational counterpart of direct products—disjoint unions of frames—is defined as usual.

Theorem 2 (i) *If $\mathfrak{G}$ is the subframe of a $\Box\Diamond$-frame $\mathfrak{F}$ generated by V then the map h defined by $h(X) = X \cap V$, for X an element in $\mathfrak{F}^+$, is a homomorphism from $\mathfrak{F}^+$ onto $\mathfrak{G}^+$.*

(ii) *If h is a homomorphism from a $\Box\Diamond$-algebra $\mathfrak{A}$ onto a $\Box\Diamond$-algebra $\mathfrak{B}$ then the map h_+ defined by $h_+(\nabla) = h^{-1}(\nabla)$, ∇ a prime filter in $\mathfrak{B}$, is an isomorphism from $\mathfrak{B}_+$ onto a generated subframe of $\mathfrak{A}_+$.*

(iii) *If f is a reduction of a $\Box\Diamond$-frame $\mathfrak{F}$ to a $\Box\Diamond$-frame $\mathfrak{G}$ then the map f^+ defined by $f^+(X) = f^{-1}(X)$, X an element in $\mathfrak{G}^+$, is an embedding of $\mathfrak{G}^+$ into $\mathfrak{F}^+$.*

(iv) *If* $\mathfrak{B}$ *is a subalgebra of a* $\Box\Diamond$*-algebra* $\mathfrak{A}$ *then the map* f *defined by* $f(\nabla) = \nabla \cap B$, ∇ *a prime filter in* $\mathfrak{A}$ *and* B *the universe of* $\mathfrak{B}$, *is a reduction of* $\mathfrak{A}_+$ *to* $\mathfrak{B}_+$.

This duality can be used for proving various results on modal definability. For instance, a class $\mathcal{C}$ of $\Box\Diamond$-frames is of the form $\mathcal{C} = \{\mathfrak{F} : \mathfrak{F} \models \Gamma\}$, for some set Γ of $\mathcal{L}_{\Box\Diamond}$-formulas, iff $\mathcal{C}$ is closed under the formation of generated subframes, reducts, disjoint unions, and both $\mathcal{C}$ and its complement are closed under the operation $\mathfrak{F} \mapsto (\mathfrak{F}^+)_+$ (see Wolter and Zakharyaschev [1996]). Moreover, one can extend Fine's theorem connecting the first order definability and $\mathcal{D}$-persistence of classical modal logics to the intuitionistic modal case:

Theorem 3 *If a* $\Box\Diamond$*-logic* L *is characterized by an elementary class of Kripke frames then* L *is* $\mathcal{D}$*-persistent.*

These results may be regarded as a justification for the relational semantics introduced in this section. However, it is not the only possible one. For example, Božić and Došen [1984] impose a weaker condition on the connection between R and $R_\Box$ in $\Box$-frames. Fisher Servi [1980] interprets **FS** in birelational Kripke frames of the form $\langle W, R, S\rangle$ in which R is a partial order, $R \circ S \subseteq S \circ R$, and

$$xRy \wedge xSz \to \exists z' \ (ySz' \wedge zRz').$$

The intuitionistic connectives are interpreted by R and the truth-conditions for $\Box$ and $\Diamond$ are defined as follows

$$\Box X = \{x \in W : \forall y, z \ (xRySz \to z \in X\},$$

$$\Diamond X = \{x \in W : \exists y \in X \ xSy\}.$$

In birelational frames for **MIPC** S is an equivalence relation and

$$xSyRz \to \exists y' \ xRy'Sz.$$

These frames were independently introduced by L. Esakia who also established duality between them and "monadic Heyting algebras".

One can argue as to which conditions on the accessibility relations are more natural, but from the technical point of view all these semantics seem to be equivalent. For instance, every frame $\langle W, R, R_\Box, R_\Diamond\rangle$ for **FS** can be transformed into a Fischer Servi's birelational frame by putting $S = R_\Box \cap R_\Diamond$. For more details we refer the reader to Wolter and Zakharyaschev [1996].

One of the reasons why we prefer our semantics is that it makes it possible to construct a rather natural translation of intuitionistic modal logics into classical polymodal logics, which is the subject of the next section.

3 There are two ways of investigating various properties of intuitionistic modal logics. One is to continue extending the well known classical methods to logics in Ext$\mathbf{IntK}_{\mathsf{M}}$. Another one uses those methods indirectly via embeddings of intuitionistic modal logics into classical ones. That such embeddings are possible was noticed by Shehtman [1979], Fischer Servi [1980, 1984], and Sotirov [1984]. Our exposition here follows Wolter and Zakharyaschev [1996, 1997]. For simplicity we confine ourselves only to considering the class Ext$\mathbf{IntK}_{\Box}$ and refer the reader to the cited papers for information about more general embeddings.

First we remind the reader that the Gödel translation T prefixing $\Box_I$ to every subformula of a given intuitionistic formula embeds each superintuitionistic logic into some normal extensions of $\mathbf{S4}$ formulated in the language $\mathcal{L}_{\Box_I}$. Let us extend T to a translation of the language $\mathcal{L}_{\Box}$ into $\mathcal{L}_{\Box_I\Box}$ by putting

$$T(\Box\varphi) = \Box_I\Box T(\varphi).$$

Thus, we are trying to embed intuitionistic modal logics in Ext$\mathbf{IntK}_{\Box}$ into classical bimodal logics with the necessity operators $\Box_I$ (of $\mathbf{S4}$) and $\Box$. Given two normal unimodal logics L_1 and L_2 (formulated in languages with different modal operators), denote by $L_1 \otimes L_2$ the smallest normal bimodal logic containing $L_1 \cup L_2$. Say that T *embeds* $L \in$ Ext$\mathbf{IntK}_{\Box}$ into $M \in$ Ext$(\mathbf{S4} \otimes \mathbf{K})$ ($\mathbf{S4}$ in $\mathcal{L}_{\Box_I}$ and $\mathbf{K}$ in $\mathcal{L}_{\Box}$) if, for all $\varphi \in \mathcal{L}_{\Box}$,

$$\varphi \in L \text{ iff } T(\varphi) \in M.$$

In this case M is called a *bimodal companion* (*BM-companion*, for short) of L.

For every logic $M \in$ Ext$(\mathbf{S4} \otimes \mathbf{K})$ put

$$\boldsymbol{\rho} M = \{\varphi \in \mathcal{L}_{\Box} : T(\varphi) \in M\},$$

and let $\boldsymbol{\sigma}$ be the map from Ext$\mathbf{IntK}_{\Box}$ into Ext$(\mathbf{S4} \otimes \mathbf{K})$ defined by

$$\boldsymbol{\sigma}(\mathbf{IntK}_{\Box} \oplus \Gamma) = (\mathbf{Grz} \otimes \mathbf{K}) \oplus \boldsymbol{mix} \oplus T(\Gamma),$$

where $\Gamma \subseteq \mathcal{L}_{\Box}$ and $\boldsymbol{mix} = \Box_I\Box\Box_I p \leftrightarrow \Box p$. (The axiom $\boldsymbol{mix}$ reflects the condition $R \circ R_{\Box} \circ R = R_{\Box}$ of $\Box$-frames.) Then we have the following extension of the embedding results of Maksimova and Rybakov [1974], Blok [1976] and Esakia [1979]:

Theorem 4 (i) *The map* $\boldsymbol{\rho}$ *is a lattice homomorphism from* Ext$(\mathbf{S4} \otimes \mathbf{K})$ *onto* Ext$\mathbf{IntK}_{\Box}$ *preserving decidability, Kripke completeness, tabularity and the finite model property.*

(ii) *Each logic* $\mathbf{IntK}_{\Box} \oplus \Gamma$ *is embedded by* T *into any logic* M *in the interval*

$$(\mathbf{S4} \otimes \mathbf{K}) \oplus T(\Gamma) \subseteq M \subseteq (\mathbf{Grz} \otimes \mathbf{K}) \oplus \boldsymbol{mix} \oplus T(\Gamma).$$

(iii) *The map* $\boldsymbol{\sigma}$ *is an isomorphism from the lattice* Ext$\mathbf{IntK}_{\Box}$ *onto the lattice* Ext$(\mathbf{Grz} \otimes \mathbf{K}) \oplus \boldsymbol{mix}$ *preserving the finite model property and tabularity.*

Note that Fischer Servi [1980] used another generalization of the Gödel translation. She defined

$$T(\Diamond\varphi) = \Diamond T(\varphi),$$
$$T(\Box\varphi) = \Box_I \Box T(\varphi)$$

and showed that this translation embeds **FS** into the logic

$$(\mathbf{S4} \otimes \mathbf{K}) \oplus \Diamond\Box_I p \to \Box_I \Diamond p \oplus \Diamond\Diamond_I p \to \Diamond_I \Diamond p.$$

It is not clear, however, whether all extensions of **FS** can be embedded into classical bimodal logics via this translation.

4 In this section we summarize known completeness results for intuitionistic modal logics. As to the standard systems $\mathbf{I}^{\triangle}$, **FS**, and **MIPC**, their finite model property (FMP, for short) can be proved by using (sometimes rather involved) filtration arguments; see Muravitskij [1981], Simpson [1994] and Grefe [1997], and Ono [1977], respectively. Further results based on the filtration method were obtained by Sotirov [1984] and Ono [1977]. However, in contrast to classical modal logic only a few general completeness results covering interesting classes of intuitionistic modal logics are known. The proofs of the following two theorems are based on the translation into classical bimodal logics discussed above.

Theorem 5 *Suppose that a superintuitionistic logic* $\mathbf{Int} + \Gamma$[1] *has one of the properties: decidability, Kripke completeness, the finite model property. Then the logics* $\mathbf{IntK}_\Box \oplus \Gamma$ *and* $\mathbf{IntK}_\Box \oplus \Gamma \oplus \Box p \to p$ *also have the same property.*

Here is a sketch of the proof. It suffices to show that there is a BM-companion of each of these systems satisfying the corresponding property. Notice that

$$\boldsymbol{\rho}((\mathbf{S4} \oplus T(\Gamma)) \otimes \mathbf{K}) = \mathbf{IntK}_\Box \oplus \Gamma,$$

$$\boldsymbol{\rho}((\mathbf{S4} \oplus T(\Gamma)) \otimes (\mathbf{K} \oplus \Box p \to p)) = \mathbf{IntK}_\Box \oplus \Gamma \oplus \Box p \to p.$$

So it remains to use the fact that if a logic $\mathbf{Int} + \Gamma$ has one of the properties under consideration then its smallest modal companion $\mathbf{S4} \oplus T(\Gamma)$ has this property as well (see Zakharyaschev [1991]), and if L_1, L_2 are unimodal logics having one of those properties then the fusion $L_1 \otimes L_2$ also enjoys the same property (see Kracht and Wolter [1991] and Wolter [1997a]).

Such a simple reduction to known results in classical modal logic is not available for logics containing $\mathbf{IntK4}_\Box = \mathbf{IntK} \oplus \Box p \to \Box\Box p$. However, by extending Fine's [1974] method of maximal points to bimodal companions of extensions of $\mathbf{IntK4}_\Box$ Wolter and Zakharyaschev [1996] proved the following:

[1]The operation $+$ presupposes taking the closure only under modus ponens and substitution.

Theorem 6 *Suppose a logic $L \supseteq$ $\mathbf{IntK4}_\Box$ has a $\mathcal{D}$-persistent BM-companion $M \supseteq (\mathbf{S4} \otimes \mathbf{K4}) \oplus \boldsymbol{mix}$ whose Kripke frames are closed under the formation of substructures. Then*

(i) *for every set Γ of intuitionistic negation and disjunction free formulas, $L \oplus \Gamma$ has FMP;*

(ii) *for every set Γ of intuitionistic disjunction free formulas and every $n \geq 1$,*

$$L \oplus \Gamma \oplus \bigvee_{i=0}^{n} (p_i \to \bigvee_{j \neq i} p_j)$$

has the finite model property.

One can use this result to show that the following intuitionistic modal logics enjoy FMP:

(1) **IntK4**;
(2) $\mathbf{IntS4} = \mathbf{IntK4} \oplus \Box p \to p$ ($R_\Box$ is reflexive);
(3) $\mathbf{IntS4.3} = \mathbf{IntS4} \oplus \Box(\Box p \to q) \vee \Box(\Box q \to p)$ ($R_\Box$ is reflexive and connected);
(4) $\mathbf{IntK4} \oplus p \vee \Box\neg\Box p$ ($R_\Box$ is symmetrical);
(5) $\mathbf{IntK4} \oplus \Box p \vee \Box\neg\Box p$ ($R_\Box$ is Euclidean);
(6) $\mathbf{IntK4} \oplus \Box p \vee \neg\Box p$ ($xRy \wedge xR_\Box z \to yR_\Box z$);
(7) $\mathbf{IntK4} \oplus p \to \Box p$ ($xR_\Box y \to xRy$);
(8) $\mathbf{IntK4} \oplus \Box(p \vee \neg p)$ ($xR_\Box y \wedge yRz \to z = y$);
(9) $\mathbf{IntK4} \oplus \Box p \vee \Box\neg p$ ($xR_\Box y \wedge xR_\Box z \to y = z$);
(10) $\mathbf{IntK4} \oplus \Box(p \to q) \vee \Box(q \to p)$ ($xR_\Box y \wedge xR_\Box z \to yRz \vee zRy$).

Recently, Aoto has proved that for any set Γ of implicational $\mathcal{L}$-formulas, the logic $\mathbf{MIPC} \oplus \Gamma$ has FMP. Bezhanishvili has established the finite model property of all logics in Ext**MIPC** of finite depth and all logics of the form $\mathbf{MIPC} \oplus L$, where L is a locally finite superintuitionistic logic.

5 We conclude the paper with some remarks on lattices of intuitionistic modal logics. Wolter [1997b] uses the duality theory discussed above to study splittings of lattices of intuitionistic modal logics. For example, he showed that each finite rooted frame splits the lattice $\mathrm{Ext}(L \oplus \Box^{\leq n} p \to \Box^{n+1} p)$, for $L = \mathbf{IntK}_\Box$ and $L = \mathbf{FS}$, and each $R_\Box$-cycle free finite rooted frame splits the lattices of extensions of $\mathbf{IntK}_\Box$ and **FS**. No positive results are known, however, for the lattice $\mathrm{Ext}\mathbf{IntK}_\Diamond$. In fact, the behavior of $\Diamond$-frames is quite different from that of frames for **FS**. For instance, in classical modal logic we have $\mathsf{RG}\mathcal{F} = \mathsf{GR}\mathcal{F}$, for each class of frames (or even $\Box$-frames) $\mathcal{F}$, where G and R are the operations of forming generated subframes and reducts, respectively. But this does not hold for $\Diamond$-frames. More precisely, there exists a finite $\Diamond$-frame $\mathfrak{G}$ such that $\mathsf{RG}\{\mathfrak{G}\} \not\supseteq \mathsf{GR}\{\mathfrak{G}\}$. In other terms, the variety of modal algebras for **K** has the *congruence extension property* (i.e., each congruence of a subalgebra of a modal algebra can be extended to a congruence of the algebra itself) but this is not the case for the variety of $\Diamond$-algebras.

Vakarelov [1981, 1985] and Wolter [1997b] investigate how logics having **Int** as their non-modal fragment are located in the lattices of intuitionistic modal logics. A particularly interesting result states that in Ext$\mathbf{IntK}_\Diamond$ the inconsistent logic has a continuum of immediate predecessors all of which have **Int** as their non-modal fragment, but no such logic exists in the lattice of extensions of $\mathbf{IntK}_\Box$.

Bezhanishvili [1997] has described all pretabular logics in Ext**MIPC**.

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