Termination Proving for Rewrite Systems and Programs

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Advanced Track

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https://www.dcs.bbk.ac.uk/~carsten/isr2026/

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- 2 special case of 1
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2011: PHP and Java issues with floating-point number parser

- http://www.exploringbinary.com/php-hangs-on-numeric-value-2-2250738585072011e-308/
- http://www.exploringbinary.com/java-hangs-when-converting-2-2250738585072012e-308/

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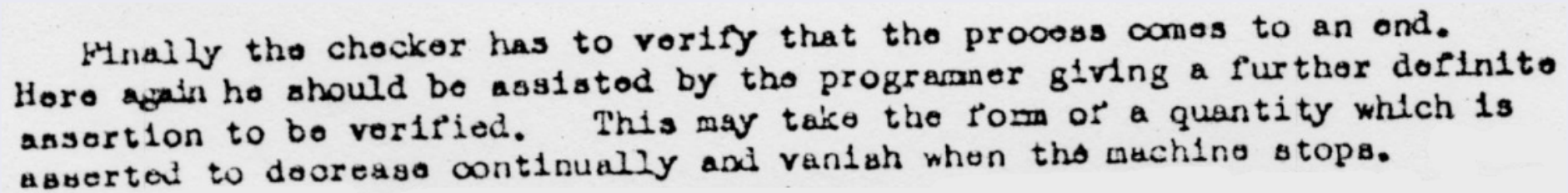
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- But, fear not ...

Termination Analysis, Classically

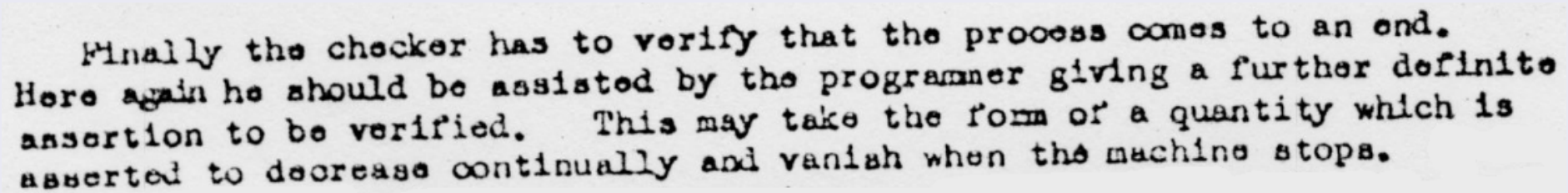
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Finally the checker has to verify that the process comes to an end. Here again he should be assisted by the programmer giving a further definite assertion to be verified. This may take the form of a quantity which is asserted to decrease continually and vanish when the machine stops.

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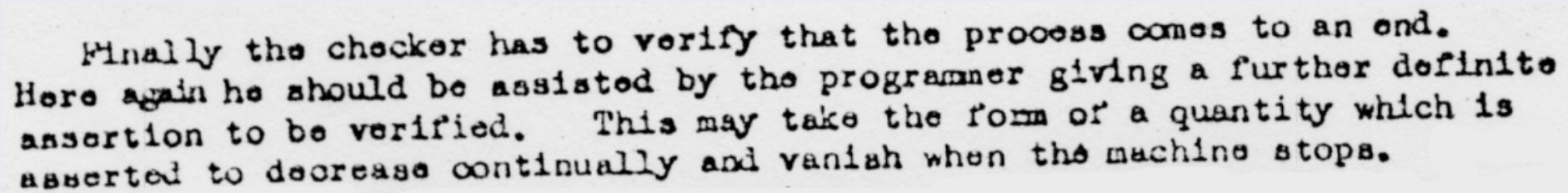

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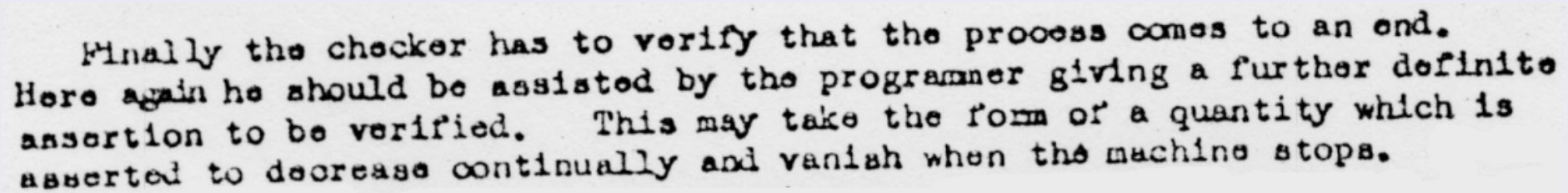

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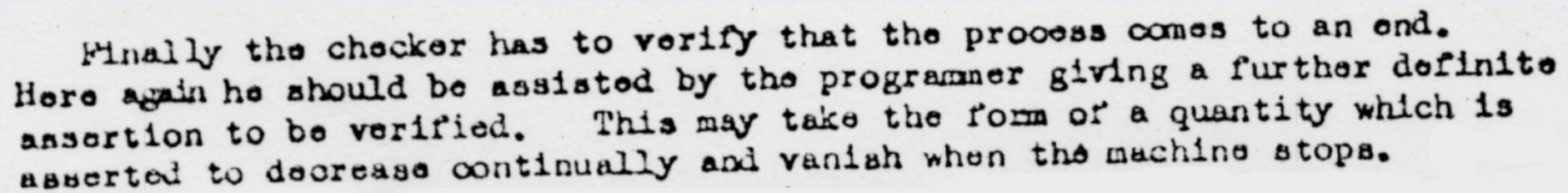

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Example (Termination can be simple)

while $x > 0$:

$x = x - 1$

Termination Analysis, in the Era of Automated Reasoning

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In practice:

- Encode only one proof **step** at a time
  $\longrightarrow$ try to prove only **part** of the program terminating
- **Repeat** until the whole program is proved terminating

The Rest of the Course

I. Termination Proving for Rewrite Systems

1. Term Rewrite Systems (TRSs)
2. Logically Constrained TRSs (LCTRSs)
3. Certification of Termination Proofs

II. Beyond Termination of Rewriting

1. Proving Program Termination via Rewrite Systems: Java
2. Finding Complexity Bounds for TRSs

I. Termination Analysis of Rewriting

I.1 Termination Analysis of Term Rewrite Systems

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- first-order (usually)
- no fixed evaluation strategy $\longrightarrow$ non-determinism!
- no fixed order of rules to apply (Haskell: top to bottom) $\longrightarrow$ non-determinism!
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  - Object-oriented programming: Java [Otto et al, *RTA '10*]

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In practice: use polynomial interpretations together with **Dependency Pairs**

Example (Division)

$$\mathcal{R} = \left\{ \begin{array}{rcl} \mathsf{minus}(x, \mathsf{0}) & \longrightarrow & x \\ \mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) & \longrightarrow & \mathsf{minus}(x, y) \\ \mathsf{quot}(\mathsf{0}, \mathsf{s}(y)) & \longrightarrow & \mathsf{0} \\ \mathsf{quot}(\mathsf{s}(x), \mathsf{s}(y)) & \longrightarrow & \mathsf{s}(\mathsf{quot}(\mathsf{minus}(x, y), \mathsf{s}(y))) \end{array} \right.$$

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Example (Constraints for Division)

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Non-linear constraints, even for linear interpretations

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3. Eliminate $\forall x, y$ by absolute positiveness criterion [Hong, Jakuš, *JAR '98*]:

   Here: $a_{\mathsf{s}}\, b_{\mathsf{m}} + a_{\mathsf{s}}\, c_{\mathsf{m}} \geq 0 \wedge b_{\mathsf{s}}\, b_{\mathsf{m}} - b_{\mathsf{m}} \geq 0 \wedge b_{\mathsf{s}}\, c_{\mathsf{m}} - c_{\mathsf{m}} \geq 0$

   Non-linear constraints, even for linear interpretations

Task: Show satisfiability of non-linear constraints over $\mathbb{N}$ ($\longrightarrow$ SMT solver!)
$\curvearrowright$ Prove termination of given term rewrite system

Extensions of Polynomial Interpretations

- Polynomials with negative coefficients and max-operator [Hirokawa, Middeldorp, *IC '07*; Fuhs et al, *SAT '07, RTA '08*]
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$$[\mathsf{a}(\mathsf{a}(x))] = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 + 2x_1 + 2x_2 \\ 2 + 2x_1 + 2x_2 \end{pmatrix}$$

$$>$$

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Matrix Interpretations and Monotone Algebras

Matrix interpretations [Endrullis, Waldmann, Zantema, *JAR '08*]

- linear interpretation to vectors over $\mathbb{N}^k$, coefficients are matrices
- useful for deeply nested terms
- automation: constraints with more complex atoms
- several flavours: plus-times-semiring, max-plus-semiring [Koprowski, Waldmann, *Acta Cyb. '09*], ...
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- if also $\succ$ should be monotone (**extended monotone algebra**):
  $a_i > b_i \Rightarrow [f](a_1, \ldots, a_i, \ldots, a_n) > [f](a_1, \ldots, b_i, \ldots, a_n)$

SAT and SMT Solving for Path Orders

Path orders: based on **precedences** on function symbols

- **Knuth-Bendix Order** (KBO) [Knuth, Bendix, *CPAA '70*]
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- **Weighted Path Order** (WPO) [Yamada, Kusakari, Sakabe, *SCP '15*]
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Dependency Graph

Example (Constraints for Division)

$$\mathcal{R} = \{ \ldots$$

$$\mathcal{P} = \left\{ \begin{array}{lcl} \mathsf{minus}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) & \gtrsim_{(\gtrsim,\gtrsim)} & \mathsf{minus}^\sharp(x,y) \\ \mathsf{quot}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) & \gtrsim_{(\gtrsim,\gtrsim)} & \mathsf{minus}^\sharp(x,y) \\ \mathsf{quot}^\sharp(\mathsf{s}(x),\mathsf{s}(y)) & \gtrsim_{(\gtrsim,\gtrsim)} & \mathsf{quot}^\sharp(\mathsf{minus}(x,y),\mathsf{s}(y)) \end{array} \right.$$

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which DPs can follow one another? [Arts, Giesl, *TCS '00*]

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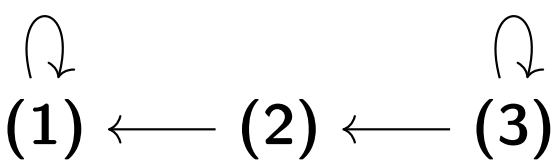
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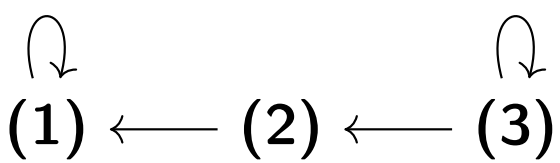
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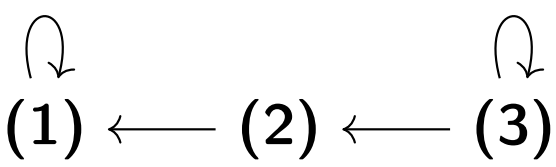
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Dependency Graph Processor

Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be the non-trivial Strongly Connected Components of the (over-approximated) dependency graph for $(\mathcal{P}, \mathcal{R})$.

Dependency Graph Processor: $(\mathcal{P}, \mathcal{R}) \vdash (\mathcal{P}_1, \mathcal{R}), \ldots, (\mathcal{P}_n, \mathcal{R})$

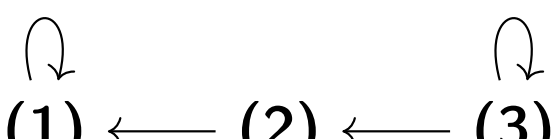
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Usable Rules

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$\Rightarrow$ instead of $\mathcal{R} \subseteq \succsim$, it suffices if **Usable Rules** $UR(\mathcal{P}, \mathcal{R}) \subseteq \succsim$

[Giesl et al, *JAR '06*; Hirokawa, Middeldorp, *IC '07*]

Usable Rules

Example (Constraints for Division)

$$UR(\mathcal{P}_2,\mathcal{R}) = \left\{ \begin{array}{rcl} \mathsf{minus}(x,\mathsf{0}) & \to & x \\ \mathsf{minus}(\mathsf{s}(x),\mathsf{s}(y)) & \to & \mathsf{minus}(x,y) \end{array} \right.$$

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Full rewriting: $\succsim$ must be "C_ε-compatible" ($\mathsf{c}(x,y) \succsim x$ and $\mathsf{c}(x,y) \succsim y$)

Not needed for termination of innermost rewriting!

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- Proving **non**-termination (an infinite run is possible) [Giesl, Thiemann, Schneider-Kamp, *FroCoS '05*; Payet, *TCS '08*; Zankl et al, *SOFSEM '10*; Emmes, Enger, Giesl, *IJCAR '12*; ...]

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- **Complexity analysis**
  [Hirokawa, Moser, *IJCAR '08*; Noschinski, Emmes, Giesl, *JAR '13*; ...]
  Can re-use termination machinery to infer and prove statements like "runtime complexity of this TRS is in $\mathcal{O}(n^3)$"
  $\longrightarrow$ more tomorrow!

SMT Solvers *from* Termination Analysis

Annual SMT-COMP, division QF_NIA (Quantifier-Free Non-linear Integer Arithmetic)

Year	Winner
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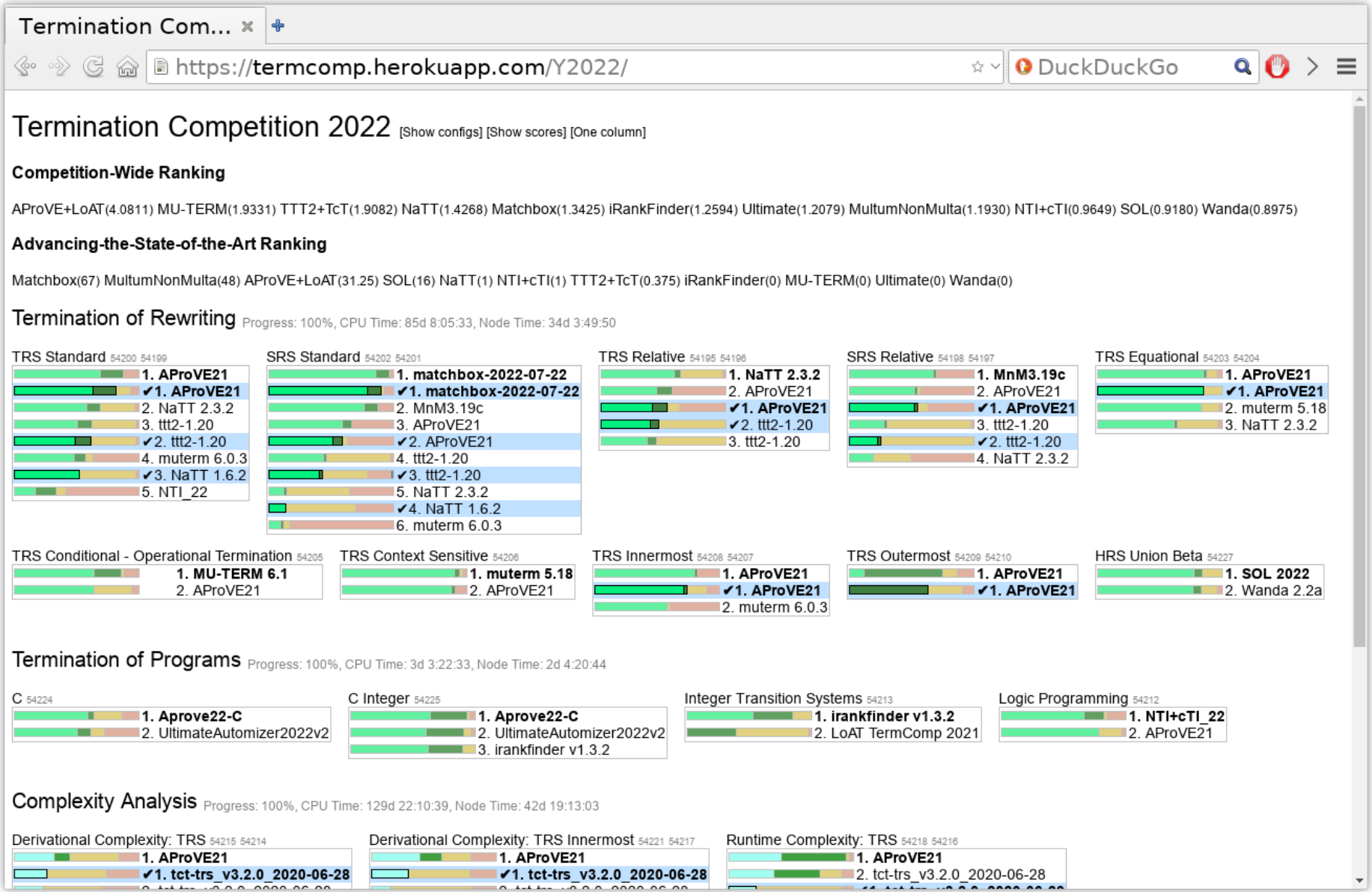
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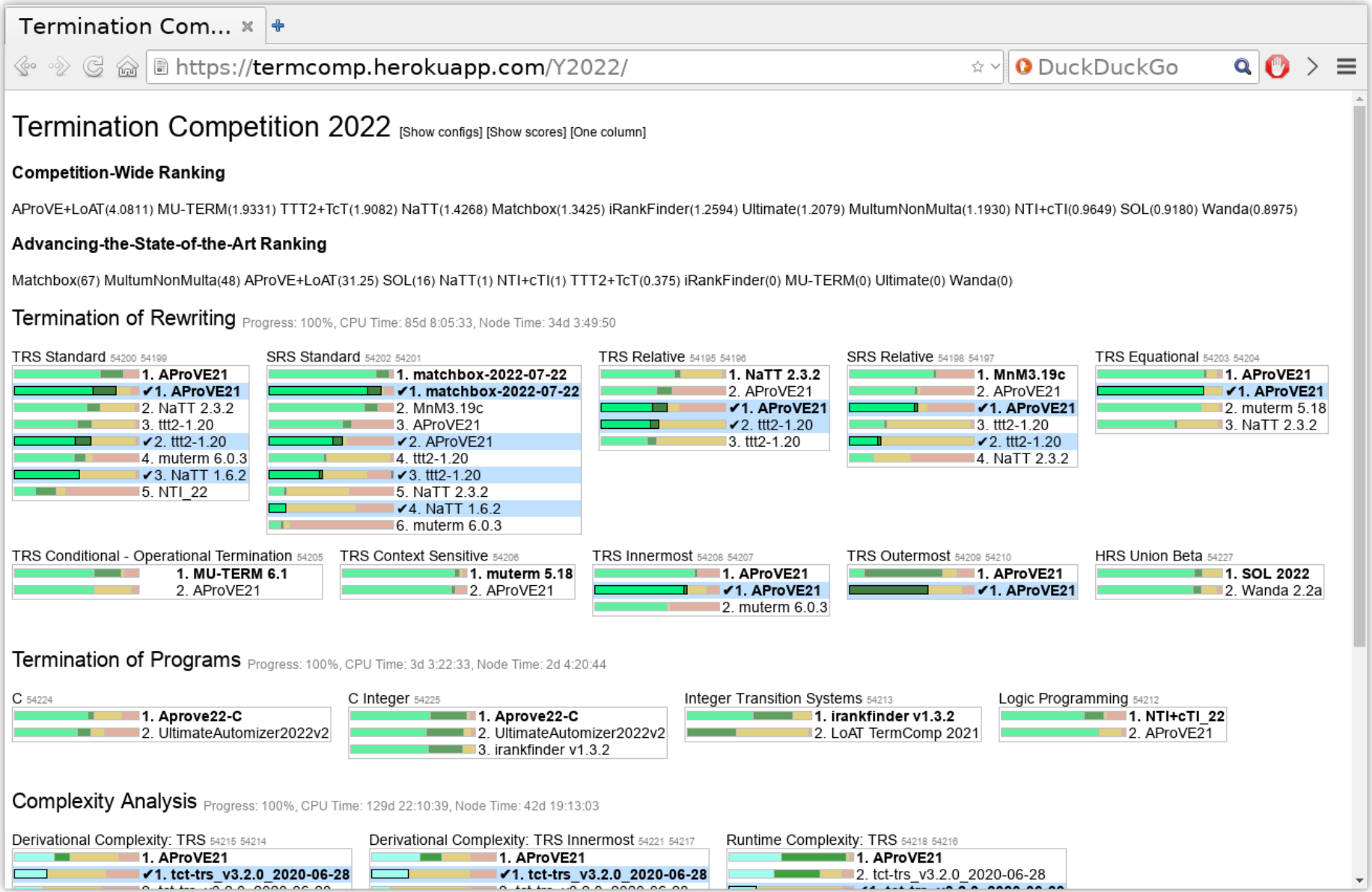
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(disclaimer: Z3 participated only *hors concours*)

The Termination Competition (termCOMP) (1/3)



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https://termination-portal.org/wiki/Termination_Competition

The Termination Competition (termCOMP) (2/3)

termCOMP 2022 participants (recent years similar)

- AProVE (RWTH Aachen, Birkbeck U London, U Innsbruck, . . .)
- iRankFinder (UC Madrid)
- LoAT (RWTH Aachen)
- Matchbox (HTWK Leipzig)
- Mu-Term (UP Valencia, UP Madrid)
- MultumNonMulta (BA Saarland)
- NaTT (AIST Tokyo)
- NTI+cTI (U Réunion)
- SOL (Gunma U)
- TcT (U Innsbruck, INRIA Sophia Antipolis)
- T_TT_2 (U Innsbruck)
- Ultimate Automizer (U Freiburg)
- Wanda (RU Nijmegen)

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- Part of the Olympic Games at the Federated Logic Conference

Input for Automated Tools

Web interfaces for termination and complexity of TRSs:

- AProVE: https://aprove.informatik.rwth-aachen.de/interface
- Mu-Term: http://zenon.dsic.upv.es/muterm/index.php/web-interface/
- TcT: https://tcs-informatik.uibk.ac.at/tools/tct/webinterface.php
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Input format for termination of TRSs:

```
(VAR x y)
(RULES
  plus(0, y) -> y
  plus(s(x), y) -> s(plus(x, y))
)
```

Remark: old format; termCOMP uses new ARI format (similar to SMT-LIB)

I.2 Termination Analysis of Rewrite Systems with Logical Constraints

Papers on termination of imperative programs often about **integers** as data

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Example (Imperative Program)

if $(x \geq 0)$
  while $(x \neq 0)$
    $x = x - 1;$

Does this program terminate?
(x ranges over $\mathbb{Z}$)

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Oh no! $\ell_1(-1) \to \ell_2(-1) \to \ell_1(-2) \to \ell_2(-2) \to \ell_1(-3) \to \cdots$

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Termination of TRSs from a given set of start terms:
Local termination
[Endrullis, de Vrijer, Waldmann, *LMCS '10*]

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Proving Termination with Invariants

Example (Transition system with invariants)

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$$[\ell_0](x) = a_0 + b_0 \cdot x, \quad [\ell_1](x) = a_1 + b_1 \cdot x, \quad \ldots$$

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More: [Podelski, Rybalchenko, *VMCAI '04*, Alias et al, *SAS '10*]

Proving Termination with Invariants

Example (Transition system with invariants)

$$\begin{array}{lcll} \ell_0(x) & \longrightarrow & \ell_1(x) & [x \geq 0] \\ \ell_1(x) & \longrightarrow & \ell_2(x) & [x \neq 0 \wedge x \geq 0] \\ \\ \ell_1(x) & \longrightarrow & \ell_3(x) & [x = 0 \wedge x \geq 0] \end{array}$$

Prove termination by ranking function $[\cdot]$ with $[\ell_0](x) = [\ell_1](x) = \cdots = x$

Automate search using parametric ranking function:

$$[\ell_0](x) = a_0 + b_0 \cdot x, \quad [\ell_1](x) = a_1 + b_1 \cdot x, \quad \ldots$$

Constraints here:

$$\begin{array}{lcll} x \geq 0 & \Rightarrow & a_2 + b_2 \cdot x > a_1 + b_1 \cdot (x-1) & \text{“decrease . . .”} \\ x \geq 0 & \Rightarrow & a_2 + b_2 \cdot x \geq 0 & \text{“. . . against a bound”} \end{array}$$

Use Farkas' Lemma to eliminate $\forall x$, solver for **linear** constraints gives model for a_i, b_i.

More: [Podelski, Rybalchenko, *VMCAI '04*, Alias et al, *SAS '10*]

Searching for Invariants Using SMT

Termination prover needs to find invariants for programs on integers

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Nowadays all SMT-based!

Extensions

- Proving **non**-termination (infinite run is possible **from initial states**) [Gupta et al, *POPL '08*, Brockschmidt et al, *FoVeOOS '11*, Chen et al, *TACAS '14*, Larraz et al, *CAV '14*, Cook et al, *FMCAD '14*, . . .]

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- Beyond sequential programs on integers:
  - structs/classes [Berdine et al, *CAV '06*; Otto et al, *RTA '10*; ...]
  - arrays (pointer arithmetic) [Ströder et al, *JAR '17*, ...]
  - multi-threaded programs [Cook et al, *PLDI '07*, ...]
  - ...

Recall: Why Care about Termination of Term Rewriting?

- Termination needed by theorem provers
- Translate program P with inductive data structures (trees) to TRS, represent data structures as terms

  $\Rightarrow$ Termination of TRS implies termination of P

  - Logic programming: Prolog
    [van Raamsdonk, *ICLP '97*; Schneider-Kamp et al, *TOCL '09*; Giesl et al, *PPDP '12*]
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What are the downsides of using TRSs here?

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Solution: use **constrained term rewriting**

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Term rewriting "with batteries included"

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- Adoption of LCTRSs by the community: 89 citations so far (Google Scholar, July 2026)

Logically Constrained TRSs: adoption

Analysis techniques for Logically Constrained TRSs:

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- Reachability / Safety [Ciobâcă, Lucanu, *IJCAR '18*; Kojima, Nishida, *JLAMP '23*]

Constrained Rewriting by Example

Example (Constrained Rewrite System)

$$
\begin{array}{rcll}
\ell_0(n, r) & \longrightarrow & \ell_1(n, r, \mathsf{Nil}) & \\
\ell_1(n, r, xs) & \longrightarrow & \ell_1(n-1, r+1, \mathsf{Cons}(r, xs)) & [n > 0] \\
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Here 7, 8, $\ldots$ are predefined constants.

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$$\begin{array}{rcll} \ell_0(n,r) & \to & \ell_1(n,r,\mathsf{Nil}) & \\ \ell_1(n,r,xs) & \to & \ell_1(n-1,r+1,\mathsf{Cons}(r,xs)) & [n>0] \\ \ell_1(n,r,xs) & \to & \ell_2(xs) & [n=0] \end{array}$$

Possible rewrite sequence:

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Here 7, 8, ... are predefined constants.

Termination: reuse techniques for TRSs and integer programs [Giesl et al, *JAR '17*]

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Techniques for LCTRSs in Ctrl [Kop, *WST '13*; Kop, Nishida, *LPAR '15*]

I.3 Termination and Complexity Proof Certification

Certification: Who Watches the Watchers?

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- solution: extract source code (Haskell, OCaml, ...) for proof checker $\longrightarrow$ CeTA tool from IsaFoR

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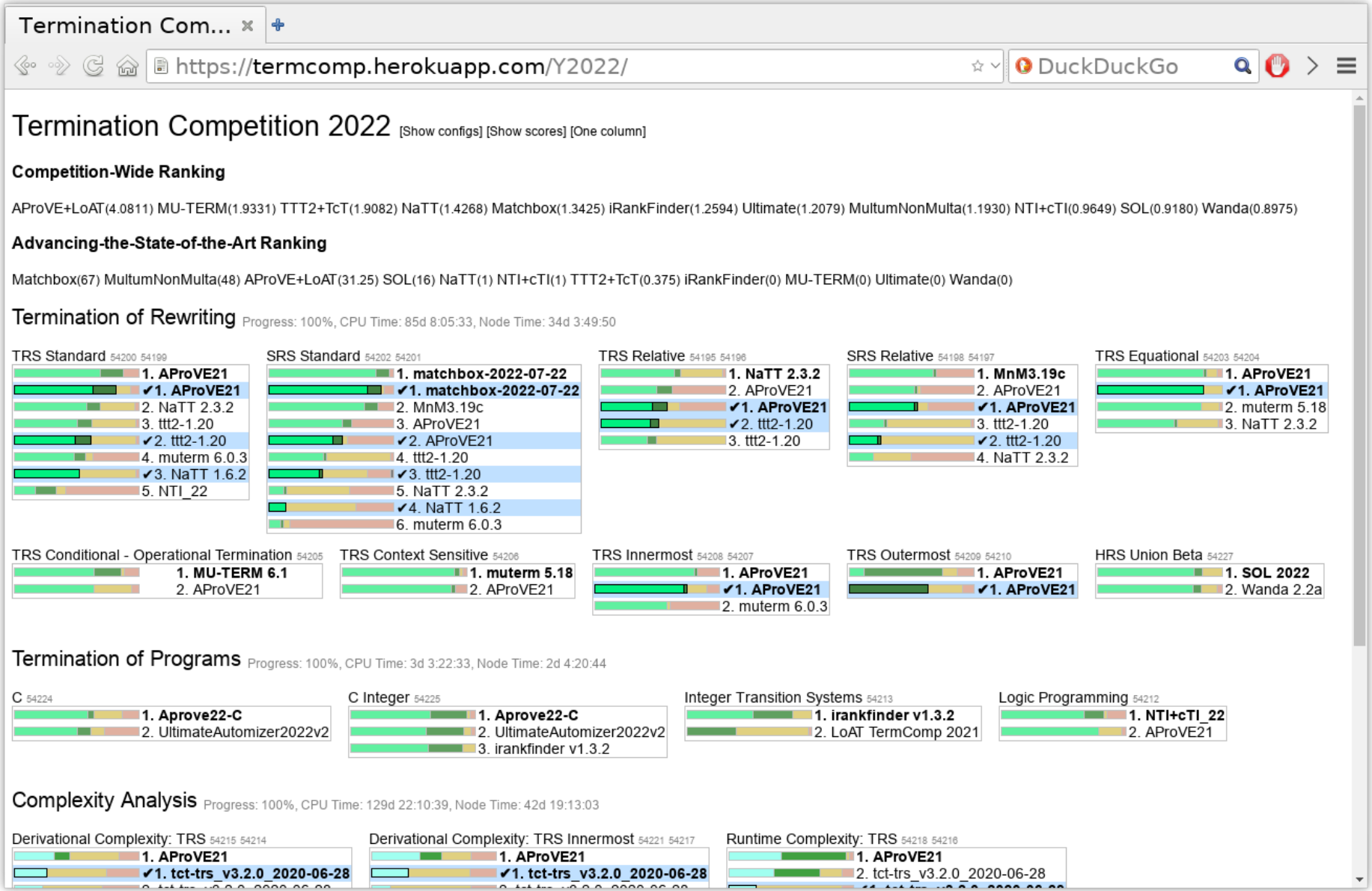
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If certification unsuccessful:
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termCOMP with Certification (✓) (1/2)



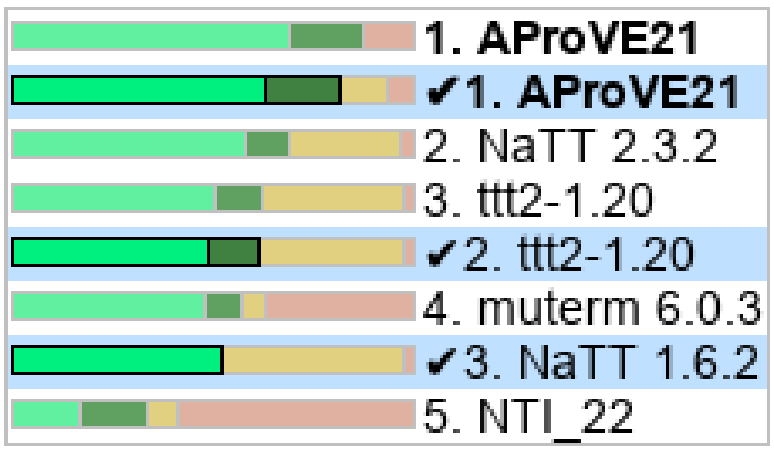
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Let's zoom in ...

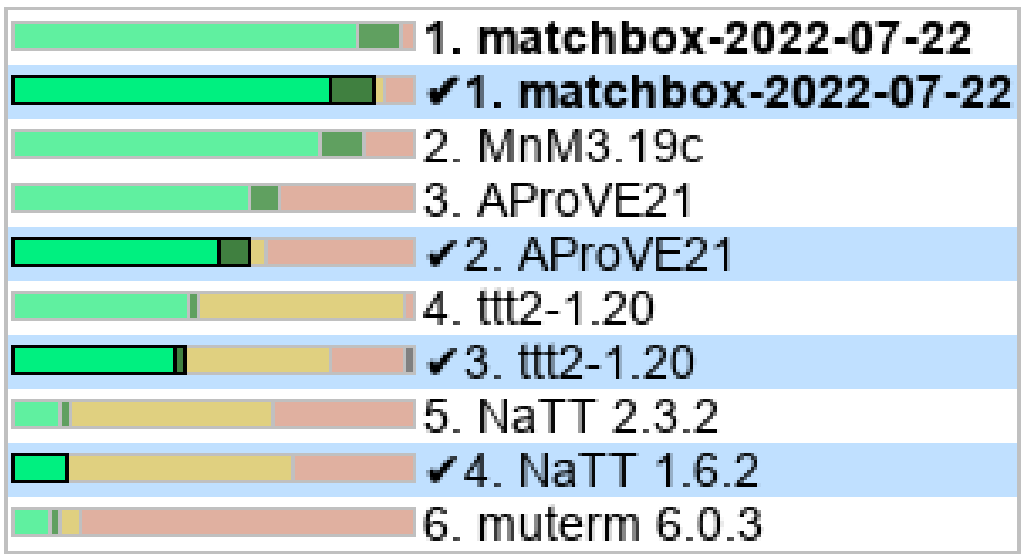
Termination of Rewriting

Progress: 100%, CPU Time: 85d 8:05:33, Node Time: 34d 3:4

TRS Standard 54200 54199

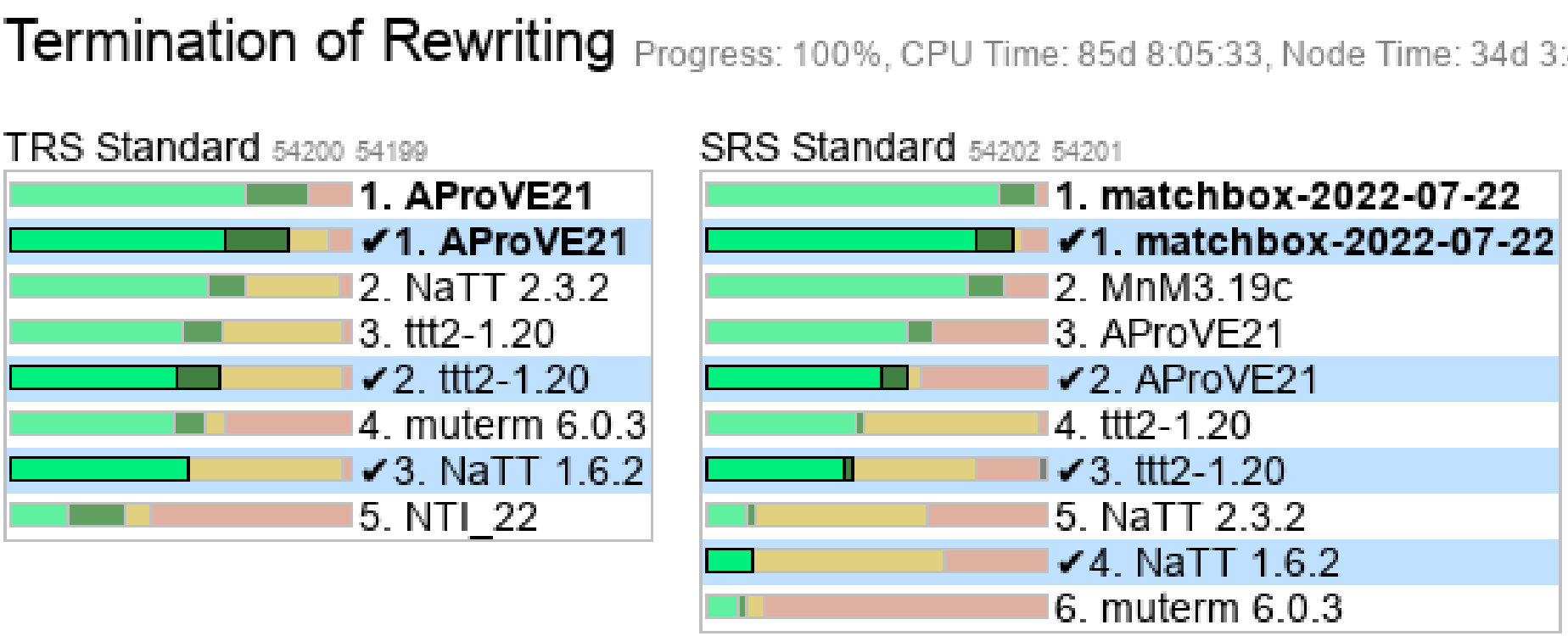


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⇒ proof certification is competitive!

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... there is a powerful SAT / SMT solver!

The Rest of the Course

I. Termination Proving for Rewrite Systems

1. Term Rewrite Systems (TRSs)
2. Logically Constrained TRSs (LCTRSs)
3. Certification of Termination Proofs

II. Beyond Termination of Rewriting

1. Proving Program Termination via Rewrite Systems: Java
2. Finding Complexity Bounds for TRSs

II. Beyond Termination of TRSs

II.1 Termination Analysis of Java Programs via TRSs

The story so far...

We have:

- tools to prove termination of TRSs
- tools to prove termination of TRSs with Logical Constraints (LCTRSs)
- (more trustworthy) tools to check termination proof certificates

We want: make use of this technology for real-world programs! How?

How to Analyse Program Termination?

Given: Program P in some language L

Goal: (Automatically) prove whether P terminates

Approach: Often in two phases

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From Program to Constrained Term Rewriting, high-level

- execute program **symbolically** from initial states of the program, handle language peculiarities here ($\longrightarrow$ Java: sharing, cyclicity analysis)

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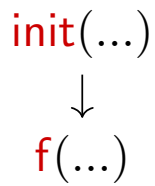
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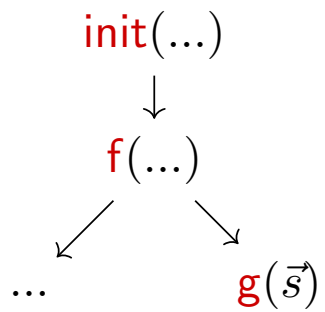
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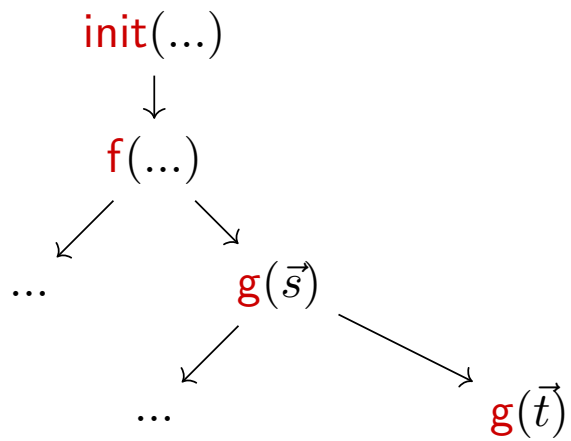
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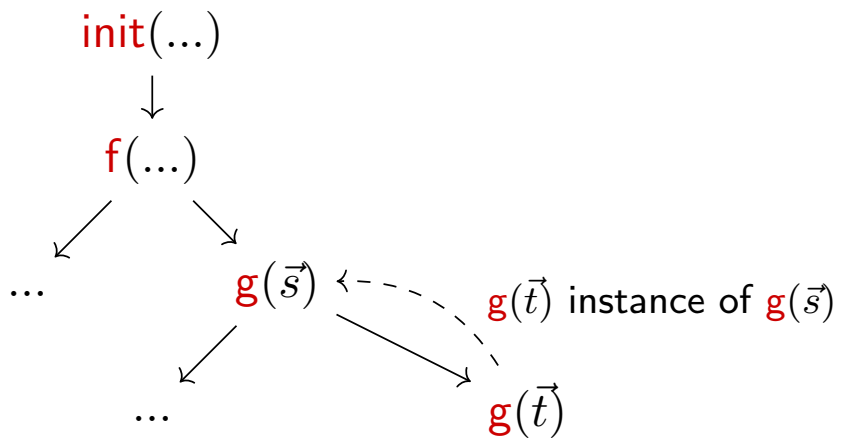
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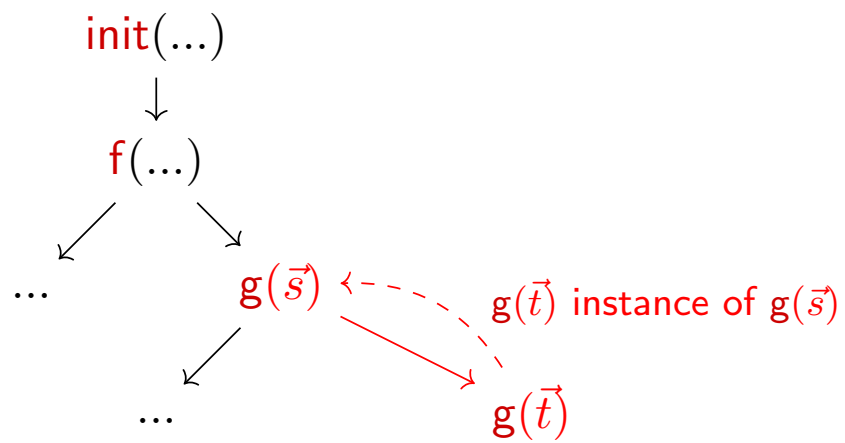
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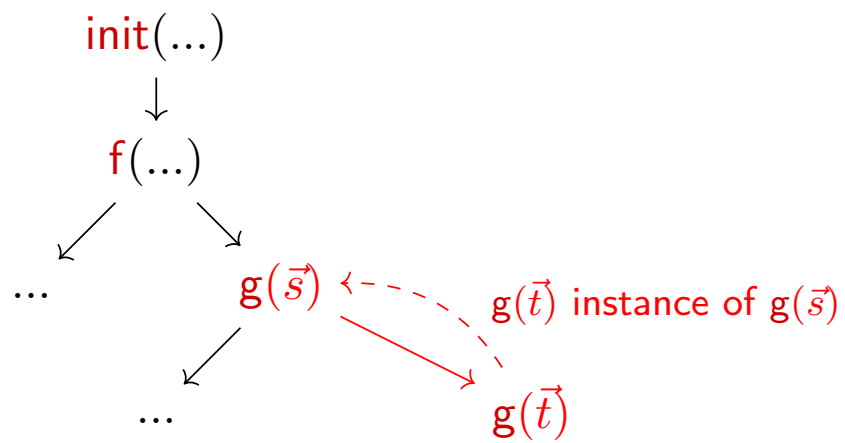
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- **extract (LC)TRS** from cycles in the representation
- if (LC)TRS terminates
  $\Rightarrow$ any **concrete program execution** can use cycles only finitely often
  $\Rightarrow$ the program **must terminate**

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- Prove **termination** of these rewrite rules
  $\Rightarrow$ implies termination of program from initial states

Java Challenges

Java: object-oriented imperative language

- sharing and aliasing (several references to the same object)
- side effects
- cyclic data objects (e.g., `list.next == list`)
- object-orientation with inheritance
- ...

Java Example

```
public class MyInt {

  // only wrap a primitive int
  private int val;

  // count "num" up to the value in "limit"
  public static void count(MyInt num, MyInt limit) {
    if (num == null || limit == null) {
      return;
    }
    // introduce sharing
    MyInt copy = num;
    while (num.val < limit.val) {
      copy.val++;
    }
  }
}
```

Does **count** terminate for all inputs? Why (not)?

(Assume that **num** and **limit** are not references to the same object.)

Approach to Termination Analysis of Java

Tailor two-stage approach to Java [Otto et al, *RTA '10*]

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Front-end: From Java to constrained rewrite system

- Build **symbolic execution graph** that over-approximates all runs of Java program (abstract interpretation)
- Symbolic execution graph has **invariants** for integers and heap object shape (trees?)
- Extract rewrite system from symbolic execution graph

Approach to Termination Analysis of Java

Tailor two-stage approach to Java [Otto et al, *RTA '10*]

Back-end: From rewrite system to termination proof

- Constrained term rewriting with integers [Giesl et al, *JAR '17*]
- Termination techniques for rewriting and for integers can be integrated

Front-end: From Java to constrained rewrite system

- Build **symbolic execution graph** that over-approximates all runs of Java program (abstract interpretation)
- Symbolic execution graph has **invariants** for integers and heap object shape (trees?)
- Extract rewrite system from symbolic execution graph

Implemented in the tool AProVE ($\longrightarrow$ web interface)

http://aprove.informatik.rwth-aachen.de/

Java: Source Code vs Bytecode

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```
00: aload_0
01: ifnull 8
04: aload_1
05: ifnonnull 9
08: return
09: aload_0
10: astore_2
11: aload_0
12: getfield val
15: aload_1
16: getfield val
19: if_icmpge 35
22: aload_2
23: aload_2
24: getfield val
27: iconst_1
28: iadd
29: putfield val
32: goto 11
35: return
```

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Here: **Java source code**

Ingredients for the Abstract Domain

1. program counter value (line number)
2. values of variables (treating `int` as $\mathbb{Z}$)
3. over-approximating info on possible variable values
   - integers: use intervals, e.g. x $\in [4,\ 7]$ or y $\in [0,\ \infty)$
   - heap memory with objects, **no sharing** unless stated otherwise
   - `MyInt`(?): maybe `null`, maybe a `MyInt` object

   Heap predicates:
   - Two references may be equal: $o_1 =^{?} o_2$

03 | num : o_1, limit : o_2
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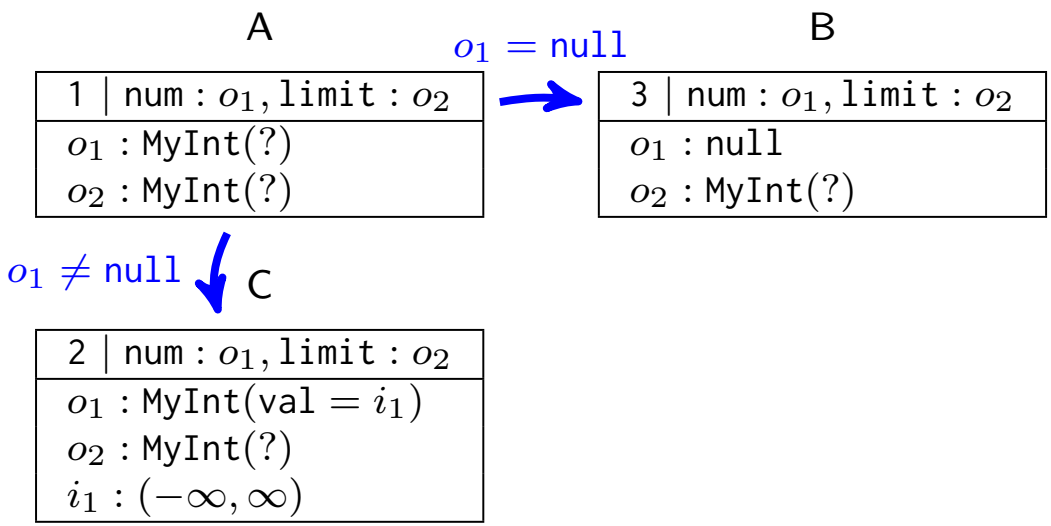
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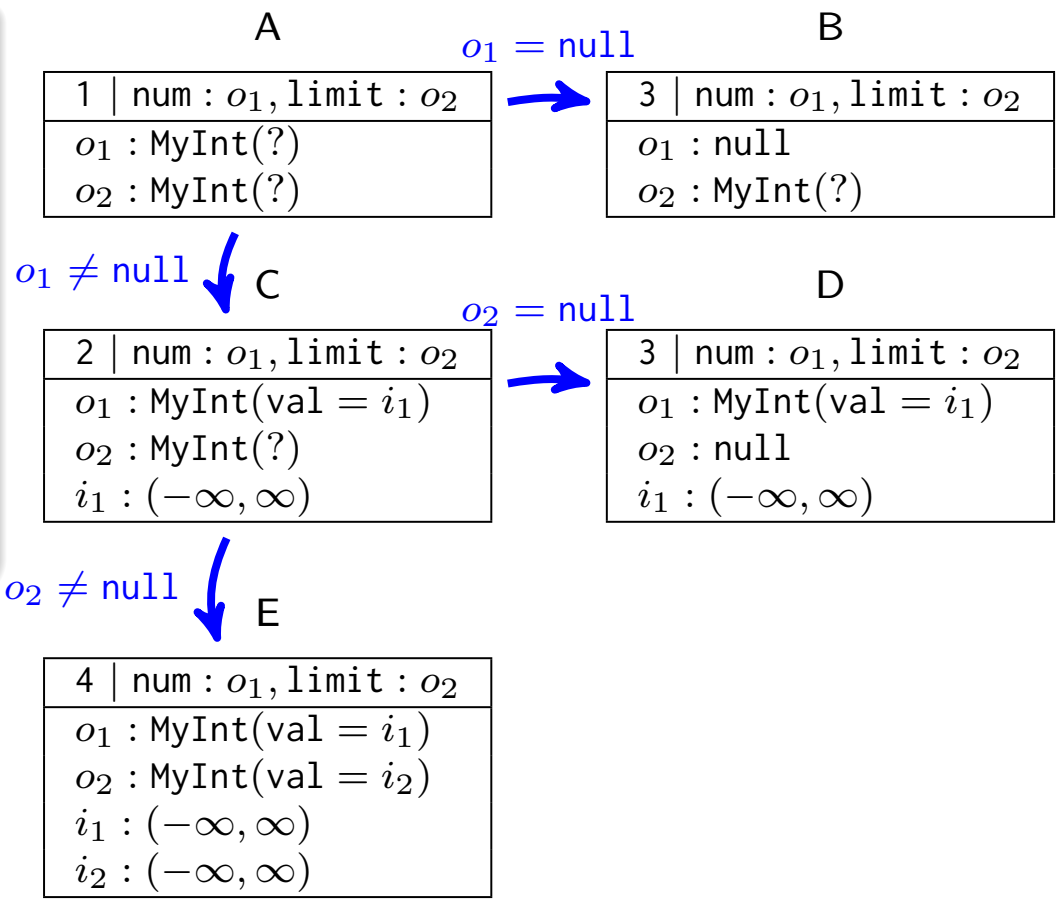


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means: refine X with $cond$, then evaluate to Y; here combined for brevity (narrowing)

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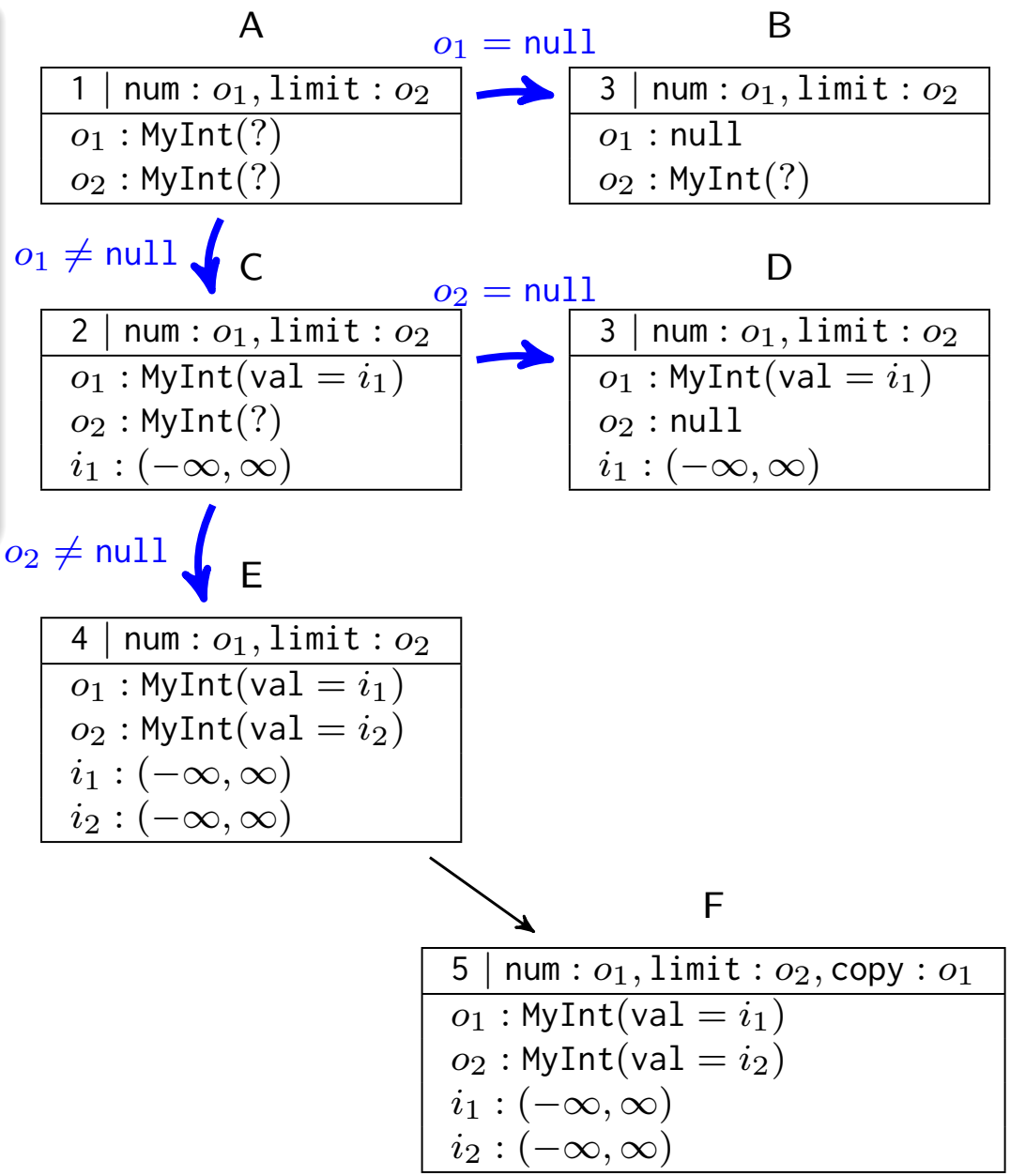


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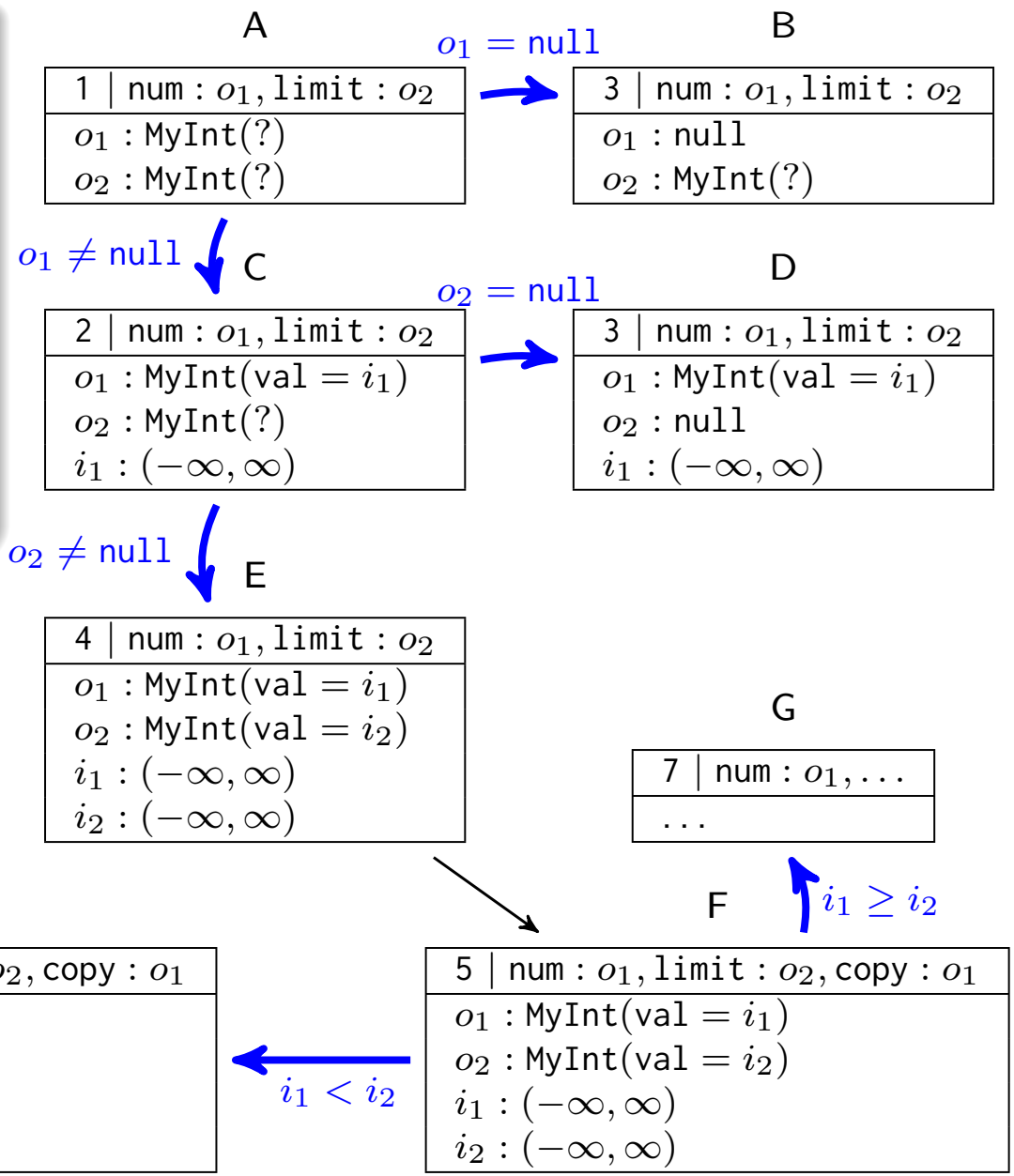


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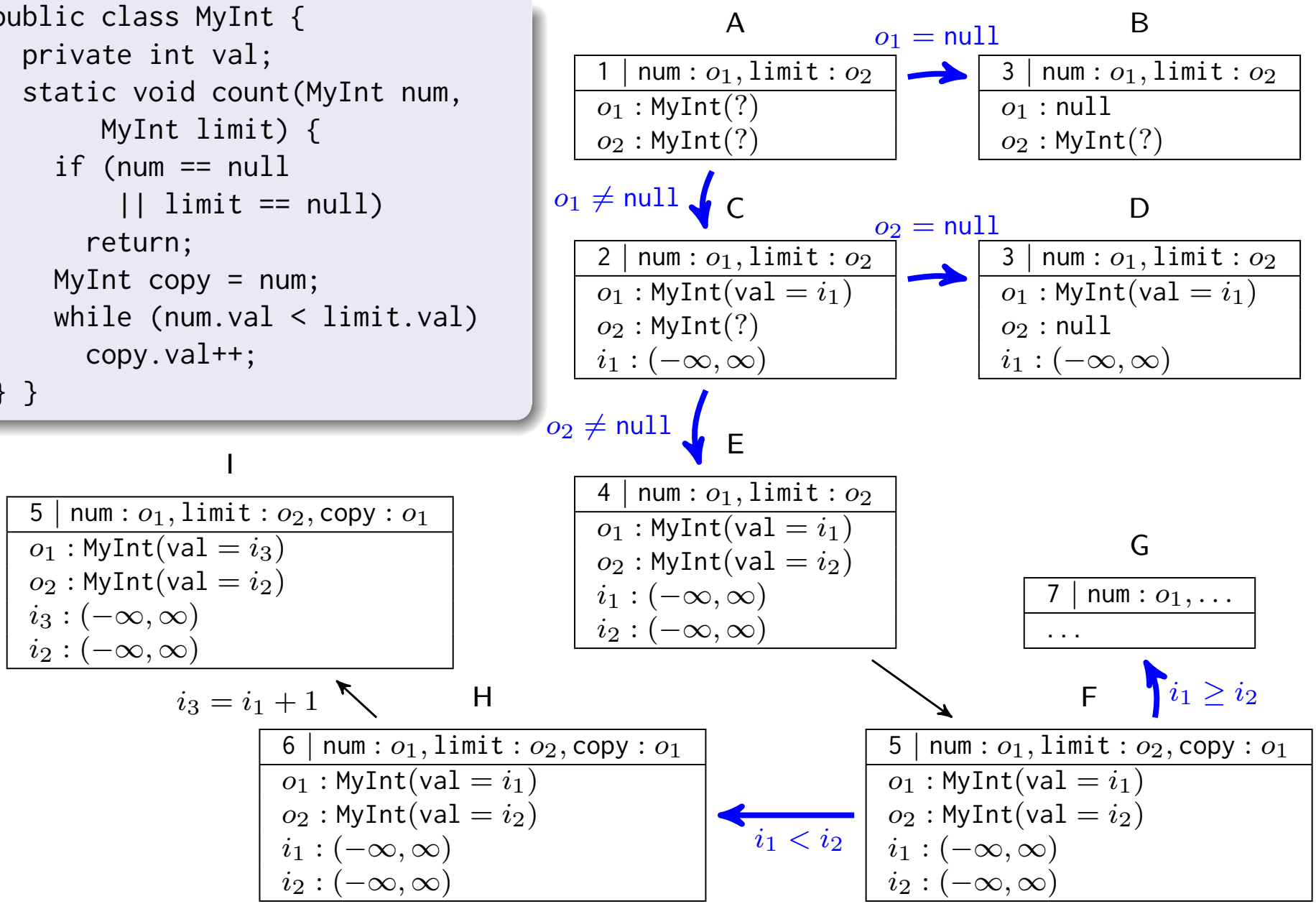
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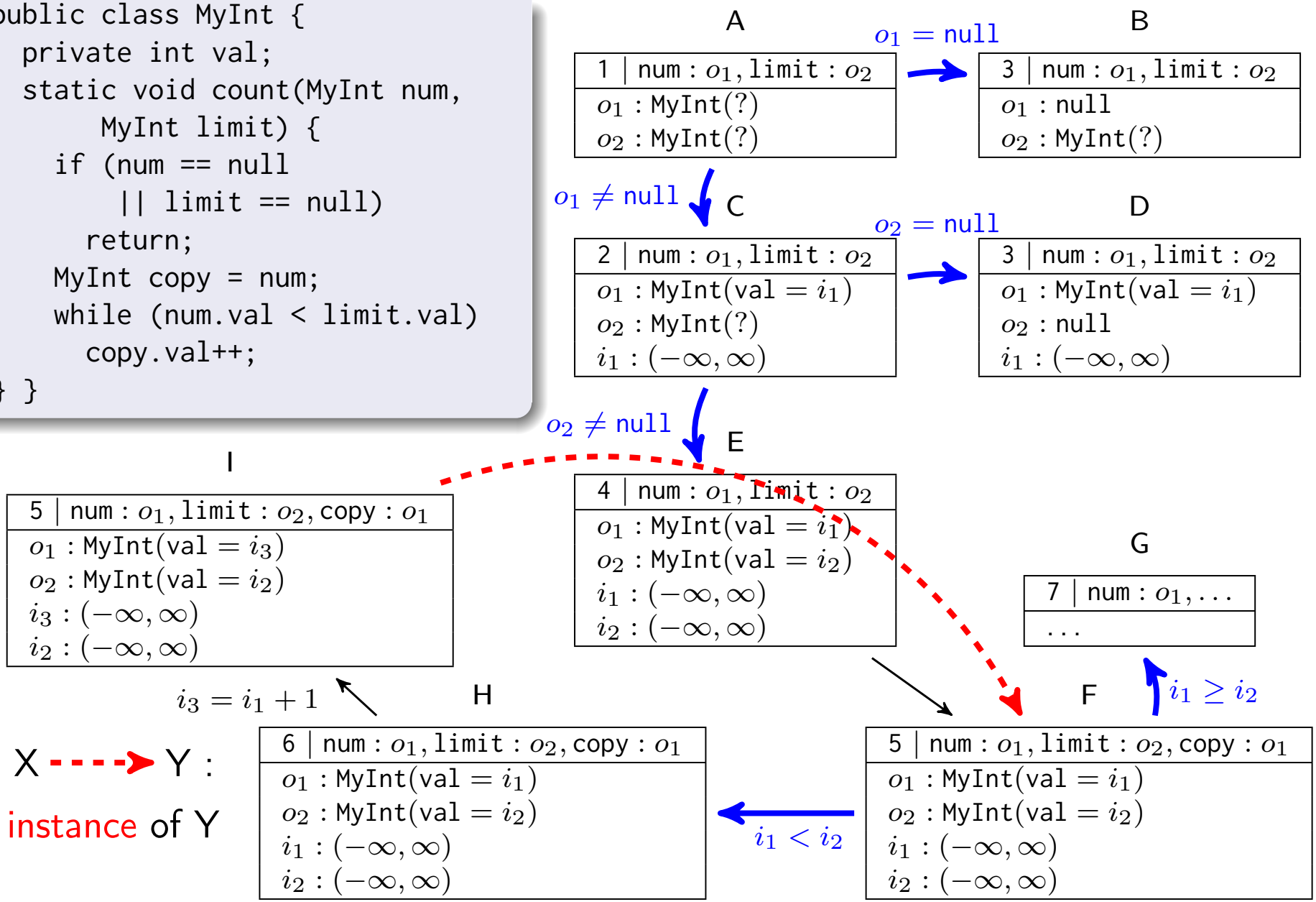
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Symbolic Execution Graphs

- symbolic over-approximation of all computations (abstract interpretation)
- expand nodes until all leaves correspond to program ends
- by suitable generalisation steps (widening), one can always get a **finite** symbolic execution graph
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From Java to Symbolic Execution Graphs

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Using Symbolic Execution Graphs for Termination Proofs

- every concrete Java computation corresponds to a **computation path** in the symbolic execution graph
- symbolic execution graph is called **terminating** iff it has no infinite computation path

Transformation of Objects to Terms (1/2)

Q

16 | num : o_1, limit : o_2, x : o_3, y : o_4, z : i_1
o_1 : MyInt(?)
o_2 : MyInt(val = i_2)
o_3 : null
o_4 : MyList(?)
o_4 !
i_1 : $[7, \infty)$
i_2 : $(-\infty, \infty)$

For every class C with n fields, introduce an n-ary function symbol C

- **term** for o_1: o_1
- **term** for o_2: MyInt(i_2)
- **term** for o_3: null
- **term** for o_4: x (new variable)
- **term** for i_1: i_1 with **side constraint** $i_1 \geq 7$
  (add invariant $i_1 \geq 7$ to constrained rewrite rules from state Q)

Transformation of Objects to Terms (2/2)

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public class A {
  int a;
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public class B extends A {
  int b;
}


...
A x = new A();
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Dealing with **subclasses**:

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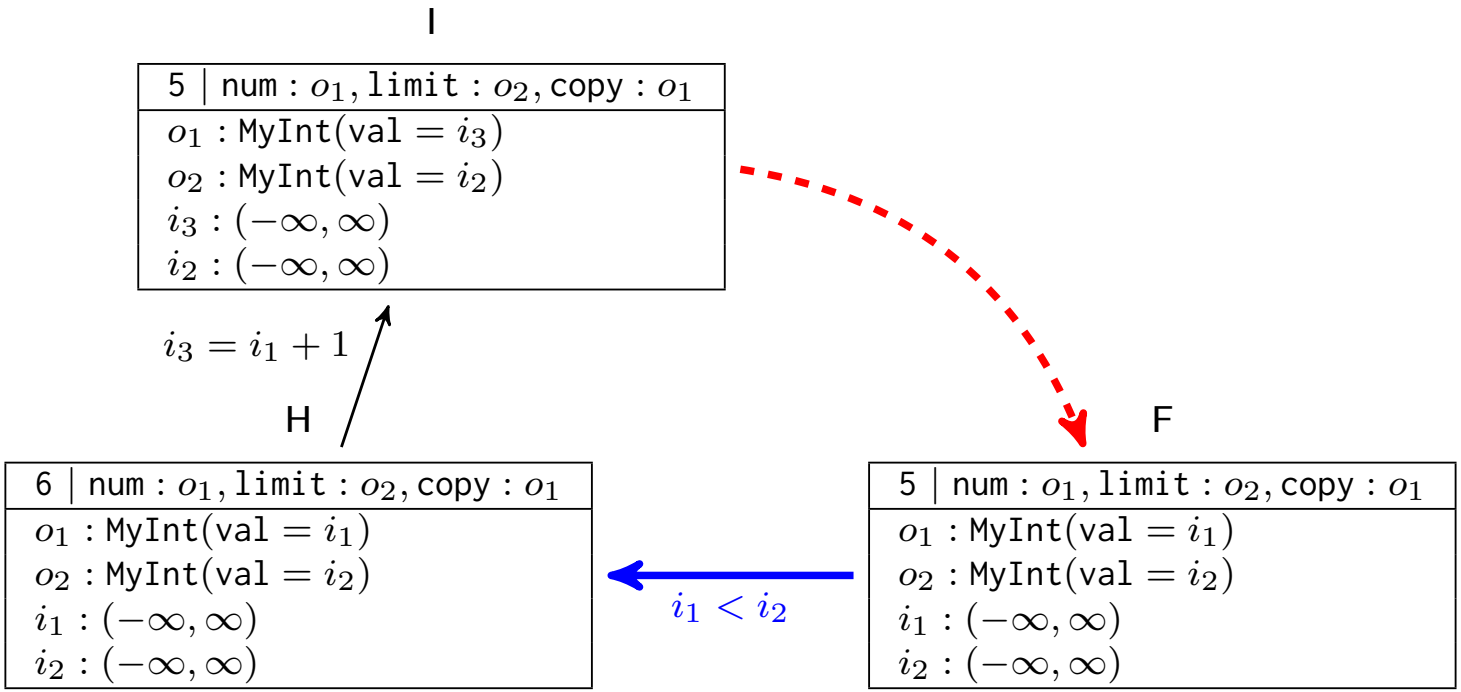
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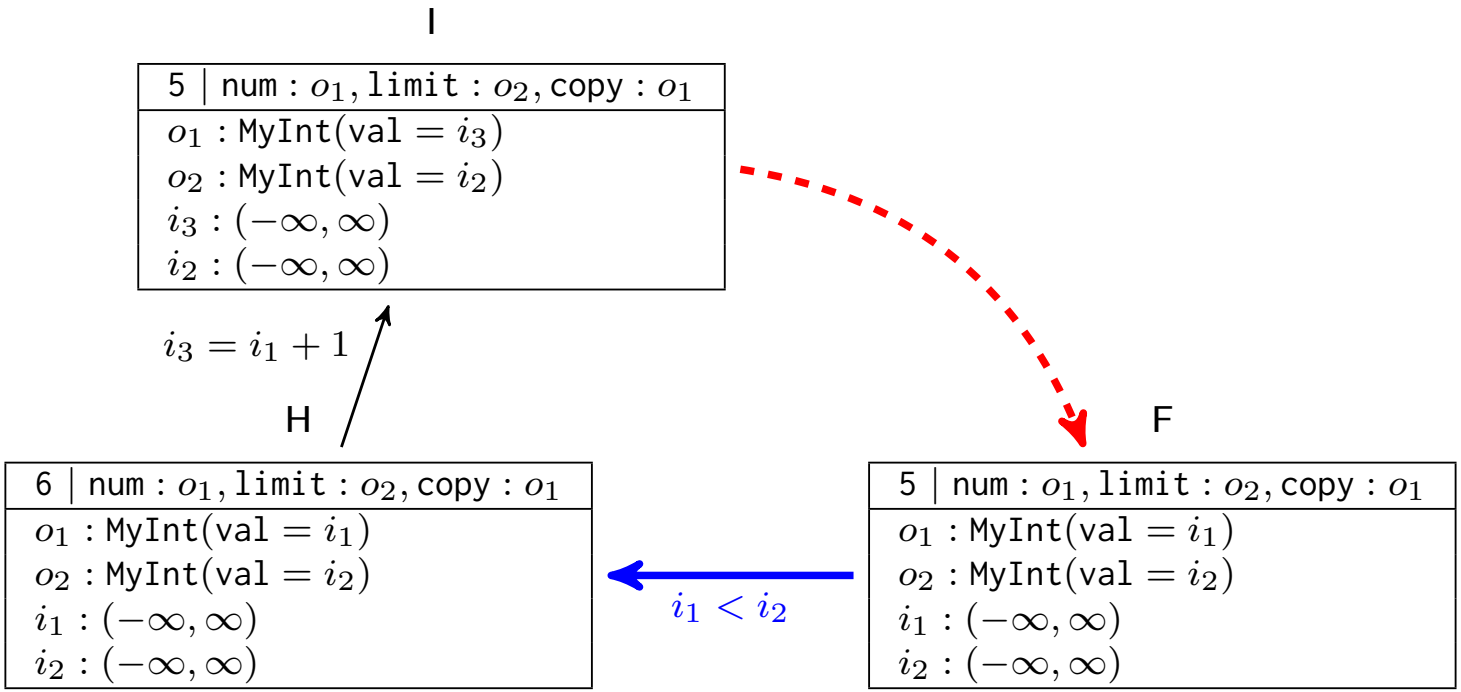
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- **term** for y: $\mathsf{jlO}(\mathsf{A}(\mathsf{B}(\mathsf{eoc}, 3), 2))$
- every class extends `Object`!
  ($\longrightarrow$ jlO $\equiv$ `java.lang.Object`)

From the Symbolic Execution Graph to Terms and Rules



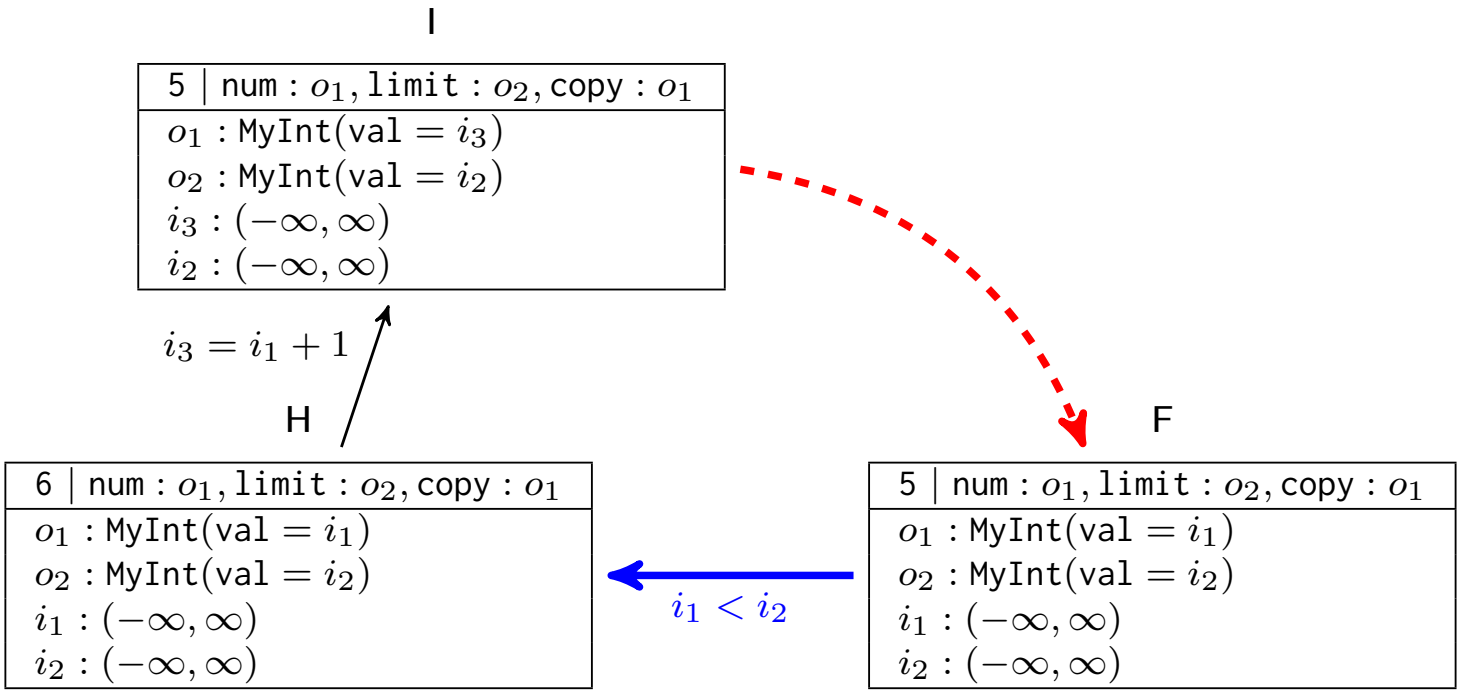
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- State F: $\ell_{\mathsf{F}}(\ \mathsf{jIO}(\mathsf{MyInt}(\mathsf{eoc}, i_1)),\ \mathsf{jIO}(\mathsf{MyInt}(\mathsf{eoc}, i_2))\ )$

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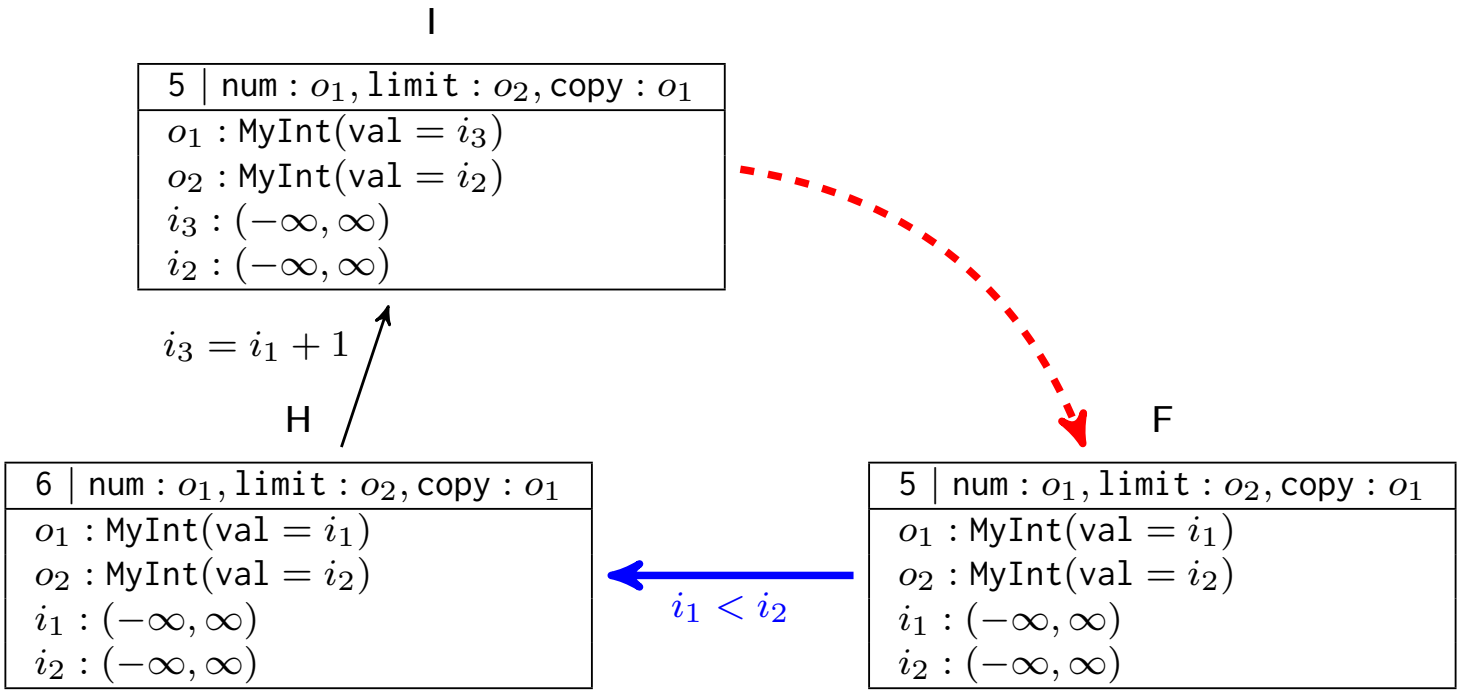


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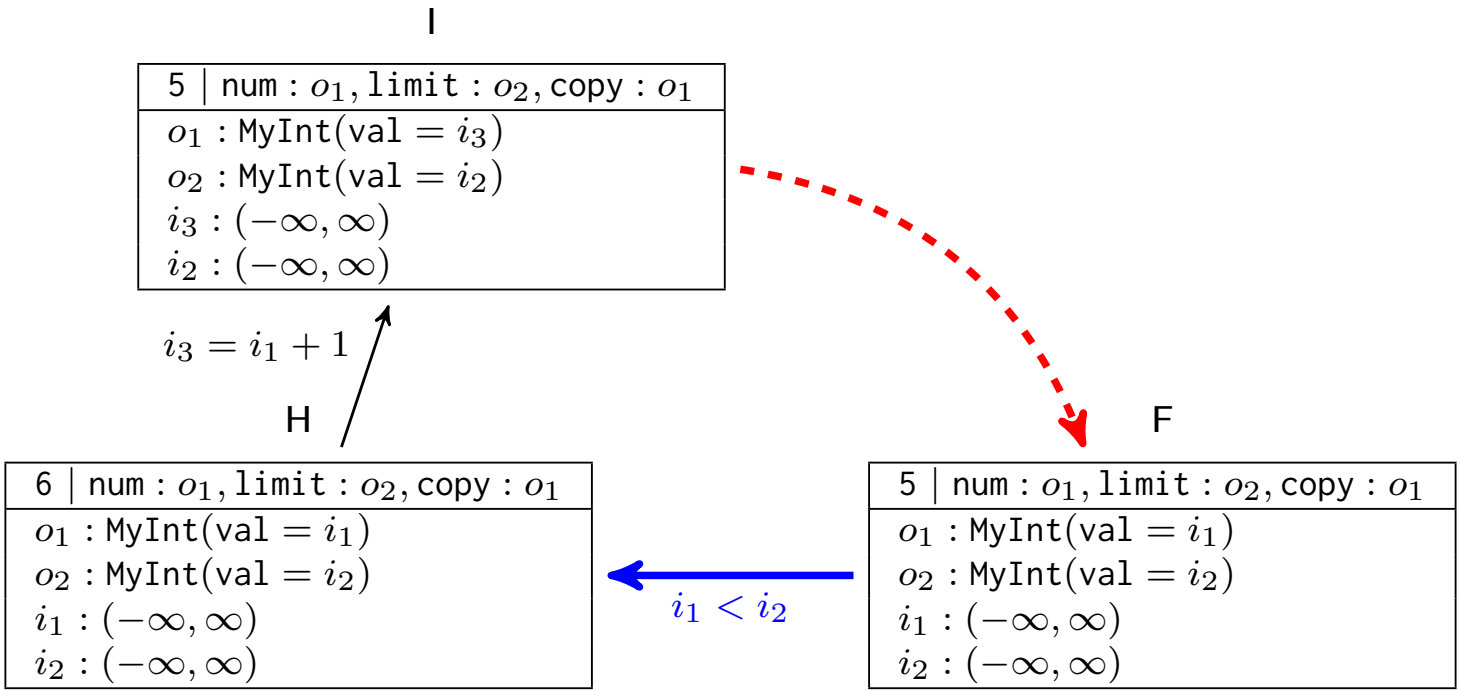
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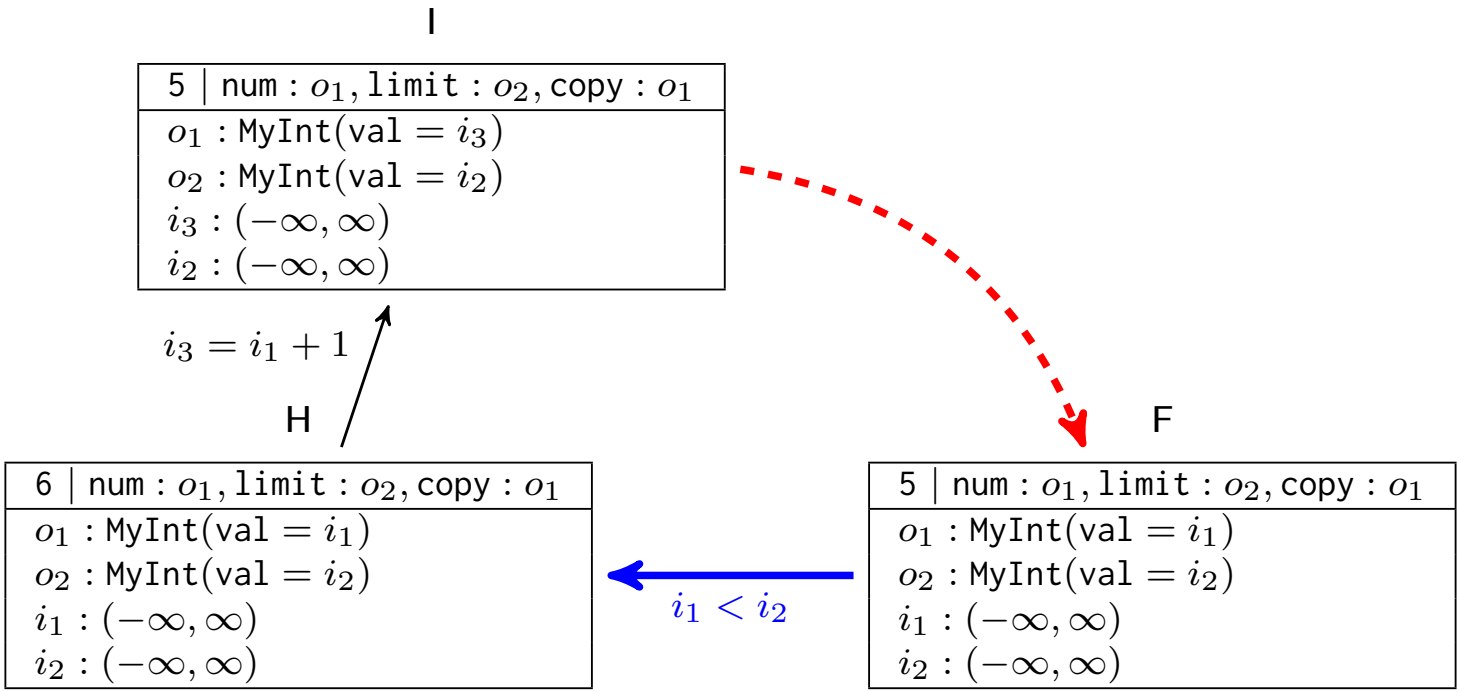
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- Termination easy to show (intuitively: $i_2 - i_1$ decreases against bound 0)

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- proving upper bounds for **time complexity** (abstracts terms to numbers) [Frohn and Giesl, *iFM '17*]

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Extensions:

- bitvector `int` semantics [Hensel et al, *JLAMP '18*]
- linked lists [Hensel, Giesl, *CADE '23*]

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- Works across paradigms: Java, C, Haskell, Prolog, ...

II.2 Complexity Analysis for Term Rewriting

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Example (Term Rewrite System (TRS) $\mathcal{R}$)

$$\mathsf{double}(\mathsf{0}) \longrightarrow \mathsf{0}$$

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Compute "double of 3 is 6":

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- $\mathrm{dc}_{\mathcal{R}}(n)$ for **equational reasoning**: cost of solving the word problem $\mathcal{E} \models s \equiv t$ by rewriting s and t via an equivalent convergent TRS $\mathcal{R}_{\mathcal{E}}$

Complexity Analysis for TRSs: Overview

1. Introduction
2. Automatically Finding Upper Bounds
3. Automatically Finding Lower Bounds
4. Transformational Techniques
5. Analysing Program Complexity via TRS Complexity
6. Current Developments

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Some Definitions

Definition (Derivation Height dh)

For a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ and a relation $\longrightarrow$, the **derivation height** is:

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Example: For $\mathcal{R}$ for double, we have $\mathrm{dc}_{\mathcal{R}}(n) \in \Theta(2^n)$.

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Goal: find **approximations** for derivational complexity

Initial focus: find upper bounds

$$\mathrm{dc}_{\mathcal{R}}(n) \in \mathcal{O}(...)$$

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Derivational Complexity from Polynomial Interpretations (1/2)

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$$\begin{aligned} \mathsf{double}(0) &\succ 0 \\ \mathsf{double}(\mathsf{s}(x)) &\succ \mathsf{s}(\mathsf{s}(\mathsf{double}(x))) \end{aligned}$$

Show $\mathrm{dc}_{\mathcal{R}}(n) < \omega$ by **termination proof** with reduction order $\succ$ on terms.

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[12] D. Lankford: *Canonical algebraic simplification in computational logic*, U Texas '75

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Example: $[\mathsf{double}](x) = 3 \cdot x, \qquad [\mathsf{s}](x) = x + 1, \qquad [0] = 1$

[12] D. Lankford: *Canonical algebraic simplification in computational logic*, U Texas '75

Derivational Complexity from Polynomial Interpretations (1/2)

Example (double)

$$\begin{aligned} \mathsf{double}(0) &\succ \mathsf{0} \\ \mathsf{double}(\mathsf{s}(x)) &\succ \mathsf{s}(\mathsf{s}(\mathsf{double}(x))) \end{aligned}$$

Show $\mathrm{dc}_{\mathcal{R}}(n) < \omega$ by **termination proof** with reduction order $\succ$ on terms.
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Extend to terms:

- $[x] = x$
- $[f(t_1, \ldots, t_n)] = [f]([t_1], \ldots, [t_n])$

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Automated search for $[\cdot]$ via SAT[13] or SMT[14] solving

[12] D. Lankford: *Canonical algebraic simplification in computational logic*, U Texas '75

[13] C. Fuhs, J. Giesl, A. Middeldorp, P. Schneider-Kamp, R. Thiemann, H. Zankl: *SAT solving for termination analysis with polynomial interpretations*, SAT '07

[14] C. Borralleras, S. Lucas, A. Oliveras, E. Rodríguez-Carbonell, A. Rubio: *SAT modulo linear arithmetic for solving polynomial constraints*, JAR '12

Derivational Complexity from Polynomial Interpretations (2/2)

Example (double)

$$\begin{array}{rcl|rcl} \mathsf{double}(0) & \succ & 0 & 3 & > & 1 \\ \mathsf{double}(\mathsf{s}(x)) & \succ & \mathsf{s}(\mathsf{s}(\mathsf{double}(x))) & 3 \cdot x + 3 & > & 3 \cdot x + 2 \end{array}$$

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This proves more than just termination...

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Theorem (Upper bounds for $\mathrm{dc}_{\mathcal{R}}(n)$ from polynomial interpretations[15])

- *Termination proof for TRS $\mathcal{R}$ with **polynomial** interpretation*

$$\Rightarrow \mathrm{dc}_{\mathcal{R}}(n) \in 2^{2^{\mathcal{O}(n)}}$$

[15] D. Hofbauer, C. Lautemann: *Termination proofs and the length of derivations*, RTA '89

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- *Termination proof for TRS $\mathcal{R}$ with **linear polynomial** interpretation*
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[15] D. Hofbauer, C. Lautemann: *Termination proofs and the length of derivations*, RTA '89

Derivational Complexity from Termination Proofs (1/2)

Termination proof for TRS $\mathcal{R}$ with ...

- match-bounds[16] $\Rightarrow \mathrm{dc}_{\mathcal{R}}(n) \in \mathcal{O}(n)$
- arctic matrix interpretations[17] $\Rightarrow \mathrm{dc}_{\mathcal{R}}(n) \in \mathcal{O}(n)$

[16] A. Geser, D. Hofbauer, J. Waldmann: *Match-bounded string rewriting systems*, AAECC '04

[17] A. Koprowski, J. Waldmann: *Max/plus tree automata for termination of term rewriting*, Acta Cyb. '09

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- matrix interpretation of spectral radius[19] ≤ 1
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[18] G. Moser, A. Schnabl, J. Waldmann: *Complexity analysis of term rewriting based on matrix and context dependent interpretations*, FSTTCS '08

[19] F. Neurauter, H. Zankl, A. Middeldorp: *Revisiting matrix interpretations for polynomial derivational complexity of term rewriting*, LPAR (Yogyakarta) '10

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- standard matrix interpretation[20] $\Rightarrow \mathrm{dc}_{\mathcal{R}}(n)$ is at most exponential

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[16] A. Geser, D. Hofbauer, J. Waldmann: *Match-bounded string rewriting systems*, AAECC '04

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[19] F. Neurauter, H. Zankl, A. Middeldorp: *Revisiting matrix interpretations for polynomial derivational complexity of term rewriting*, LPAR (Yogyakarta) '10

[20] J. Endrullis, J. Waldmann, and H. Zantema: *Matrix interpretations for proving termination of term rewriting*, JAR '08

Derivational Complexity from Termination Proofs (2/2)

Termination proof for TRS $\mathcal{R}$ with ...

- Lexicographic Path Order[21] $\Rightarrow \mathrm{dc}_{\mathcal{R}}(n)$ is at most multiple recursive[22]

[21] S. Kamin, J.-J. Lévy: *Two generalizations of the recursive path ordering*, U Illinois '80

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- Dependency Pairs framework[25][26] with dependency graphs, reduction pairs, subterm criterion $\Rightarrow \mathrm{dc}_{\mathcal{R}}(n)$ is at most multiple recursive[27]

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[25] J. Giesl, R. Thiemann, P. Schneider-Kamp, S. Falke: *Mechanizing and improving dependency pairs*, JAR '06

[26] N. Hirokawa and A. Middeldorp: *Tyrolean Termination Tool: Techniques and features*, IC '07

[27] G. Moser, A. Schnabl: *Termination proofs in the dependency pair framework may induce multiple recursive derivational complexity*, RTA '11

Runtime Complexity

- So far: upper bounds for derivational complexity

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Definition (Basic Term[28])

For defined symbols $\mathcal{D}$ and constructor symbols $\mathcal{C}$, the term

$$f(t_1, \ldots, t_n)$$

is in the set $\mathcal{T}_{\text{basic}}$ of **basic terms** iff $f \in \mathcal{D}$ and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{C}, \mathcal{V})$.

[28] N. Hirokawa, G. Moser: *Automated complexity analysis based on the dependency pair method*, IJCAR '08

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For a TRS $\mathcal{R}$, the **runtime complexity** is:

$$\mathrm{rc}_{\mathcal{R}}(n) \;=\; \sup\{\, \mathrm{dh}(t, \to_{\mathcal{R}}) \mid t \in \mathcal{T}_{\text{basic}}, |t| \leq n \,\}$$

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$\mathrm{rc}_{\mathcal{R}}(n)$: like derivational complexity... but for basic terms only!

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Runtime Complexity from Polynomial Interpretations

Polynomial interpretations can induce upper bounds to runtime complexity:[29]

Definition (Strongly linear polynomial, restricted interpretation)

- Polynomial p is **strongly linear** iff
  $p(x_1, \ldots, x_n) = x_1 + \cdots + x_n + a$ for some $a \in \mathbb{N}$.
- Polynomial interpretation $[\cdot]$ is **restricted** iff
  for all constructor symbols f, $[f](x_1, \ldots, x_n)$ is strongly linear.

Idea: $[t] \leq c \cdot |t|$ for fixed $c \in \mathbb{N}$.

[29] G. Bonfante, A. Cichon, J. Marion, H. Touzet: *Algorithms with polynomial interpretation termination proof*, JFP '01

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Theorem (Upper bounds for $\mathrm{rc}_{\mathcal{R}}(n)$ from restricted interpretations)

*Termination proof for TRS $\mathcal{R}$ with **restricted** interpretation $[\cdot]$ of degree at most d for $[f]$* $\Rightarrow \mathrm{rc}_{\mathcal{R}}(n) \in \mathcal{O}(n^d)$

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Example: $[\mathsf{double}](x) = 3 \cdot x, [\mathsf{s}](x) = x + 1, [\mathsf{0}] = 1$ is restricted, degree 1

$\Rightarrow \mathrm{rc}_{\mathcal{R}}(n) \in \mathcal{O}(n)$ for TRS $\mathcal{R}$ for double

[29] G. Bonfante, A. Cichon, J. Marion, H. Touzet: *Algorithms with polynomial interpretation termination proof*, JFP '01

Dependency Tuples for *Innermost* Runtime Complexity irc

Here: innermost rewriting ($\approx$ call-by-value)

Example (reverse)

$$\begin{array}{rcl|rcl} \mathsf{app}(\mathsf{nil}, y) & \to & y & \mathsf{app}(\mathsf{add}(n,x),y) & \to & \mathsf{add}(n,\mathsf{app}(x,y)) \\ \mathsf{reverse}(\mathsf{nil}) & \to & \mathsf{nil} & \mathsf{reverse}(\mathsf{add}(n,x)) & \to & \mathsf{app}(\mathsf{reverse}(x),\mathsf{add}(n,\mathsf{nil})) \end{array}$$

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For rule $\ell \to r$, eval of ℓ costs 1 + eval of all function calls in r **together**:

[30] L. Noschinski, F. Emmes, J. Giesl: *Analyzing innermost runtime complexity of term rewriting by dependency pairs*, JAR '13

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$$\begin{aligned} \mathsf{app}^\sharp(\mathsf{nil}, y) &\to \mathsf{Com}_0 \\ \mathsf{app}^\sharp(\mathsf{add}(n, x), y) &\to \mathsf{Com}_1(\mathsf{app}^\sharp(x, y)) \\ \mathsf{reverse}^\sharp(\mathsf{nil}) &\to \mathsf{Com}_0 \\ \mathsf{reverse}^\sharp(\mathsf{add}(n, x)) &\to \mathsf{Com}_2(\mathsf{app}^\sharp(\mathsf{reverse}(x), \mathsf{add}(n, \mathsf{nil})), \mathsf{reverse}^\sharp(x)) \end{aligned}$$

- Function calls to count marked with $\sharp$
- Compound symbols Com_k group function calls together

[30] L. Noschinski, F. Emmes, J. Giesl: *Analyzing innermost runtime complexity of term rewriting by dependency pairs*, JAR '13

Polynomial Interpretations for Dependency Tuples

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$[\mathsf{app}](x_1, x_2) = x_1 + x_2$ $\quad$ $[\mathsf{reverse}](x_1) = x_1$ (bounds helper fct. result size)

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$[\mathsf{add}](x_1, x_2) = x_2 + 1$ ($\leq$ restricted interpret.)
$[\mathsf{reverse}](x_1) = x_1$ (bounds helper fct. result size)
$[\mathsf{reverse}^\sharp](x_1) = x_1^2 + x_1 + 1$ (complexity of fct.)

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\mathsf{app}^\sharp(\mathsf{add}(n, x), y) &\to \mathsf{Com}_1(\mathsf{app}^\sharp(x, y)) \\
\mathsf{reverse}^\sharp(\mathsf{nil}) &\to \mathsf{Com}_0 \\
\mathsf{reverse}^\sharp(\mathsf{add}(n, x)) &\to \mathsf{Com}_2(\mathsf{app}^\sharp(\mathsf{reverse}(x), \mathsf{add}(n, \mathsf{nil})), \mathsf{reverse}^\sharp(x))
\end{aligned}
$$

$$
\begin{array}{l|l}
\mathsf{app}(\mathsf{nil}, y) \to y & \mathsf{app}(\mathsf{add}(n, x), y) \to \mathsf{add}(n, \mathsf{app}(x, y)) \\
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\end{array}
$$

Use interpretation $[\cdot]$ with $[\mathsf{Com}_k](x_1, \ldots, x_k) = x_1 + \cdots + x_k$ and

$[\mathsf{nil}] = 0$ $\quad$ $[\mathsf{add}](x_1, x_2) = x_2 + 1$ ($\leq$ restricted interpret.)

$[\mathsf{app}](x_1, x_2) = x_1 + x_2$ $\quad$ $[\mathsf{reverse}](x_1) = x_1$ (bounds helper fct. result size)

$[\mathsf{app}^\sharp](x_1, x_2) = x_1 + 1$ $\quad$ $[\mathsf{reverse}^\sharp](x_1) = x_1^2 + x_1 + 1$ (complexity of fct.)

to show $[\ell] \geq [r]$ for all rules and $[\ell] \geq 1 + [r]$ for all Dependency Tuples

Polynomial Interpretations for Dependency Tuples

Example (reverse, Dependency Tuples for reverse)

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Maximum degree of $[f^\sharp]$ is 2 $\Rightarrow \mathrm{irc}_\mathcal{R}(n) \in \mathcal{O}(n^2)$

Dependency Tuple (DT) Framework (simplified)

Initial set of DTs for $\mathcal{R}$: $DT(\mathcal{R})$.

$\mathcal{S} := DT(\mathcal{R})$ // must still be counted
while $\mathcal{S} \neq \emptyset$:

- Find (at most) restricted interpretation $[\cdot]$ with
  - $[\mathsf{Com}_k](x_1, \ldots, x_k) = x_1 + \cdots + x_k$
  - for all $s \longrightarrow t \in DT(\mathcal{R}) \cup UR(DT(\mathcal{R}), \mathcal{R})$: $[s] \gtrsim [t]$
- Delete DTs $s \longrightarrow t$ with $[s] \gtrsim [t] + 1$ from $\mathcal{S}$

If k is the maximum degree of any $[f^\sharp]$, $\mathrm{irc}_{\mathcal{R}}(n) \in \mathcal{O}(n^k)$!

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- Extensions by polynomial path orders[32], usable replacement maps[33], a combination framework for complexity analysis[34], ...

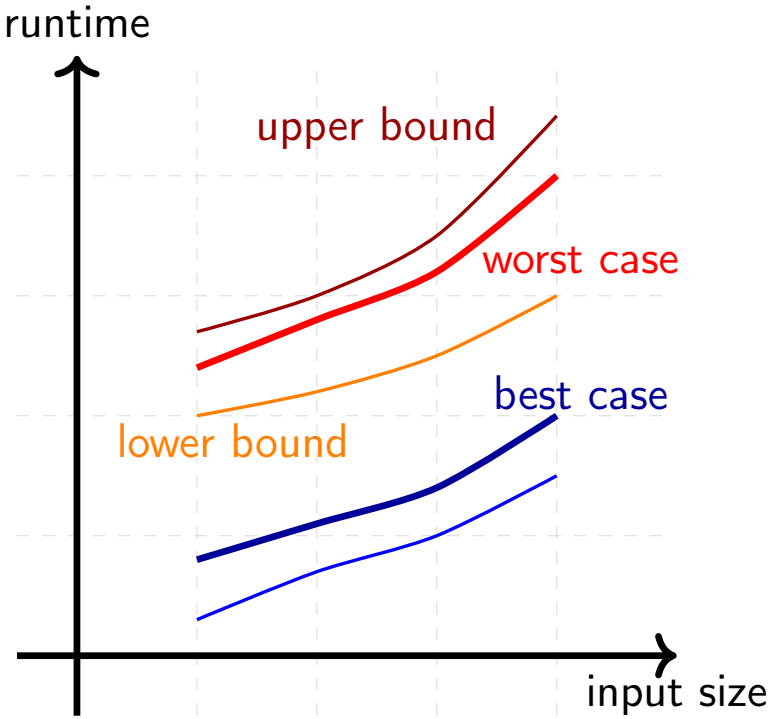
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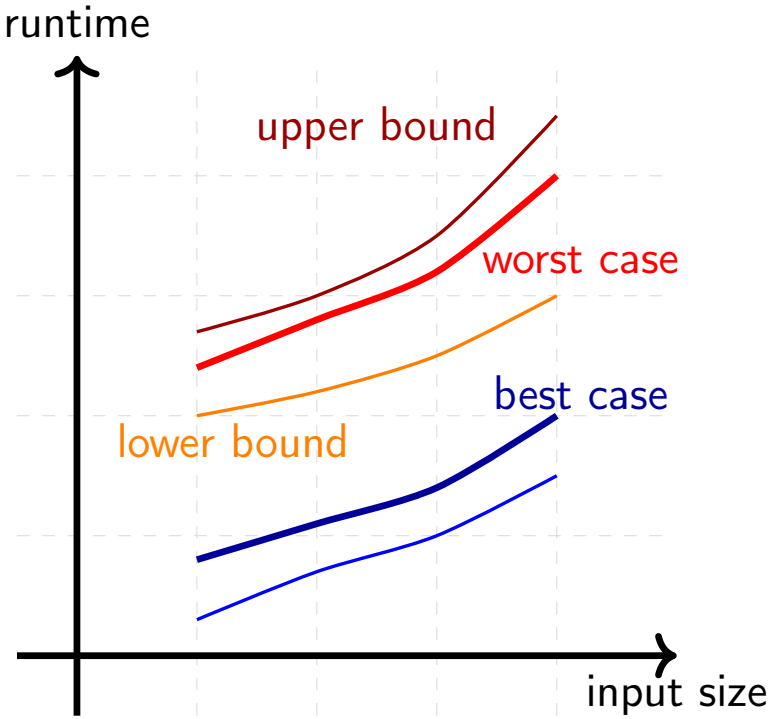
[33] N. Hirokawa, G. Moser: *Automated complexity analysis based on context-sensitive rewriting*, RTA-TLCA '14

[34] M. Avanzini, G. Moser: *A combination framework for complexity*, IC '16

How about *Lower* Bounds for Runtime Complexity?



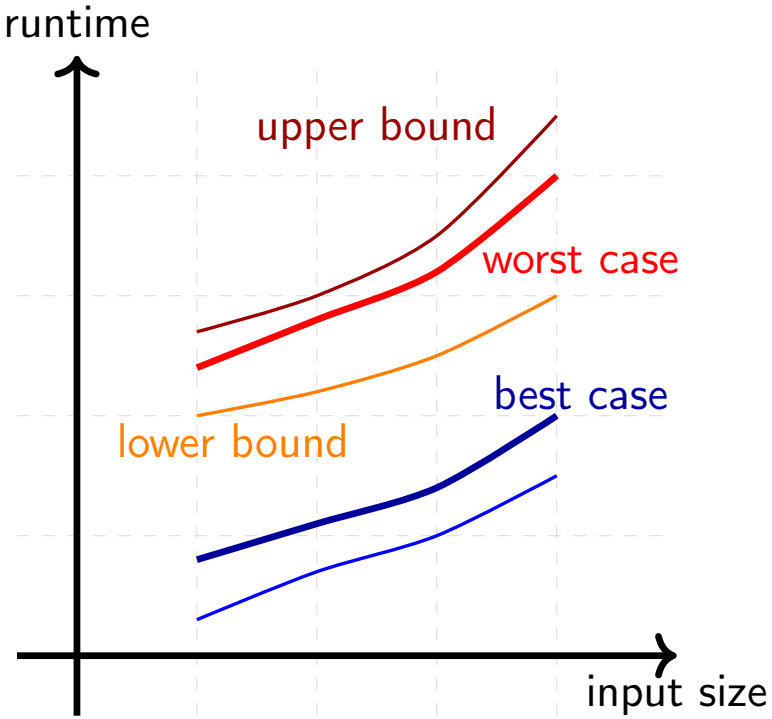
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- can indicate implementation bugs
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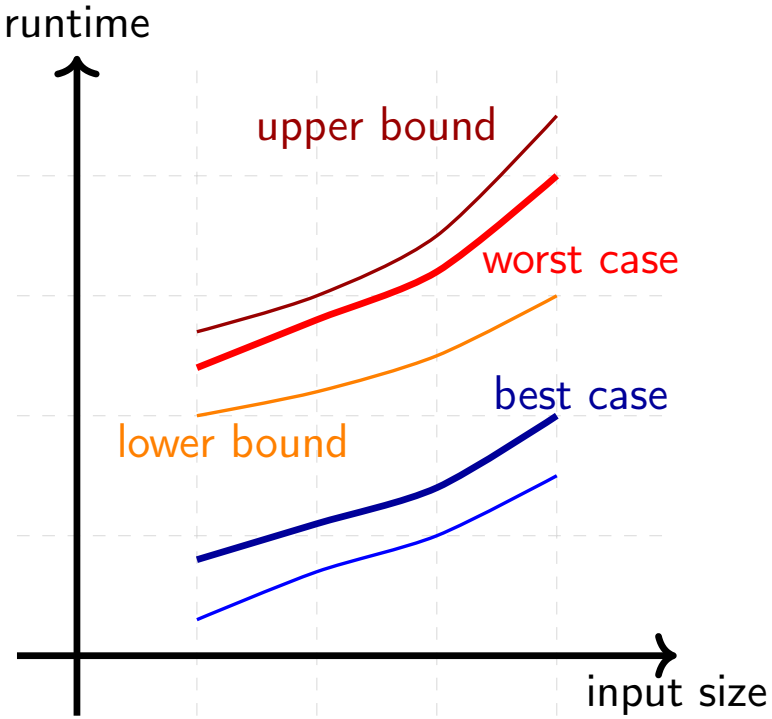
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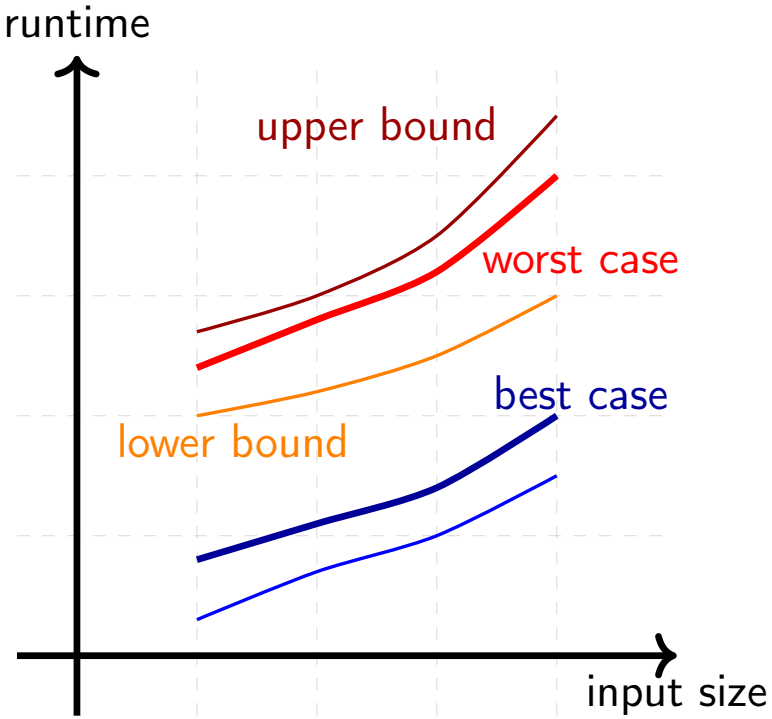
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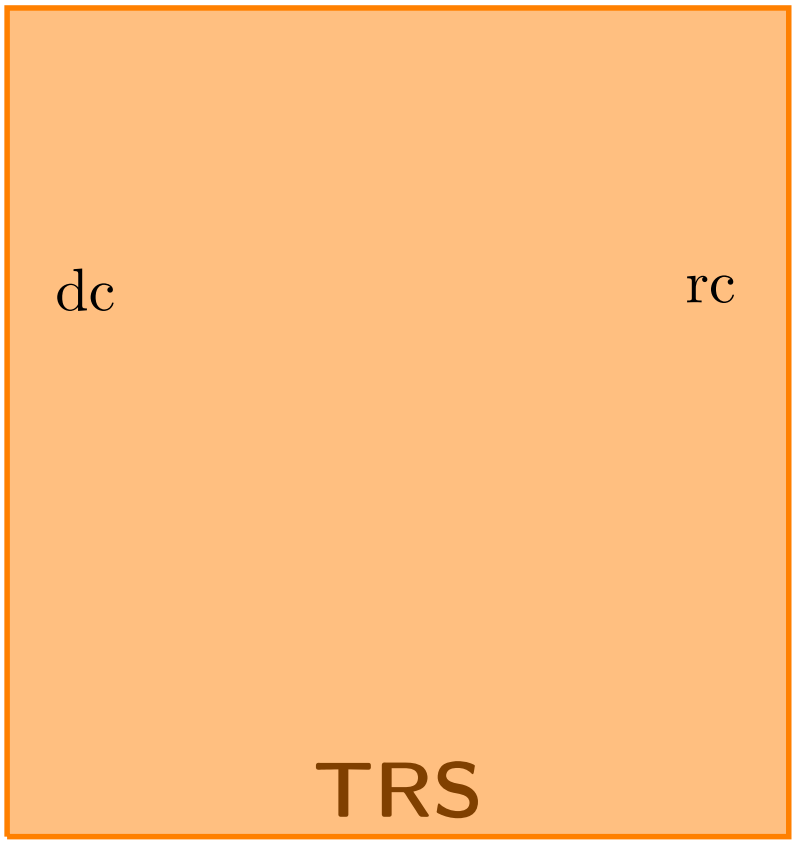
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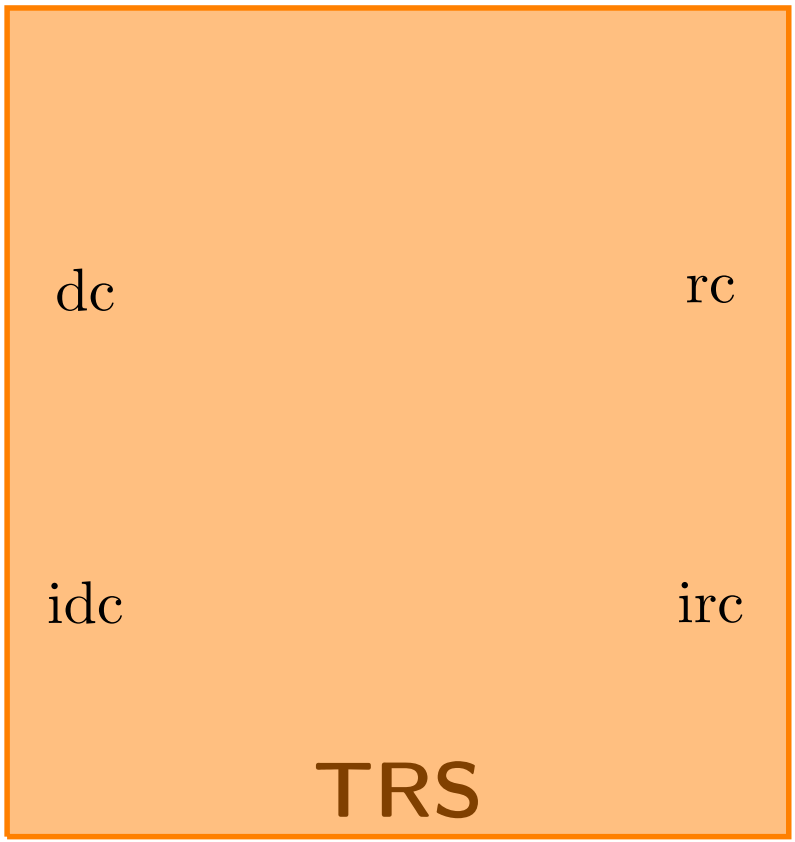
1. Speculate infinite family of basic terms as **witnesses**, run on small inputs to speculate polynomial lower bound, prove bound by induction
2. Decreasing loops: rules like $\mathsf{f}(\mathsf{s}(x)) \longrightarrow \mathsf{f}(x)$ remove **fixed context** $\Rightarrow$ linear lower bound

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A Landscape of Complexity Properties and Transformations

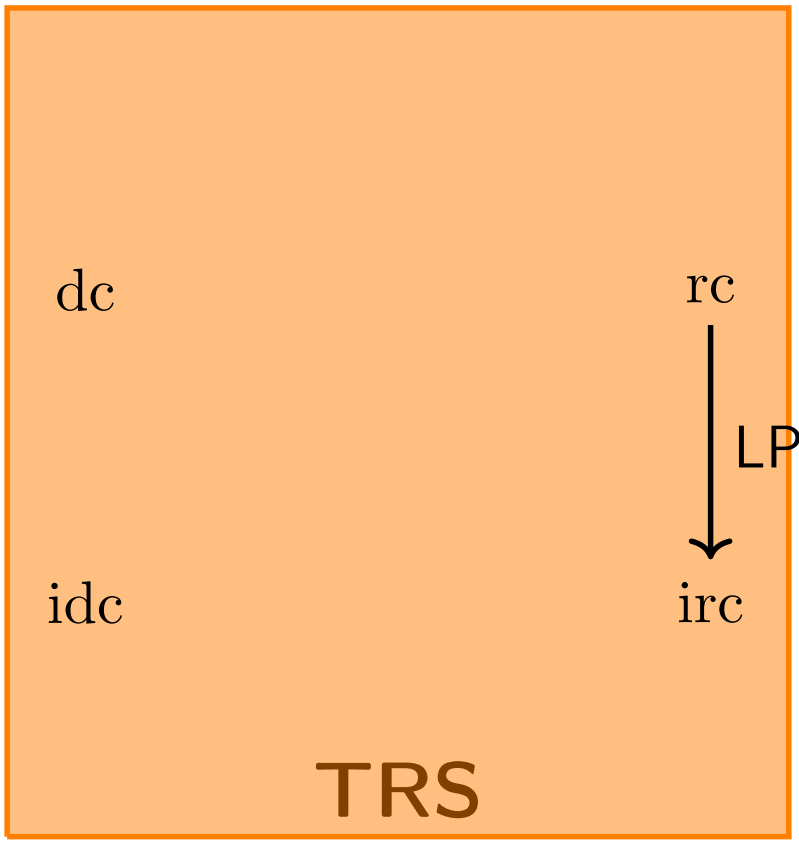


A Landscape of Complexity Properties and Transformations



idc, irc: like dc, rc,
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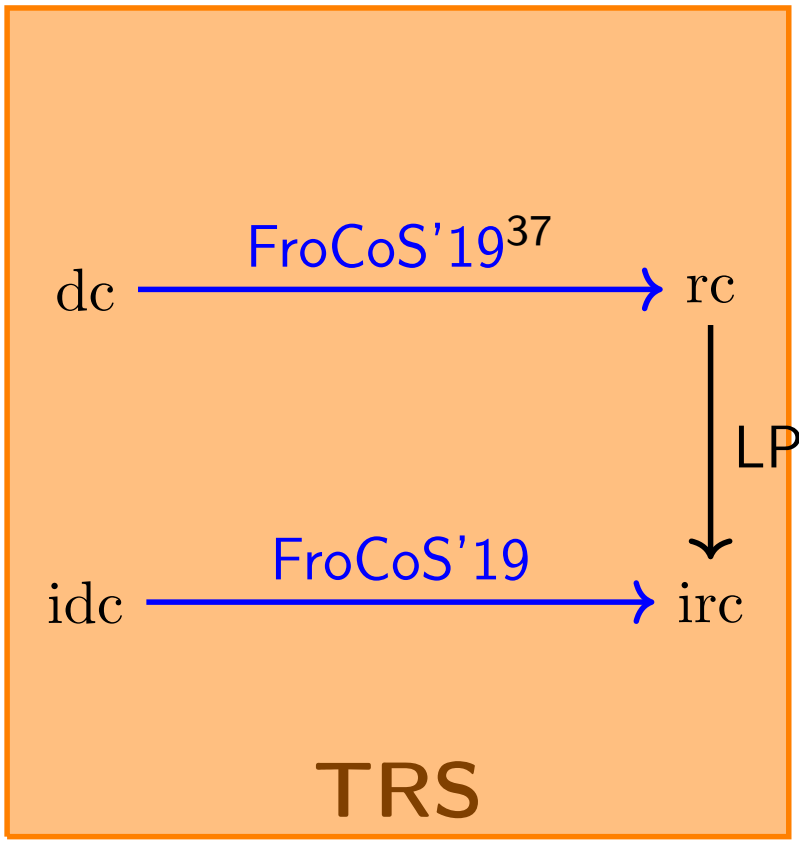
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[36] F. Frohn, J. Giesl: *Analyzing runtime complexity via innermost runtime complexity*, LPAR '17

[37] C. Fuhs: *Transforming Derivational Complexity of Term Rewriting to Runtime Complexity*, FroCoS '19

Transforming Derivational Complexity to Runtime Complexity

The big picture:

- **Have:** Tool for automated analysis of runtime complexity $\mathrm{rc}_{\mathcal{R}}$

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- **Benefits:**
  - Get analysis of derivational complexity "for free"
  - Progress in runtime complexity analysis automatically improves derivational complexity analysis

From dc to rc: Results

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- **evaluated** successfully on TPDB[38] relative to state of the art TcT

[38] Termination Problem DataBase, standard benchmark source for annual Termination Competition (termCOMP) with 1000s of problems, http://termination-portal.org/wiki/TPDB

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- more generally: transform $\mathcal{R}/\mathcal{S}$ to $\mathcal{R}/(\mathcal{S}\cup\mathcal{G})$
  (input may contain relative rules $\mathcal{S}$, too)

From dc to rc: Correctness

Theorem (Derivational Complexity via Runtime Complexity)

Let $\mathcal{R}/\mathcal{S}$ be a relative TRS, let $\mathcal{G}$ be the generator rules for $\mathcal{R}/\mathcal{S}$. Then

1. $\mathrm{dc}_{\mathcal{R}/\mathcal{S}}(n) = \mathrm{rc}_{\mathcal{R}/(\mathcal{S}\cup\mathcal{G})}(n)$ *(arbitrary rewrite strategies)*
2. $\mathrm{idc}_{\mathcal{R}/\mathcal{S}}(n) = \mathrm{irc}_{\mathcal{R}/(\mathcal{S}\cup\mathcal{G})}(n)$ *(innermost rewriting)*

Note: equalities hold also non-asymptotically!

From (i)dc to (i)rc: Experiments

Experiments on TPDB, compare with state of the art in TcT:

- upper bounds idc: both AProVE and TcT with transformation are stronger than standard TcT
- upper bounds dc: TcT stronger than AProVE and TcT with transformation, but AProVE still solves some new examples
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- lower bounds idc and dc: heuristics do not seem to benefit much

$\Rightarrow$ Transformation-based approach should be part of the portfolio of analysis tools for derivational complexity

Derivational Complexity: Future Work

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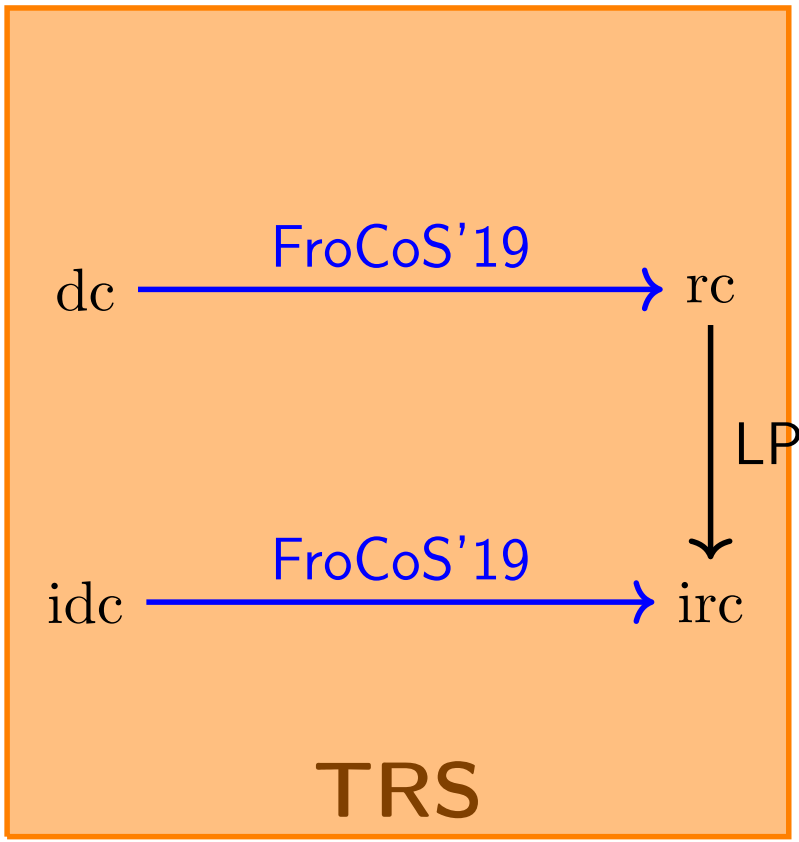
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- Want to adapt **techniques** from runtime complexity analysis to derivational complexity! How?
  - (Useful) adaptation of Dependency Pairs?
  - Abstractions to numbers?
  - ...

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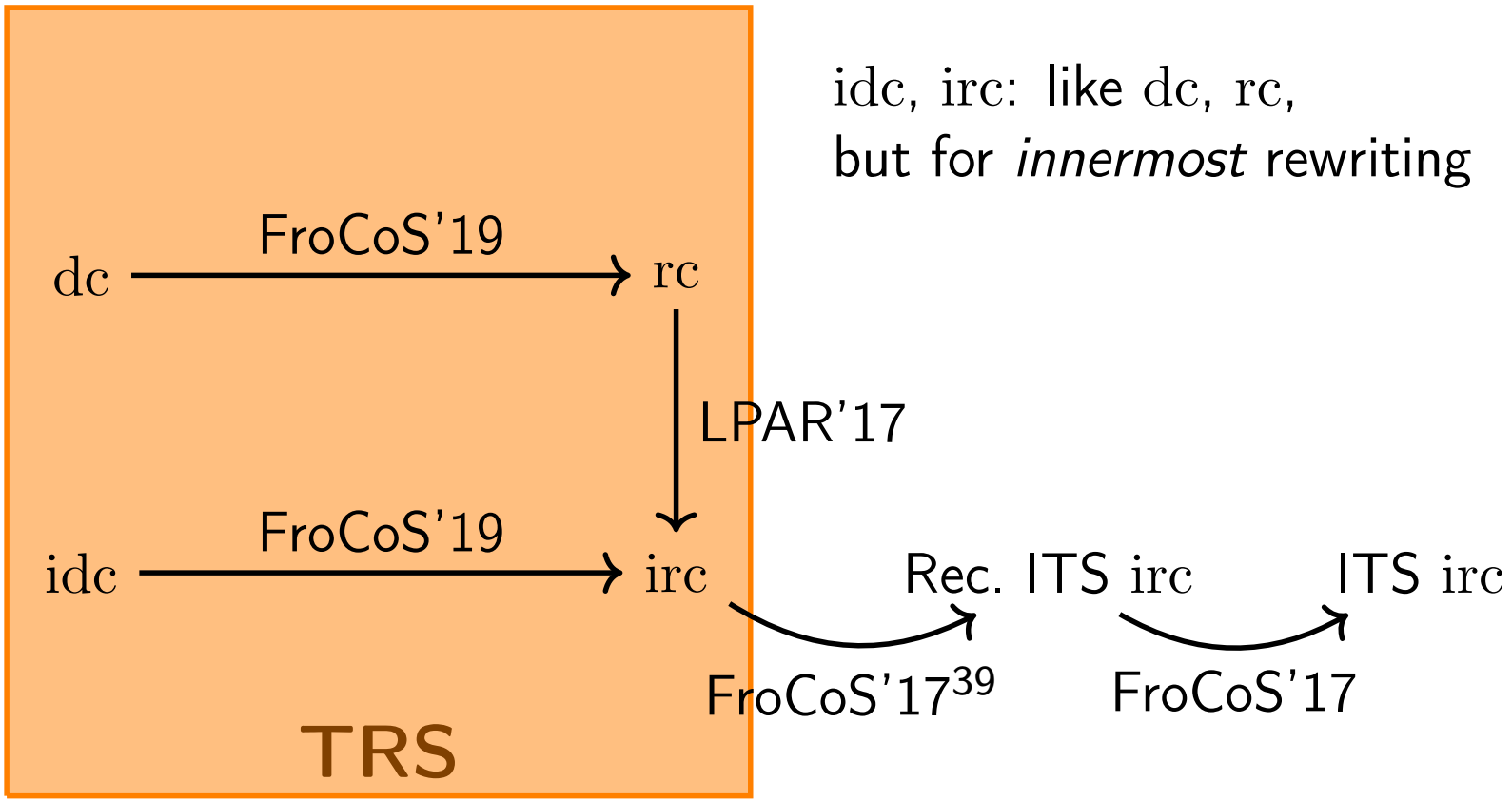


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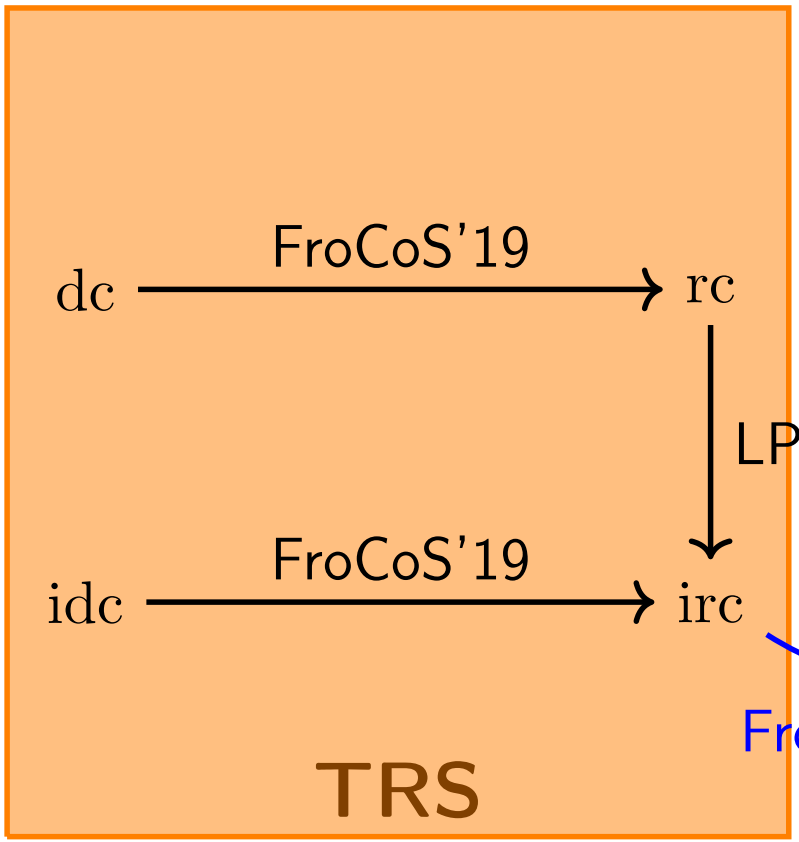
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Bottom-Up Complexity Analysis for TRSs

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Goal: use these tools to find upper bounds for TRSs in a modular way

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- CoFloCo[40]
- KoAT[41] [42]
- PUBS[43]

Goal: use these tools to find upper bounds for TRSs in a modular way

Works well in practice after resolving some technical pitfalls

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Abstract a list to its length, its size, its maximum element, . . . ?

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\begin{array}{rcl|rcl}
\mathsf{app}(\mathsf{nil}, y) & \to & y & \mathsf{app}(\mathsf{add}(n, x), y) & \to & \mathsf{add}(n, \mathsf{app}(x, y)) \\
\mathsf{reverse}(\mathsf{nil}) & \to & \mathsf{nil} & \mathsf{reverse}(\mathsf{add}(n, x)) & \to & \mathsf{app}(\mathsf{reverse}(x), \mathsf{add}(n, \mathsf{nil})) \\
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AProVE finds (tight) upper bound $\mathcal{O}(n^4)$ for $\mathrm{dc}_{\mathcal{R}}$:

Derivational_Complexity_Full_Rewriting/AG01/#3.12, TPDB

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5. Upper bound $\mathcal{O}(n^4)$ for RITS complexity carries over to $\mathrm{dc}_{\mathcal{R}}$ of input!

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AProVE finds lower bound $\Omega(n^3)$ for $\mathrm{dc}_{\mathcal{R}}$ using induction technique.

Input for Automated Tools (1/4)

Automated tools for TRS Complexity at the Termination Competition 2022:

- AProVE: https://aprove.informatik.rwth-aachen.de/
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Web interfaces available:

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- TcT: http://colo6-c703.uibk.ac.at/tct/tct-trs/

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Input format for runtime complexity:[44]

```
(VAR x y)
(GOAL COMPLEXITY)
(STARTTERM CONSTRUCTOR-BASED)
(RULES
  plus(0, y) -> y
  plus(s(x), y) -> s(plus(x, y))
)
```

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Input for Automated Tools (2/4)

Innermost runtime complexity:

```
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(STRATEGY INNERMOST)
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)
```

Input for Automated Tools (3/4)

Derivational complexity:

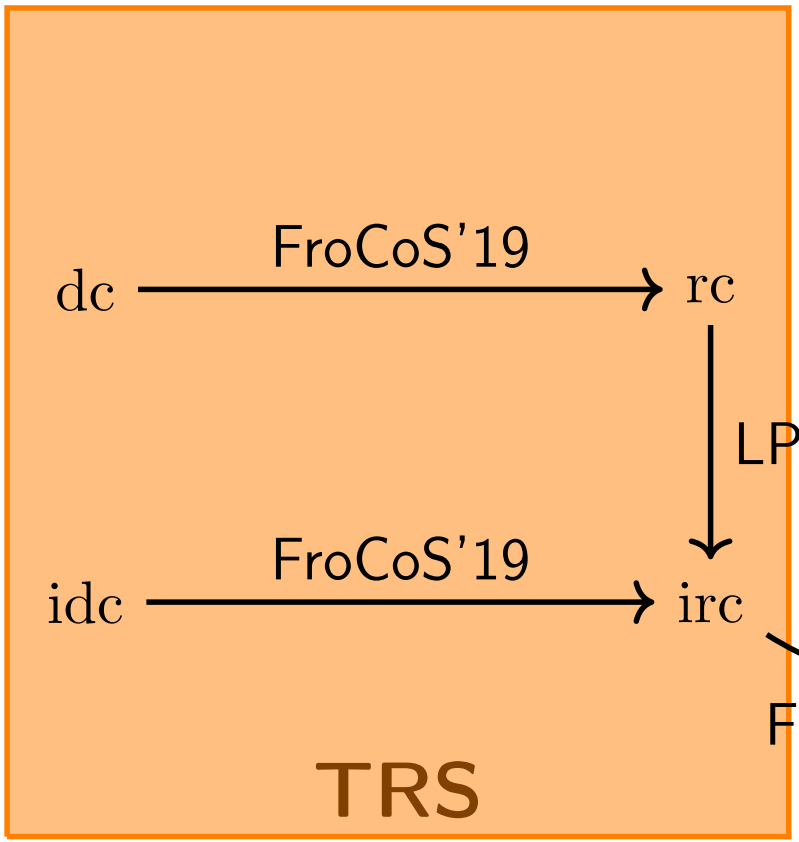
```
(VAR x y)
(GOAL COMPLEXITY)
(STARTTERM UNRESTRICTED)
(RULES
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```

Input for Automated Tools (4/4)

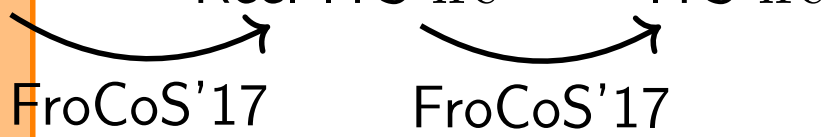
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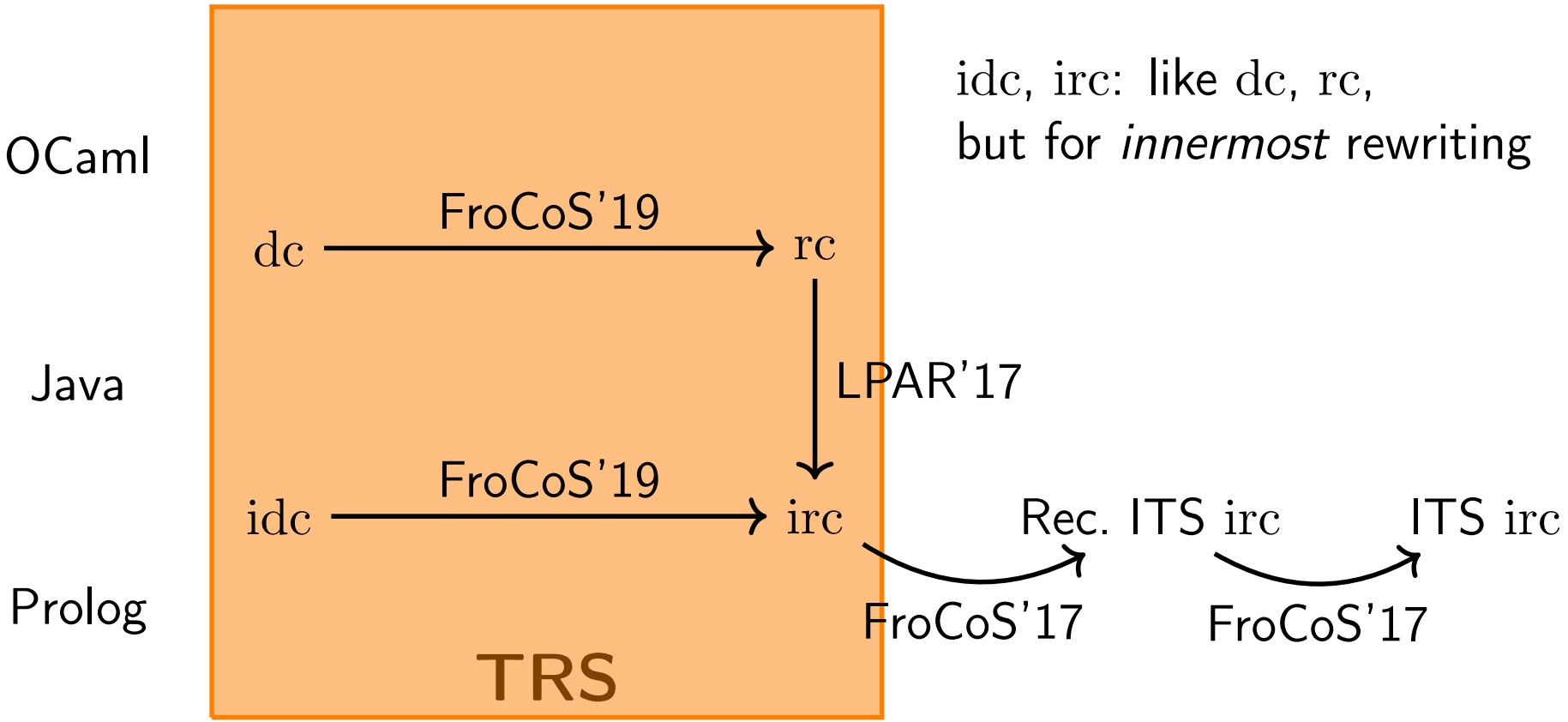
A Landscape of Complexity Properties and Transformations



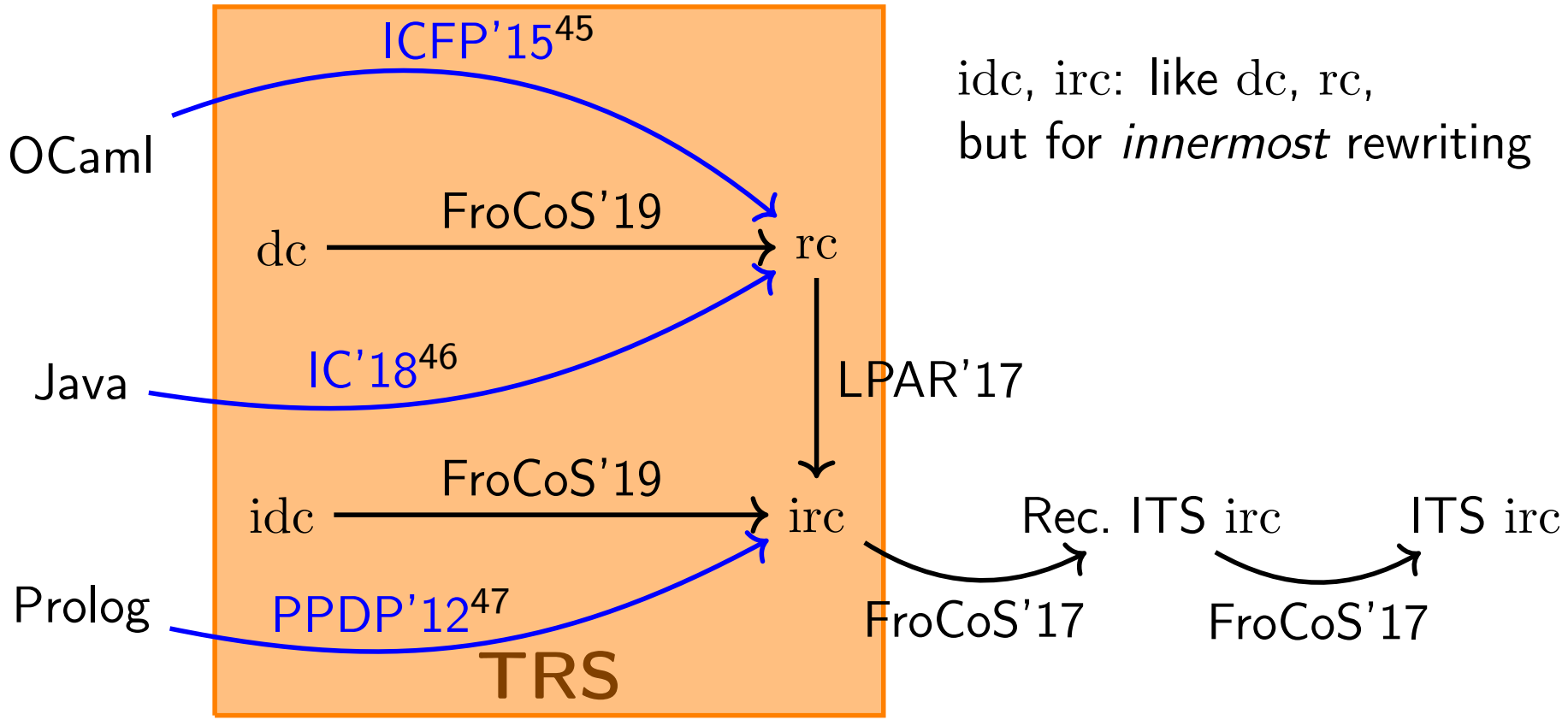
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[45] M. Avanzini, U. Dal Lago, G. Moser: *Analysing the Complexity of Functional Programs: Higher-Order Meets First-Order*, ICFP '15

[46] G. Moser, M. Schaper: *From Jinja bytecode to term rewriting: A complexity reflecting transformation*, IC '18

[47] J. Giesl, T. Ströder, P. Schneider-Kamp, F. Emmes, C. Fuhs: *Symbolic evaluation graphs and term rewriting: A general methodology for analyzing logic programs*, PPDP '12

Program Complexity Analysis via Term Rewriting: OCaml

Complexity analysis for functional programs (OCaml) by translation to term rewriting

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Challenge for translation to TRS: OCaml is higher-order – functions can take functions as arguments: $\mathsf{map}(F, xs)$

Solution:

- Defunctionalisation to: $\mathsf{a}(\mathsf{a}(\mathsf{map}, F), xs)$
- Analyse start term with non-functional parameter types, then partially evaluate functions to instantiate higher-order variables
- Further program transformations

$\Rightarrow$ First-order TRS $\mathcal{R}$ with $\mathrm{rc}_{\mathcal{R}}(n)$ an upper bound for the complexity of the OCaml program

Program Complexity Analysis via Term Rewriting: Prolog and Java

Complexity analysis for Prolog programs and for Java programs by translation to term rewriting

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Complexity analysis for Prolog programs and for Java programs by translation to term rewriting

Common ideas:

- Analyse program via symbolic execution and generalisation (a form of abstract interpretation[48])
- Deal with language specifics in program analysis
- Extract TRS $\mathcal{R}$ such that $\mathrm{rc}_{\mathcal{R}}(n)$ is provably at least as high as runtime of program on input of size n
- Can represent tree structures of program as terms in TRS!

[48] P. Cousot, R. Cousot: *Abstract interpretation: a unified lattice model for static analysis of programs by construction or approximation of fixpoints*, POPL '77

Current Developments

- **amortised** complexity analysis for term rewriting[49]

[49] G. Moser, M. Schneckenreither: *Automated amortised resource analysis for term rewrite systems*, SCP '20

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[51] J.-C. Kassing, L. V. Spitzer, J. Giesl: *Dependency Pairs for Expected Innermost Runtime Complexity and Strong Almost-Sure Termination of Probabilistic Term Rewriting*, PPDP '25

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- analysis of **parallel**-innermost runtime complexity[54]

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[54] T. Baudon, C. Fuhs, L. Gonnord: *On Complexity Bounds and Confluence of Parallel Term Rewriting*, Fundam. Inf. '24

Termination and Complexity: Conclusion

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Thanks a lot for your attention!

References I


E. Albert, P. Arenas, S. Genaim, and G. Puebla. Closed-form upper bounds in static cost analysis. *Journal of Automated Reasoning*, 46(2): 161–203, 2011.


C. Alias, A. Darte, P. Feautrier, and L. Gonnord. Multi-dimensional rankings, program termination, and complexity bounds of flowchart programs. In *SAS '10*, pages 117–133, 2010.


T. Arts and J. Giesl. Termination of term rewriting using dependency pairs. *Theoretical Computer Science*, 236(1-2):133–178, 2000.


M. Avanzini and G. Moser. Dependency pairs and polynomial path orders. In *RTA '09*, pages 48–62, 2009.


M. Avanzini and G. Moser. A combination framework for complexity. *Information and Computation*, 248:22–55, 2016.

References II


M. Avanzini, U. Dal Lago, and G. Moser. Analysing the complexity of functional programs: higher-order meets first-order. In *ICFP '15*, pages 152–164, 2015.


M. Avanzini, G. Moser, and M. Schaper. TcT: Tyrolean Complexity Tool. In *TACAS '16*, pages 407–423, 2016.


M. Avanzini, U. Dal Lago, and A. Yamada. On probabilistic term rewriting. *Science of Computer Programming*, 185, 2020.


T. Baudon, C. Fuhs, and L. Gonnord. On complexity bounds and confluence of parallel term rewriting. *Fundam. Informaticae*, 192(2): 121–166, 2024.


J. Berdine, B. Cook, D. Distefano, and P. W. O'Hearn. Automatic termination proofs for programs with shape-shifting heaps. In *CAV '06*, pages 386–400, 2006.

References III


F. Blanqui and A. Koprowski. CoLoR: a Coq library on well-founded rewrite relations and its application to the automated verification of termination certificates. *Mathematical Structures in Computer Science*, 21(4):827–859, 2011.


G. Bonfante, A. Cichon, J. Marion, and H. Touzet. Algorithms with polynomial interpretation termination proof. *Journal of Functional Programming*, 11(1):33–53, 2001.


C. Borralleras, S. Lucas, A. Oliveras, E. Rodríguez-Carbonell, and A. Rubio. SAT modulo linear arithmetic for solving polynomial constraints. *Journal of Automated Reasoning*, 48(1):107–131, 2012.


M. Brockschmidt, C. Otto, and J. Giesl. Modular termination proofs of recursive Java Bytecode programs by term rewriting. In *RTA '11*, pages 155–170, 2011.

References IV


M. Brockschmidt, R. Musiol, C. Otto, and J. Giesl. Automated termination proofs for Java programs with cyclic data. In *CAV '12*, pages 105–122, 2012a.


M. Brockschmidt, T. Ströder, C. Otto, and J. Giesl. Automated detection of non-termination and `NullPointerExceptions` for Java Bytecode. In *FoVeOOS '11*, pages 123–141, 2012b.


M. Brockschmidt, B. Cook, and C. Fuhs. Better termination proving through cooperation. In *CAV '13*, pages 413–429, 2013.


M. Brockschmidt, B. Cook, S. Ishtiaq, H. Khlaaf, and N. Piterman. T2: temporal property verification. In *TACAS '16*, pages 387–393, 2016a.


M. Brockschmidt, F. Emmes, S. Falke, C. Fuhs, and J. Giesl. Analyzing runtime and size complexity of integer programs. *ACM Transactions on Programming Languages and Systems*, 38(4), 2016b.

References V


M. Brockschmidt, S. J. C. Joosten, R. Thiemann, and A. Yamada. Certifying safety and termination proofs for integer transition systems. In *CADE '17*, pages 454–471, 2017.


H.-Y. Chen, B. Cook, C. Fuhs, K. Nimkar, and P. W. O'Hearn. Proving nontermination via safety. In *TACAS '14*, pages 156–171, 2014.


Ş. Ciobâcă and D. Lucanu. A coinductive approach to proving reachability properties in logically constrained term rewriting systems. In *IJCAR '18*, pages 295–311, 2018.


Ş. Ciobâcă, D. Lucanu, and A. Buruiana. Operationally-based program equivalence proofs using LCTRSs. *Journal of Logical and Algebraic Methods in Programming*, 135:100894, 2023.

References VI


M. Codish, J. Giesl, P. Schneider-Kamp, and R. Thiemann. SAT solving for termination proofs with recursive path orders and dependency pairs. *Journal of Automated Reasoning*, 49(1):53–93, 2012.


E. Contejean, P. Courtieu, J. Forest, O. Pons, and X. Urbain. Automated certified proofs with C*i*ME3. In *RTA '11*, pages 21–30, 2011.


B. Cook, A. Podelski, and A. Rybalchenko. Terminator: Beyond safety. In *CAV '06*, pages 415–418, 2006a.


B. Cook, A. Podelski, and A. Rybalchenko. Termination proofs for systems code. In *PLDI '06*, pages 415–426, 2006b.


B. Cook, A. Podelski, and A. Rybalchenko. Proving thread termination. In *PLDI '07*, pages 320–330, 2007.

References VII


B. Cook, C. Fuhs, K. Nimkar, and P. W. O'Hearn. Disproving termination with overapproximation. In *FMCAD '14*, pages 67–74, 2014.


B. Cook, H. Khlaaf, and N. Piterman. Verifying increasingly expressive temporal logics for infinite-state systems. *Journal of the ACM*, 64(2): 15:1–15:39, 2017.


P. Cousot and R. Cousot. Abstract interpretation: a unified lattice model for static analysis of programs by construction or approximation of fixpoints. In *POPL '77*, pages 238–252, 1977.


N. Dershowitz. Orderings for term-rewriting systems. *Theoretical Computer Science*, 17(3):279–301, 1982.


N. Dershowitz and Z. Manna. Proving termination with multiset orderings. *Communications of the ACM*, 22(8):465–476, 1979.

References VIII


F. Emmes, T. Enger, and J. Giesl. Proving non-looping non-termination automatically. In *IJCAR '12*, pages 225–240, 2012.


J. Endrullis, J. Waldmann, and H. Zantema. Matrix interpretations for proving termination of term rewriting. *Journal of Automated Reasoning*, 40(2–3):195–220, 2008.


J. Endrullis, R. C. de Vrijer, and J. Waldmann. Local termination: theory and practice. *Logical Methods in Computer Science*, 6(3), 2010.


A. Flores-Montoya and R. Hähnle. Resource analysis of complex programs with cost equations. In *APLAS '14*, pages 275–295, 2014.


F. Frohn and J. Giesl. Analyzing runtime complexity via innermost runtime complexity. In *LPAR '17*, pages 249–268, 2017.

References IX


F. Frohn, J. Giesl, J. Hensel, C. Aschermann, and T. Ströder. Lower bounds for runtime complexity of term rewriting. *Journal of Automated Reasoning*, 59(1):121–163, 2017.


C. Fuhs. Transforming derivational complexity of term rewriting to runtime complexity. In *FroCoS '19*, pages 348–364, 2019.


C. Fuhs, J. Giesl, A. Middeldorp, P. Schneider-Kamp, R. Thiemann, and H. Zankl. SAT solving for termination analysis with polynomial interpretations. In *SAT '07*, pages 340–354, 2007.


C. Fuhs, J. Giesl, A. Middeldorp, P. Schneider-Kamp, R. Thiemann, and H. Zankl. Maximal termination. In *RTA '08*, pages 110–125, 2008a.


C. Fuhs, R. Navarro-Marset, C. Otto, J. Giesl, S. Lucas, and P. Schneider-Kamp. Search techniques for rational polynomial orders. In *AISC '08*, pages 109–124, 2008b.

References X


C. Fuhs, C. Kop, and N. Nishida. Verifying procedural programs via constrained rewriting induction. *ACM Transactions on Computational Logic*, 18(2):14:1–14:50, 2017.


A. Geser, D. Hofbauer, and J. Waldmann. Match-bounded string rewriting systems. *Applicable Algebra in Engineering, Communication and Computing*, 15(3–4):149–171, 2004.


A. Geser, D. Hofbauer, J. Waldmann, and H. Zantema. On tree automata that certify termination of left-linear term rewriting systems. *Information and Computation*, 205(4):512–534, 2007.


J. Giesl, R. Thiemann, and P. Schneider-Kamp. Proving and disproving termination of higher-order functions. In *FroCoS '05*, pages 216–231, 2005.

References XI


J. Giesl, R. Thiemann, P. Schneider-Kamp, and S. Falke. Mechanizing and improving dependency pairs. *Journal of Automated Reasoning*, 37 (3):155–203, 2006.


J. Giesl, M. Raffelsieper, P. Schneider-Kamp, S. Swiderski, and R. Thiemann. Automated termination proofs for Haskell by term rewriting. *ACM Transactions on Programming Languages and Systems*, 33(2):1–39, 2011. See also `http://aprove.informatik.rwth-aachen.de/eval/Haskell/`.


J. Giesl, T. Ströder, P. Schneider-Kamp, F. Emmes, and C. Fuhs. Symbolic evaluation graphs and term rewriting: A general methodology for analyzing logic programs. In *PPDP '12*, pages 1–12, 2012.

References XII


J. Giesl, C. Aschermann, M. Brockschmidt, F. Emmes, F. Frohn, C. Fuhs, J. Hensel, C. Otto, M. Plücker, P. Schneider-Kamp, T. Ströder, S. Swiderski, and R. Thiemann. Analyzing program termination and complexity automatically with AProVE. *Journal of Automated Reasoning*, 58(1):3–31, 2017.


L. Guo, K. Hagens, C. Kop, and D. Vale. Higher-order constrained dependency pairs for (universal) computability. In *MFCS '24*, pages 57:1–57:15, 2024.


A. Gupta, T. A. Henzinger, R. Majumdar, A. Rybalchenko, and R.-G. Xu. Proving non-termination. In *POPL '08*, pages 147–158, 2008.


M. W. Haslbeck and R. Thiemann. An Isabelle/HOL formalization of AProVE's termination method for LLVM IR. In *CPP '21*, pages 238–249, 2021.

References XIII


J. Hensel and J. Giesl. Proving termination of C programs with lists. In *CADE '23*, pages 266–285, 2023.


J. Hensel, J. Giesl, F. Frohn, and T. Ströder. Termination and complexity analysis for programs with bitvector arithmetic by symbolic execution. *Journal of Logical and Algebraic Methods in Programming*, 97:105–130, 2018.


N. Hirokawa and A. Middeldorp. Tyrolean Termination Tool: Techniques and features. *Information and Computation*, 205(4): 474–511, 2007.


N. Hirokawa and G. Moser. Automated complexity analysis based on the dependency pair method. In *IJCAR '08*, pages 364–379, 2008.


N. Hirokawa and G. Moser. Automated complexity analysis based on context-sensitive rewriting. In *RTA-TLCA '14*, pages 257–271, 2014.

References XIV


D. Hofbauer and C. Lautemann. Termination proofs and the length of derivations. In *RTA '89*, pages 167–177, 1989.


J. Hoffmann and Z. Shao. Type-based amortized resource analysis with integers and arrays. *Journal of Functional Programming*, 25, 2015.


H. Hong and D. Jakuš. Testing positiveness of polynomials. *Journal of Automated Reasoning*, 21(1):23–38, 1998.


S. Kamin and J.-J. Lévy. Two generalizations of the recursive path ordering. Unpublished Manuscript, University of Illinois, Urbana, IL, USA, 1980.


J. Kassing and J. Giesl. Proving almost-sure innermost termination of probabilistic term rewriting using dependency pairs. In *CADE '23*, pages 344–364, 2023.

References XV


J. Kassing, L. V. Spitzer, and J. Giesl. Dependency pairs for expected innermost runtime complexity and strong almost-sure termination of probabilistic term rewriting. In *PPDP '25*, pages 10:1–10:14, 2025.


D. E. Knuth and P. B. Bendix. Simple word problems in universal algebras. *Computational Problems in Abstract Algebra*, pages 263–297, 1970.


M. Kojima and N. Nishida. Reducing non-occurrence of specified runtime errors to all-path reachability problems of constrained rewriting. *Journal of Logical and Algebraic Methods in Programming*, 135:100903, 2023.


C. Kop. *Higher Order Termination*. PhD thesis, VU Amsterdam, 2012.


C. Kop. Termination of LCTRSs. In *WST '13*, pages 59–63, 2013.

References XVI

- C. Kop and N. Nishida. Term rewriting with logical constraints. In *FroCoS '13*, pages 343–358, 2013.
- C. Kop and N. Nishida. Constrained Term Rewriting tooL. In *LPAR '15*, pages 549–557, 2015.
- C. Kop and D. Vale. Tuple interpretations for higher-order complexity. In *FSCD '21*, pages 31:1–31:22, 2021.
- A. Koprowski and J. Waldmann. Max/plus tree automata for termination of term rewriting. *Acta Cybernetica*, 19(2):357–392, 2009.
- K. Korovin and A. Voronkov. Orienting rewrite rules with the Knuth-Bendix order. *Information and Computation*, 183(2):165–186, 2003.
- M. Korp and A. Middeldorp. Match-bounds revisited. *Information and Computation*, 207(11):1259–1283, 2009.

References XVII


M. Korp, C. Sternagel, H. Zankl, and A. Middeldorp. Tyrolean Termination Tool 2. In *RTA '09*, pages 295–304, 2009.


D. S. Lankford. Canonical algebraic simplification in computational logic. Technical Report ATP-25, University of Texas, 1975.


D. Larraz, A. Oliveras, E. Rodríguez-Carbonell, and A. Rubio. Proving termination of imperative programs using Max-SMT. In *FMCAD '13*, pages 218–225, 2013.


D. Larraz, K. Nimkar, A. Oliveras, E. Rodríguez-Carbonell, and A. Rubio. Proving non-termination using Max-SMT. In *CAV '14*, pages 779–796, 2014.


N. Lommen, J. Giesl, and É. Meyer. KoAT: Automatic complexity and termination analysis of integer programs. In *CAV '26*, 2026. To appear.

References XVIII


S. Lucas. Polynomials over the reals in proofs of termination: from theory to practice. *RAIRO - Theoretical Informatics and Applications*, 39(3):547–586, 2005.


S. Lucas. Context-sensitive rewriting. *ACM Computing Surveys*, 53(4): 78:1–78:36, 2020.


A. Matsumi, N. Nishida, M. Kojima, and D. Shin. On singleton self-loop removal for termination of LCTRSs with bit-vector arithmetic. In *WST '23*, 2023.


J. McCarthy. Recursive functions of symbolic expressions and their computation by machine, part I. *Communications of the ACM*, 3(4): 184–195, 1960.


G. Moser and M. Schaper. From Jinja bytecode to term rewriting: A complexity reflecting transformation. *Information and Computation*, 261:116–143, 2018.

References XIX


G. Moser and A. Schnabl. The derivational complexity induced by the dependency pair method. *Logical Methods in Computer Science*, 7(3), 2011a.


G. Moser and A. Schnabl. Termination proofs in the dependency pair framework may induce multiple recursive derivational complexity. In *RTA '11*, pages 235–250, 2011b.


G. Moser and M. Schneckenreither. Automated amortised resource analysis for term rewrite systems. *Science of Computer Programming*, 185, 2020.


G. Moser, A. Schnabl, and J. Waldmann. Complexity analysis of term rewriting based on matrix and context dependent interpretations. In *FSTTCS '08*, pages 304–315, 2008.

References XX


M. Naaf, F. Frohn, M. Brockschmidt, C. Fuhs, and J. Giesl. Complexity analysis for term rewriting by integer transition systems. In *FroCoS '17*, pages 132–150, 2017.


F. Neurauter, H. Zankl, and A. Middeldorp. Revisiting matrix interpretations for polynomial derivational complexity of term rewriting. In *LPAR (Yogyakarta) '10*, pages 550–564, 2010.


N. Nishida and S. Winkler. Loop detection by logically constrained term rewriting. In *VSTTE '18*, pages 309–321, 2018.


L. Noschinski, F. Emmes, and J. Giesl. Analyzing innermost runtime complexity of term rewriting by dependency pairs. *Journal of Automated Reasoning*, 51(1):27–56, 2013.


C. Otto, M. Brockschmidt, C. v. Essen, and J. Giesl. Automated termination analysis of Java Bytecode by term rewriting. In *RTA '10*, pages 259–276, 2010.

References XXI


É. Payet. Loop detection in term rewriting using the eliminating unfoldings. *Theoretical Computer Science*, 403(2-3), 2008.


A. Podelski and A. Rybalchenko. A complete method for the synthesis of linear ranking functions. In *VMCAI '04*, pages 239–251, 2004.


A. Schnabl and J. G. Simonsen. The exact hardness of deciding derivational and runtime complexity. In *CSL '11*, pages 481–495, 2011.


P. Schneider-Kamp, J. Giesl, A. Serebrenik, and R. Thiemann. Automated termination proofs for logic programs by term rewriting. *ACM Transactions on Computational Logic*, 11(1):1–52, 2009.


J. Schöpf and A. Middeldorp. Confluence criteria for logically constrained rewrite systems. In *CADE '23*, pages 474–490, 2023.

References XXII


J. Schöpf, F. Mitterwallner, and A. Middeldorp. Confluence of logically constrained rewrite systems revisited. In *IJCAR '24*, pages 298–316, 2024.


T. Ströder, F. Emmes, P. Schneider-Kamp, J. Giesl, and C. Fuhs. A linear operational semantics for termination and complexity analysis of ISO Prolog. In *LOPSTR '11*, pages 237–252, 2012.


T. Ströder, J. Giesl, M. Brockschmidt, F. Frohn, C. Fuhs, J. Hensel, P. Schneider-Kamp, and C. Aschermann. Automatically proving termination and memory safety for programs with pointer arithmetic. *Journal of Automated Reasoning*, 58(1):33–65, 2017.


R. Thiemann. *The DP Framework for Proving Termination of Term Rewriting*. PhD thesis, RWTH Aachen, 2007.


R. Thiemann and C. Sternagel. Certification of termination proofs using CeTA. In *TPHOLs '09*, pages 452–468, 2009.

References XXIII


A. M. Turing. On computable numbers, with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*, 42(2):230–265, 1936.


A. M. Turing. Checking a large routine. In *Report of a Conference on High Speed Automatic Calculating Machines*, pages 67–69, 1949.


F. van Raamsdonk. Translating logic programs into conditional rewriting systems. In *ICLP '97*, pages 168–182, 1997.


S. G. Vorobyov. Conditional rewrite rule systems with built-in arithmetic and induction. In *RTA '89*, pages 492–512, 1989.


J. Waldmann. *Automatic* termination. In *RTA '09*, pages 1–16, 2009.


A. Weiermann. Termination proofs for term rewriting systems by lexicographic path orderings imply multiply recursive derivation lengths. *Theoretical Computer Science*, 139(1&2):355–362, 1995.

References XXIV


S. Winkler and G. Moser. Runtime complexity analysis of logically constrained rewriting. In *LOPSTR '20*, pages 37–55, 2020.


A. Yamada. Tuple interpretations for termination of term rewriting. *Journal of Automated Reasoning*, 66(4):667–688, 2022.


A. Yamada, K. Kusakari, and T. Sakabe. A unified ordering for termination proving. *Science of Computer Programming*, 111:110–134, 2015.


H. Zankl and A. Middeldorp. Satisfiability of non-linear (ir)rational arithmetic. In *LPAR (Dakar) '10*, pages 481–500, 2010.


H. Zankl, N. Hirokawa, and A. Middeldorp. KBO orientability. *Journal of Automated Reasoning*, 43(2):173–201, 2009.


H. Zankl, C. Sternagel, D. Hofbauer, and A. Middeldorp. Finding and certifying loops. In *SOFSEM '10*, pages 755–766, 2010.