Automated Complexity Analysis for Term Rewriting

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Course at the International School on Rewriting 2021
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https://www.dcs.bbk.ac.uk/~carsten/isr2021/

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[1] virtually

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Example (Term Rewrite System (TRS) $\mathcal{R}$)

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- $\mathrm{dc}_{\mathcal{R}}(n)$ for **equational reasoning**: cost of solving the word problem $\mathcal{E} \models s \equiv t$ by rewriting s and t via an equivalent convergent TRS $\mathcal{R}_{\mathcal{E}}$

Overview

1. Introduction
2. Automatically Finding Upper Bounds
3. Automatically Finding Lower Bounds
4. Transformational Techniques
5. Analysing Program Complexity via TRS Complexity
6. Current Developments

A Short Timeline (1/2)

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[5] M. Avanzini, G. Moser, M. Schaper: *TcT: Tyrolean Complexity Tool*, TACAS '16, `https://tcs-informatik.uibk.ac.at/tools/tct/`

[6] M. Korp, C. Sternagel, H. Zankl, A. Middeldorp: *Tyrolean Termination Tool 2*, RTA '09, `http://cl-informatik.uibk.ac.at/software/cat/`

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A Short Timeline (2/2)

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2021: Termination Competition 2021 with complexity analysis tools AProVE[7], TcT in July 2021

https://termcomp.github.io/Y2021-1

First run just finished!

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[7] J. Giesl, C. Aschermann, M. Brockschmidt, F. Emmes, F. Frohn, C. Fuhs, J. Hensel, C. Otto, M. Plücker, P. Schneider-Kamp, T. Ströder, S. Swiderski, R. Thiemann: *Analyzing Program Termination and Complexity Automatically with AProVE*, JAR '17, `http://aprove.informatik.rwth-aachen.de/`

Some Definitions

Definition (Derivation Height dh)

For a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ and a relation $\longrightarrow$, the **derivation height** is:

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Definition (Derivational Complexity dc)

For a TRS $\mathcal{R}$, the **derivational complexity** is:

$$\mathrm{dc}_{\mathcal{R}}(n) \;=\; \sup \{\, \mathrm{dh}(t, \longrightarrow_{\mathcal{R}}) \mid t \in \mathcal{T}(\mathcal{F}, \mathcal{V}), |t| \leq n \,\}$$

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$\mathrm{dh}(t, \longrightarrow)$: length of the longest $\longrightarrow$-sequence from t.

Example: $\mathrm{dh}(\,\mathsf{double}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{0})))),\ \longrightarrow_{\mathcal{R}}\,) = 4$

Definition (Derivational Complexity dc)

For a TRS $\mathcal{R}$, the **derivational complexity** is:

$$\mathrm{dc}_{\mathcal{R}}(n) \;=\; \sup \{ \, \mathrm{dh}(t, \longrightarrow_{\mathcal{R}}) \mid t \in \mathcal{T}(\mathcal{F}, \mathcal{V}), |t| \leq n \, \}$$

$\mathrm{dc}_{\mathcal{R}}(n)$: length of the longest $\longrightarrow_{\mathcal{R}}$-sequence from a term of size at most n

Some Definitions

Definition (Derivation Height dh)

For a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ and a relation $\longrightarrow$, the **derivation height** is:

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If t starts an infinite $\longrightarrow$-sequence, we set $\mathrm{dh}(t, \longrightarrow) = \omega$.

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$\mathrm{dc}_{\mathcal{R}}(n)$: length of the longest $\longrightarrow_{\mathcal{R}}$-sequence from a term of size at most n

Example: For $\mathcal{R}$ for double, we have $\mathrm{dc}_{\mathcal{R}}(n) \in \Theta(2^n)$.

Upper Bounds

The Bad News for automation:

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- $\mathrm{dc}_{\mathcal{R}}(n)$ polynomially bounded?[8]

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[8] A. Schnabl and J. G. Simonsen: *The exact hardness of deciding derivational and runtime complexity*, CSL '11

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- $\mathrm{dc}_{\mathcal{R}}(n) = \omega$ for some n? ($\longrightarrow$ termination!)
- $\mathrm{dc}_{\mathcal{R}}(n)$ polynomially bounded?[8]

Goal: find **approximations** for derivational complexity

Initial focus: find upper bounds

$$\mathrm{dc}_{\mathcal{R}}(n) \in \mathcal{O}(...)$$

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Derivational Complexity from Polynomial Interpretations (1/2)

Example (double)

$$\begin{array}{rcl}\mathsf{double}(\mathsf{0}) & \rightarrow & \mathsf{0} \\ \mathsf{double}(\mathsf{s}(x)) & \rightarrow & \mathsf{s}(\mathsf{s}(\mathsf{double}(x)))\end{array}$$

Derivational Complexity from Polynomial Interpretations (1/2)

Example (double)

$$\begin{aligned} \mathsf{double}(\mathsf{0}) &\succ \mathsf{0} \\ \mathsf{double}(\mathsf{s}(x)) &\succ \mathsf{s}(\mathsf{s}(\mathsf{double}(x))) \end{aligned}$$

Show $\mathrm{dc}_{\mathcal{R}}(n) < \omega$ by **termination proof** with reduction order $\succ$ on terms.

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[9] D. Lankford: *Canonical algebraic simplification in computational logic*, U Texas '75

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Example: $[\mathsf{double}](x) = 3 \cdot x, \qquad [\mathsf{s}](x) = x + 1, \qquad [\mathsf{0}] = 1$

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Extend to terms:

- $[x] = x$
- $[f(t_1, \ldots, t_n)] = [f]([t_1], \ldots, [t_n])$

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Derivational Complexity from Polynomial Interpretations (1/2)

Example (double)

$$\begin{array}{rcl|rcl} \mathsf{double}(\mathsf{0}) & \succ & \mathsf{0} & 3 & > & 1 \\ \mathsf{double}(\mathsf{s}(x)) & \succ & \mathsf{s}(\mathsf{s}(\mathsf{double}(x))) & 3 \cdot x + 3 & > & 3 \cdot x + 2 \end{array}$$

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$\mathsf{double}(\mathsf{0})$	$\succ$	$\mathsf{0}$	3	$>$	1
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Automated search for $[\,\cdot\,]$ via SAT[10] or SMT[11] solving

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[9] D. Lankford: *Canonical algebraic simplification in computational logic*, U Texas '75

[10] C. Fuhs, J. Giesl, A. Middeldorp, P. Schneider-Kamp, R. Thiemann, H. Zankl: *SAT solving for termination analysis with polynomial interpretations*, SAT '07

[11] C. Borralleras, S. Lucas, A. Oliveras, E. Rodríguez-Carbonell, A. Rubio: *SAT modulo linear arithmetic for solving polynomial constraints*, JAR '12

Derivational Complexity from Polynomial Interpretations (2/2)

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This proves more than just termination...

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Theorem (Upper bounds for $\mathrm{dc}_{\mathcal{R}}(n)$ from polynomial interpretations[12])

- *Termination proof for TRS $\mathcal{R}$ with* ***polynomial*** *interpretation*

$$\Rightarrow \mathrm{dc}_{\mathcal{R}}(n) \in 2^{2^{\mathcal{O}(n)}}$$

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[12] D. Hofbauer, C. Lautemann: *Termination proofs and the length of derivations*, RTA '89

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- *Termination proof for TRS $\mathcal{R}$ with **polynomial** interpretation*
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- *Termination proof for TRS $\mathcal{R}$ with **linear polynomial** interpretation*
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Derivational Complexity from Termination Proofs (1/2)

Termination proof for TRS $\mathcal{R}$ with . . .

- matchbounds[13] $\Rightarrow \mathrm{dc}_{\mathcal{R}}(n) \in \mathcal{O}(n)$
- arctic matrix interpretations[14] $\Rightarrow \mathrm{dc}_{\mathcal{R}}(n) \in \mathcal{O}(n)$

---

[13] A. Geser, D. Hofbauer, J. Waldmann: *Match-bounded string rewriting systems*, AAECC '04

[14] A. Koprowski, J. Waldmann: *Max/plus tree automata for termination of term rewriting*, Acta Cyb. '09

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Termination proof for TRS $\mathcal{R}$ with ...

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- triangular matrix interpretation[15] $\Rightarrow \mathrm{dc}_{\mathcal{R}}(n)$ is at most polynomial
- matrix interpretation of spectral radius[16] ≤ 1
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[15] G. Moser, A. Schnabl, J. Waldmann: *Complexity analysis of term rewriting based on matrix and context dependent interpretations*, FSTTCS '08

[16] F. Neurauter, H. Zankl, A. Middeldorp: *Revisiting matrix interpretations for polynomial derivational complexity of term rewriting*, LPAR (Yogyakarta) '10

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[17] J. Endrullis, J. Waldmann, and H. Zantema: *Matrix interpretations for proving termination of term rewriting*, JAR '08

Derivational Complexity from Termination Proofs (2/2)

Termination proof for TRS $\mathcal{R}$ with ...

- lexicographic path order[18] $\Rightarrow \mathrm{dc}_{\mathcal{R}}(n)$ is at most multiple recursive[19]

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[18] S. Kamin, J.-J. Lévy: *Two generalizations of the recursive path ordering*, U Illinois '80

[19] A. Weiermann: *Termination proofs for term rewriting systems by lexicographic path orderings imply multiply recursive derivation lengths*, TCS '95

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[22] J. Giesl, R. Thiemann, P. Schneider-Kamp, S. Falke: *Mechanizing and improving dependency pairs*, JAR '06

[23] N. Hirokawa and A. Middeldorp: *Tyrolean Termination Tool: Techniques and features*, IC '07

[24] G. Moser, A. Schnabl: *Termination proofs in the dependency pair framework may induce multiple recursive derivational complexity*, RTA '11

Runtime Complexity

- So far: upper bounds for derivational complexity

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Definition (Basic Term[25])

For defined symbols $\mathcal{D}$ and constructor symbols $\mathcal{C}$, the term

$$f(t_1, \ldots, t_n)$$

is in the set $\mathcal{T}_{\text{basic}}$ of **basic terms** iff $f \in \mathcal{D}$ and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{C}, \mathcal{V})$.

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[25] N. Hirokawa, G. Moser: *Automated complexity analysis based on the dependency pair method*, IJCAR '08

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$\mathrm{rc}_{\mathcal{R}}(n)$: like derivational complexity… but for basic terms only!

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Runtime Complexity from Polynomial Interpretations

Polynomial interpretations can induce upper bounds to runtime complexity:[26]

Definition (Strongly linear polynomial, restricted interpretation)

- Polynomial p is **strongly linear** iff $p(x_1, \ldots, x_n) = x_1 + \cdots + x_n + a$ for some $a \in \mathbb{N}$.
- Polynomial interpretation $[\,\cdot\,]$ is **restricted** iff for all constructor symbols f, $[f](x_1, \ldots, x_n)$ is strongly linear.

Idea: $[t] \leq c \cdot |t|$ for fixed $c \in \mathbb{N}$.

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[26] G. Bonfante, A. Cichon, J. Marion, H. Touzet: *Algorithms with polynomial interpretation termination proof*, JFP '01

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*Termination proof for TRS $\mathcal{R}$ with **restricted** interpretation $[\,\cdot\,]$ of degree at most d for $[f]$* $\Rightarrow \mathrm{rc}_{\mathcal{R}}(n) \in \mathcal{O}(n^d)$

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*Termination proof for TRS $\mathcal{R}$ with **restricted** interpretation $[\,\cdot\,]$ of degree at most d for $[f]$* $\Rightarrow \mathrm{rc}_{\mathcal{R}}(n) \in \mathcal{O}(n^d)$

Example: $[\mathsf{double}](x) = 3 \cdot x$, $[\mathsf{s}](x) = x + 1$, $[\mathsf{0}] = 1$ is restricted, degree 1

$\Rightarrow \mathrm{rc}_{\mathcal{R}}(n) \in \mathcal{O}(n)$ for TRS $\mathcal{R}$ for double

---

[26] G. Bonfante, A. Cichon, J. Marion, H. Touzet: *Algorithms with polynomial interpretation termination proof*, JFP '01

Dependency Tuples for *Innermost* Runtime Complexity irc

Here: innermost rewriting (≈ call-by-value)

Example (reverse)

$$\begin{array}{rcl|rcl} \mathsf{app}(\mathsf{nil}, y) & \to & y & \mathsf{app}(\mathsf{add}(n, x), y) & \to & \mathsf{add}(n, \mathsf{app}(x, y)) \\ \mathsf{reverse}(\mathsf{nil}) & \to & \mathsf{nil} & \mathsf{reverse}(\mathsf{add}(n, x)) & \to & \mathsf{app}(\mathsf{reverse}(x), \mathsf{add}(n, \mathsf{nil})) \end{array}$$

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For rule $\ell \to r$, eval of ℓ costs 1 + eval of all function calls in r **together**:

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[27] L. Noschinski, F. Emmes, J. Giesl: *Analyzing innermost runtime complexity of term rewriting by dependency pairs*, JAR '13

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For rule $\ell \to r$, eval of ℓ costs 1 + eval of all function calls in r **together**:

Example (Dependency Tuples[27] for reverse)

$$\begin{array}{rcl} \mathsf{app}^\sharp(\mathsf{nil}, y) & \to & \mathsf{Com}_0 \\ \mathsf{app}^\sharp(\mathsf{add}(n, x), y) & \to & \mathsf{Com}_1(\mathsf{app}^\sharp(x, y)) \\ \mathsf{reverse}^\sharp(\mathsf{nil}) & \to & \mathsf{Com}_0 \\ \mathsf{reverse}^\sharp(\mathsf{add}(n, x)) & \to & \mathsf{Com}_2(\mathsf{app}^\sharp(\mathsf{reverse}(x), \mathsf{add}(n, \mathsf{nil})), \mathsf{reverse}^\sharp(x)) \end{array}$$

- Function calls to count marked with $\sharp$
- Compound symbols Com_k group function calls together

---

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Polynomial Interpretations for Dependency Tuples

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$$\begin{array}{rcl|rcl}
\mathsf{app}(\mathsf{nil}, y) & \to & y & \mathsf{app}(\mathsf{add}(n, x), y) & \to & \mathsf{add}(n, \mathsf{app}(x, y))\\
\mathsf{reverse}(\mathsf{nil}) & \to & \mathsf{nil} & \mathsf{reverse}(\mathsf{add}(n, x)) & \to & \mathsf{app}(\mathsf{reverse}(x), \mathsf{add}(n, \mathsf{nil}))
\end{array}$$

Use interpretation $[\,\cdot\,]$ with $[\mathsf{Com}_k](x_1, \ldots, x_k) = x_1 + \cdots + x_k$ and

$[\mathsf{nil}] = 0$ $\quad$ $[\mathsf{add}](x_1, x_2) = x_2 + 1$ ($\leq$ restricted interpret.)

$[\mathsf{app}](x_1, x_2) = x_1 + x_2$ $\quad$ $[\mathsf{reverse}](x_1) = x_1$ (bounds helper fct. result size)

$[\mathsf{app}^\sharp](x_1, x_2) = x_1 + 1$ $\quad$ $[\mathsf{reverse}^\sharp](x_1) = x_1^2 + x_1 + 1$ (complexity of fct.)

to show $[\ell] \geq [r]$ for all rules and $[\ell] \geq 1 + [r]$ for all Dependency Tuples

Maximum degree of $[\,\cdot\,]$ is 2 $\Rightarrow \mathrm{irc}_{\mathcal{R}}(n) \in \mathcal{O}(n^2)$

Related Techniques

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[28] N. Hirokawa, G. Moser: *Automated complexity analysis based on the dependency pair method*, IJCAR '08

Related Techniques

- Dependency Tuples are an adaptation of Dependency Pairs (DPs) from termination analysis to complexity analysis, allow for **incremental** complexity proofs with several techniques
- Further adaptation of DPs (incomparable): Weak (Innermost) Dependency Pairs for (innermost) runtime complexity[28]
- Extensions by polynomial path orders[29], usable replacement maps[30], a combination framework for complexity analysis[31], ...

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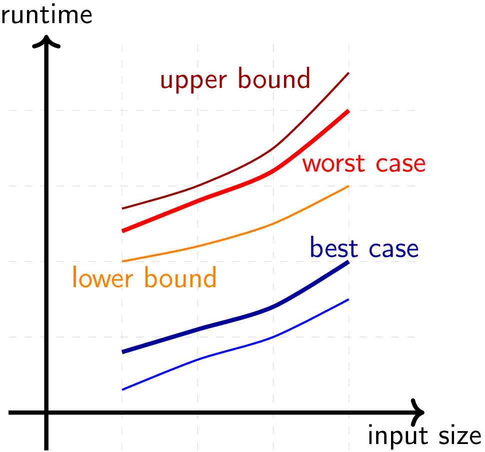
[28] N. Hirokawa, G. Moser: *Automated complexity analysis based on the dependency pair method*, IJCAR '08

[29] M. Avanzini, G. Moser: *Dependency pairs and polynomial path orders*, RTA '09

[30] N. Hirokawa, G. Moser: *Automated complexity analysis based on context-sensitive rewriting*, RTA-TLCA '14

[31] M. Avanzini, G. Moser: *A combination framework for complexity*, IC '16

How about Lower Bounds for Complexity?



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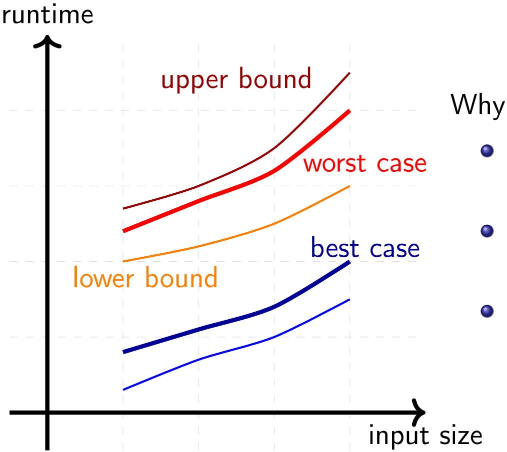

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- get **tight bounds** with upper bounds
- can indicate implementation bugs
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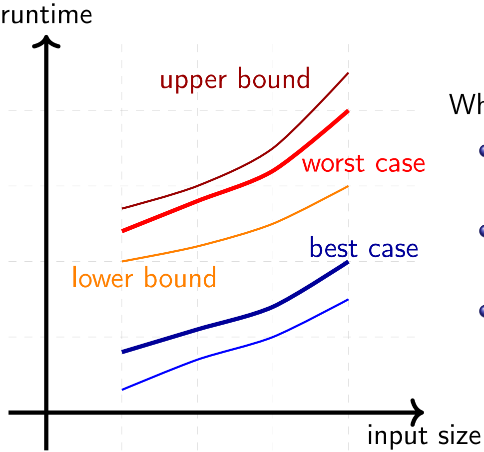

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Here: Two techniques for finding lower bounds[32] inspired by proving **non-termination**

---

[32] F. Frohn, J. Giesl, J. Hensel, C. Aschermann, and T. Ströder: *Lower bounds for runtime complexity of term rewriting*, JAR '17

Finding Lower Bounds by Induction

(1) Induction technique, inspired by **non-looping** non-termination[33]

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- Generate infinite family $\mathcal{T}_{\text{witness}}$ of basic terms as witnesses in

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[33] F. Emmes, T. Enger, J. Giesl: *Proving non-looping non-termination automatically*, IJCAR '12

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- Get lower bound for $\mathrm{rc}_{\mathcal{R}}(n)$ from $p(n)$ in rewrite lemma and $q(n)$

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Finding Lower Bounds by Induction: Example

Example (quicksort)

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\begin{array}{rcl}
\mathsf{qs}(\mathsf{nil}) & \to & \mathsf{nil} \\
\mathsf{qs}(\mathsf{cons}(x, xs)) & \to & \mathsf{qs}(\mathsf{low}(x, xs)) \mathbin{++} \mathsf{cons}(x, \mathsf{qs}(\mathsf{low}(x, xs))) \\
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\mathsf{low}(x, \mathsf{cons}(y, ys)) & \to & \mathsf{if}(x \leq y, x, \mathsf{cons}(y, ys)) \\
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Speculate and prove rewrite lemma:

$$\mathsf{qs}(\mathsf{cons}(\mathsf{zero}, \dots, \mathsf{cons}(\mathsf{zero}, \mathsf{nil}))) \to^{3n^2+2n+1} \mathsf{cons}(\mathsf{zero}, \dots, \mathsf{cons}(\mathsf{zero}, \mathsf{nil}))$$

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$\mathrm{rc}_{\mathcal{R}}(2n + 2) \geq 3n^2 + 2n + 1$ and $\mathrm{rc}_{\mathcal{R}}(n) \in \Omega(n^2)$.

Finding Linear Lower Bounds by Decreasing Loops

(2) Decreasing loops, inspired by **looping** non-termination with

$$s \to_{\mathcal{R}}^{+} C[s\sigma] \to_{\mathcal{R}}^{+} C[C\sigma[s\sigma^2]] \to_{\mathcal{R}}^{+} \cdots$$

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Intuition for **linear** lower bounds:
some fixed context D is **removed** in an argument of recursive call, other arguments may grow, sequence can be repeated (loop)

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for *base term* $s = \mathsf{plus}(x, y)$, *pumping substitution* $\theta = [x \mapsto \mathsf{s}(x)]$, and *result substitution* $\sigma = [y \mapsto \mathsf{s}(y)]$:

$$s\theta \to_{\mathcal{R}}^{+} C[s\sigma]$$

Implies $\mathrm{rc}(n) \in \Omega(n)$!

Finding Exponential Lower Bounds by Decreasing Loops

Exponential lower bounds: several "compatible" parallel recursive calls:

- **Example:** $\mathsf{fib}(\mathsf{s}(\mathsf{s}(n))) \to \mathsf{plus}(\mathsf{fib}(\mathsf{s}(n)), \mathsf{fib}(n))$ has 2 decreasing loops:

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  Has **linear** complexity. But:

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Automation for decreasing loops: **narrowing**.

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Benefits of Induction Technique:

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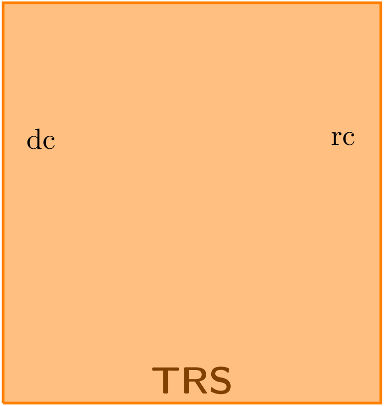
Both techniques can be adapted to innermost runtime complexity!

A Landscape of Complexity Properties and Transformations



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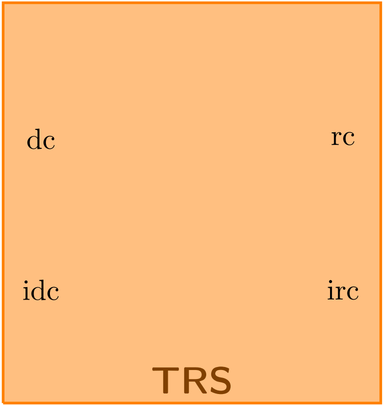

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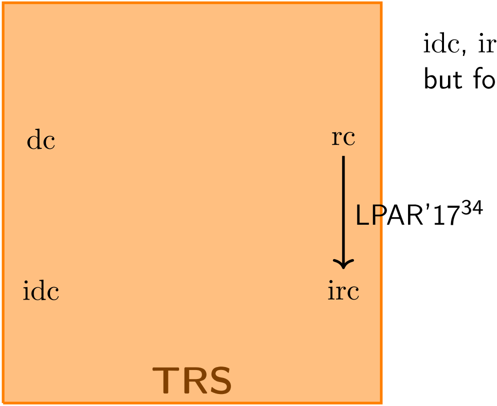

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[34] F. Frohn, J. Giesl: *Analyzing runtime complexity via innermost runtime complexity*, LPAR '17

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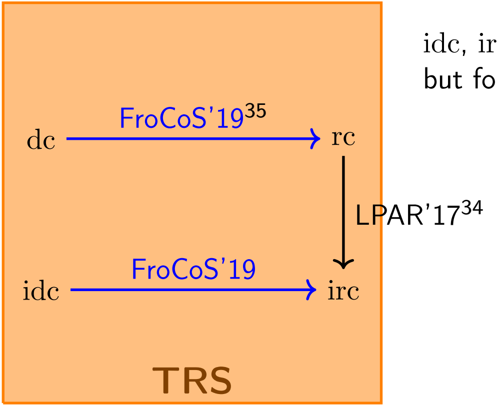

$$\begin{tikzcd}
\mathrm{dc} \arrow[r, "\text{FroCoS'19}^{35}"] & \mathrm{rc} \arrow[d, "\text{LPAR'17}^{34}"] \\
\mathrm{idc} \arrow[r, "\text{FroCoS'19}"] & \mathrm{irc}
\end{tikzcd}$$

TRS

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[34] F. Frohn, J. Giesl: *Analyzing runtime complexity via innermost runtime complexity*, LPAR '17

[35] C. Fuhs: *Transforming Derivational Complexity of Term Rewriting to Runtime Complexity*, FroCoS '19

Transforming Derivational Complexity to Runtime Complexity

The big picture:

- **Have:** Tool for automated analysis of runtime complexity $\mathrm{rc}_{\mathcal{R}}$

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- **Have:** Tool for automated analysis of runtime complexity $\mathrm{rc}_{\mathcal{R}}$
- **Want:** Tool for automated analysis of derivational complexity $\mathrm{dc}_{\mathcal{R}}$
- **Idea:**

  "$\mathrm{rc}_{\mathcal{R}}$ analysis tool + transformation on TRS $\mathcal{R}$ = $\mathrm{dc}_{\mathcal{R}}$ analysis tool"

Transforming Derivational Complexity to Runtime Complexity

The big picture:

- **Have:** Tool for automated analysis of runtime complexity $\mathrm{rc}_{\mathcal{R}}$
- **Want:** Tool for automated analysis of derivational complexity $\mathrm{dc}_{\mathcal{R}}$
- **Idea:**

  "$\mathrm{rc}_{\mathcal{R}}$ analysis tool + transformation on TRS $\mathcal{R}$ = $\mathrm{dc}_{\mathcal{R}}$ analysis tool"
- **Benefits:**
  - Get analysis of derivational complexity "for free"
  - Progress in runtime complexity analysis automatically improves derivational complexity analysis

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- program transformation such that runtime complexity of transformed TRS is **identical** to derivational complexity of original TRS

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- program transformation such that runtime complexity of transformed TRS is **identical** to derivational complexity of original TRS
- transformation correct also from idc to irc
- **implemented** in program analysis tool AProVE
- **evaluated** successfully on TPDB[36] relative to state of the art TcT

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[36] Termination Problem Data Base, standard benchmark source for annual Termination and Complexity Competition (TermComp) with 1000s of problems, `http://termination-portal.org/wiki/TPDB`

From dc to rc: Transformation

Issue:

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by **basic variant**

$$\mathrm{bv}(t) = \mathsf{enc}_{\mathsf{double}}(\mathsf{c}_{\mathsf{double}}(\mathsf{c}_{\mathsf{double}}(\mathsf{s}(0))))$$

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Example (Generator rules $\mathcal{G}$)

$$\begin{aligned}
\mathsf{enc}_{\mathsf{double}}(x) &\to \mathsf{double}(\mathsf{argenc}(x)) \\
\mathsf{enc}_{\mathsf{0}} &\to \mathsf{0} \\
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Then:

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- $\mathrm{bv}(t) \to_{\mathcal{G}}^{*} t$

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Solution:

- add $\mathcal{G}$ as **relative** rewrite rules:
  $\to_{\mathcal{G}}$ steps are **not counted** for complexity analysis!
- transform $\mathcal{R}$ to $\mathcal{R}/\mathcal{G}$ ($\to_{\mathcal{R}}$ steps are counted, $\to_{\mathcal{G}}$ steps are not)

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- transform $\mathcal{R}$ to $\mathcal{R}/\mathcal{G}$ ($\to_{\mathcal{R}}$ steps are counted, $\to_{\mathcal{G}}$ steps are not)
- more generally: transform $\mathcal{R}/\mathcal{S}$ to $\mathcal{R}/(\mathcal{S} \cup \mathcal{G})$
  (input may contain relative rules $\mathcal{S}$, too)

From dc to rc: Correctness

Theorem (Derivational Complexity via Runtime Complexity)

Let $\mathcal{R}/\mathcal{S}$ be a relative TRS, let $\mathcal{G}$ be the generator rules for $\mathcal{R}/\mathcal{S}$. Then

1. $\mathrm{dc}_{\mathcal{R}/\mathcal{S}}(n) = \mathrm{rc}_{\mathcal{R}/(\mathcal{S}\cup\mathcal{G})}(n)$ *(arbitrary rewrite strategies)*
2. $\mathrm{idc}_{\mathcal{R}/\mathcal{S}}(n) = \mathrm{irc}_{\mathcal{R}/(\mathcal{S}\cup\mathcal{G})}(n)$ *(innermost rewriting)*

Note: equalities hold also non-asymptotically!

From (i)dc to (i)rc: Experiments

Experiments on TPDB, compare with state of the art in TcT:

- upper bounds idc: both AProVE and TcT with transformation are stronger than standard TcT
- upper bounds dc: TcT stronger than AProVE and TcT with transformation, but AProVE still solves some new examples
- lower bounds idc and dc: heuristics do not seem to benefit much

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- upper bounds dc: TcT stronger than AProVE and TcT with transformation, but AProVE still solves some new examples
- lower bounds idc and dc: heuristics do not seem to benefit much

⇒ Transformation-based approach should be part of the portfolio of analysis tools for derivational complexity

Derivational Complexity: Future Work

- **Possible applications**
  - compiler simplifications
  - SMT solver preprocessing

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- Go **between** derivational and runtime complexity
  - So far: encode *full* term universe $\mathcal{T}$ via basic terms $\mathcal{T}_{\mathrm{basic}}$
  - Generalise: write relative rules to generate **arbitrary** set $\mathcal{U}$ of terms "between" basic and all terms ($\mathcal{T}_{\mathrm{basic}} \subseteq \mathcal{U} \subseteq \mathcal{T}$).

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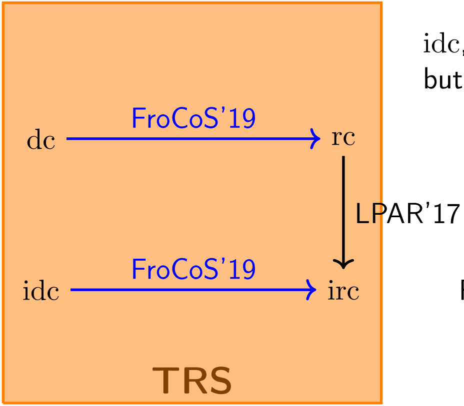
- Want to adapt **techniques** from runtime complexity analysis to derivational complexity! How?
  - (Useful) adaptation of Dependency Pairs?
  - Abstractions to numbers?
  - ...

A Landscape of Complexity Properties and Transformations

idc, irc: like dc, rc,
but for *innermost* rewriting



$$\begin{tikzcd}
\mathrm{dc} \arrow[r, "\text{FroCoS'19}"] & \mathrm{rc} \arrow[d, "\text{LPAR'17}"] \\
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\end{tikzcd}$$

TRS

Rec. ITS irc

ITS irc

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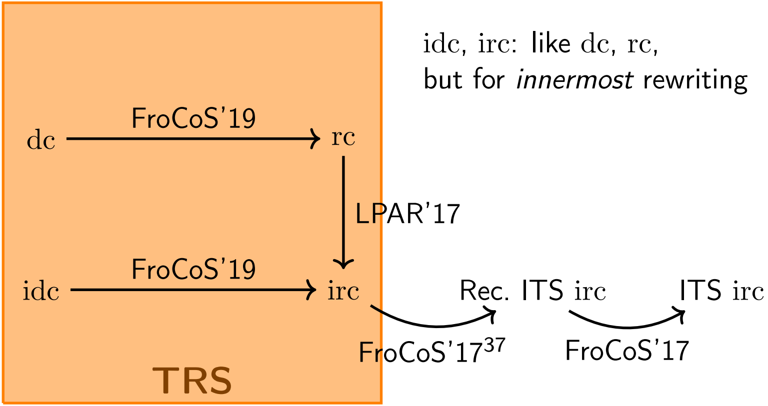

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[37] M. Naaf, F. Frohn, M. Brockschmidt, C. Fuhs, J. Giesl: *Complexity analysis for term rewriting by integer transition systems*, FroCoS '17

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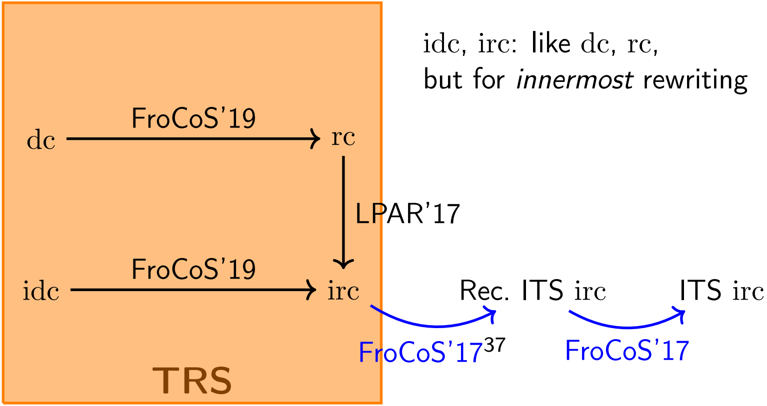

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Bottom-Up Complexity Analysis for Imperative Programs

Recently significant progress in complexity analysis tools for **Integer Transition Systems (ITSs)**:

- CoFloCo[38]
- KoAT[39]
- PUBS[40]

Goal: use these tools to find upper bounds for TRS complexity

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[38] A. Flores-Montoya, R. Hähnle: *Resource analysis of complex programs with cost equations*, APLAS '14, `https://github.com/aeflores/CoFloCo`

[39] M. Brockschmidt, F. Emmes, S. Falke, C. Fuhs, J. Giesl: *Analyzing Runtime and Size Complexity of Integer Programs*, TOPLAS '16, `https://github.com/s-falke/kittel-koat`

[40] E. Albert, P. Arenas, S. Genaim, G. Puebla: *Closed-Form Upper Bounds in Static Cost Analysis*, JAR '11, `https://costa.fdi.ucm.es/pubs/`

Analysing irc of Insertion Sort by Hand: Bottom-Up

Example

$$\begin{aligned}
\mathsf{isort}(\mathsf{nil}, ys) &\quad\longrightarrow\quad ys \\
\mathsf{isort}(\mathsf{cons}(x, xs), ys) &\quad\longrightarrow\quad \mathsf{isort}(xs, \mathsf{insert}(x, ys)) \\
\mathsf{insert}(x, \mathsf{nil}) &\quad\longrightarrow\quad \mathsf{cons}(x, \mathsf{nil}) \\
\mathsf{insert}(x, \mathsf{cons}(y, ys)) &\quad\longrightarrow\quad \mathsf{if}(\mathsf{gt}(x, y), x, \mathsf{cons}(y, ys)) \\
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- $\mathsf{rt}(\mathsf{gt}(x, y)) \in \mathcal{O}(1)$ ("$\xrightarrow{=}$" for relative rules)

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- $\mathsf{rt}(\mathsf{gt}(x, y)) \in \mathcal{O}(1)$ ("$\xrightarrow{=}$" for relative rules)
- $\mathsf{rt}(\mathsf{insert}(x, ys)) \in \mathcal{O}(\mathsf{length}(ys))$

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- $\mathsf{rt}(\mathsf{isort}(xs, ys)) \in \mathcal{O}(\mathsf{length}(xs) \cdot (\mathsf{length}(xs) + \mathsf{length}(ys)))$

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Using Dependency Tuples: Top-Down

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  - note: requires reasoning about isort, insert, and if rules!

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\mathsf{gt}(\mathsf{0}, y) & \xrightarrow{=} & \mathsf{false} \\
\mathsf{gt}(\mathsf{s}(x), \mathsf{0}) & \xrightarrow{=} & \mathsf{true} \\
\mathsf{gt}(\mathsf{s}(x), \mathsf{s}(y)) & \xrightarrow{=} & \mathsf{gt}(x, y)
\end{array}$$

- the recursive isort rule is at most applied linearly often
- the recursive insert rule is at most applied quadratically often
  - note: requires reasoning about isort, insert, and if rules!
  - found via quadratic polynomial interpretation

Using Dependency Tuples: Top-Down

Example

$$\begin{array}{rcl}
\mathsf{isort}(\mathsf{nil}, ys) & \longrightarrow & ys \\
\mathsf{isort}(\mathsf{cons}(x, xs), ys) & \longrightarrow & \mathsf{isort}(xs, \mathsf{insert}(x, ys)) \\
\mathsf{insert}(x, \mathsf{nil}) & \longrightarrow & \mathsf{cons}(x, \mathsf{nil}) \\
\mathsf{insert}(x, \mathsf{cons}(y, ys)) & \longrightarrow & \mathsf{if}(\mathsf{gt}(x, y), x, \mathsf{cons}(y, ys)) \\
\mathsf{if}(\mathsf{true}, x, \mathsf{cons}(y, ys)) & \longrightarrow & \mathsf{cons}(y, \mathsf{insert}(x, ys)) \\
\mathsf{if}(\mathsf{false}, x, \mathsf{cons}(y, ys)) & \longrightarrow & \mathsf{cons}(x, \mathsf{cons}(y, ys)) \\
\mathsf{gt}(\mathsf{0}, y) & \xrightarrow{=} & \mathsf{false} \\
\mathsf{gt}(\mathsf{s}(x), \mathsf{0}) & \xrightarrow{=} & \mathsf{true} \\
\mathsf{gt}(\mathsf{s}(x), \mathsf{s}(y)) & \xrightarrow{=} & \mathsf{gt}(x, y)
\end{array}$$

- the recursive isort rule is at most applied linearly often
- the recursive insert rule is at most applied quadratically often
  - note: requires reasoning about isort, insert, and if rules!
  - found via quadratic polynomial interpretation
- the recursive if rule is applied as often as the recursive insert rule

Bird's Eye View of the Transformation

Example

$$
\begin{array}{rcl}
\mathsf{isort}(\mathsf{nil}, ys) & \to & ys \\
\mathsf{isort}(\mathsf{cons}(x, xs), ys) & \to & \mathsf{isort}(xs, \mathsf{insert}(x, ys)) \\
\mathsf{insert}(x, \mathsf{nil}) & \to & \mathsf{cons}(x, \mathsf{nil}) \\
\mathsf{insert}(x, \mathsf{cons}(y, ys)) & \to & \mathsf{if}(\mathsf{gt}(x, y), x, \mathsf{cons}(y, ys)) \\
\mathsf{if}(\mathsf{true}, x, \mathsf{cons}(y, ys)) & \to & \mathsf{cons}(y, \mathsf{insert}(x, ys)) \\
\mathsf{if}(\mathsf{false}, x, \mathsf{cons}(y, ys)) & \to & \mathsf{cons}(x, \mathsf{cons}(y, ys)) \\
\mathsf{gt}(\mathsf{0}, y) & \xrightarrow{=} & \mathsf{false} \\
\mathsf{gt}(\mathsf{s}(x), \mathsf{0}) & \xrightarrow{=} & \mathsf{true} \\
\mathsf{gt}(\mathsf{s}(x), \mathsf{s}(y)) & \xrightarrow{=} & \mathsf{gt}(x, y)
\end{array}
$$

1. abstract terms to integers

Bird's Eye View of the Transformation

Example

$$
\begin{array}{rcll}
\mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & \mid \quad xs' = 1 \\
\mathsf{isort}(\mathsf{cons}(x, xs), ys) & \to & \mathsf{isort}(xs, \mathsf{insert}(x, ys)) & \\
\mathsf{insert}(x, \mathsf{nil}) & \to & \mathsf{cons}(x, \mathsf{nil}) & \\
\mathsf{insert}(x, \mathsf{cons}(y, ys)) & \to & \mathsf{if}(\mathsf{gt}(x, y), x, \mathsf{cons}(y, ys)) & \\
\mathsf{if}(\mathsf{true}, x, \mathsf{cons}(y, ys)) & \to & \mathsf{cons}(y, \mathsf{insert}(x, ys)) & \\
\mathsf{if}(\mathsf{false}, x, \mathsf{cons}(y, ys)) & \to & \mathsf{cons}(x, \mathsf{cons}(y, ys)) & \\
\mathsf{gt}(0, y) & \xrightarrow{=} & \mathsf{false} & \\
\mathsf{gt}(\mathsf{s}(x), 0) & \xrightarrow{=} & \mathsf{true} & \\
\mathsf{gt}(\mathsf{s}(x), \mathsf{s}(y)) & \xrightarrow{=} & \mathsf{gt}(x, y) &
\end{array}
$$

1. abstract terms to integers

Bird's Eye View of the Transformation

Example

$$\begin{array}{rcll}
\mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & \mid \quad xs' = 1 \\
\mathsf{isort}(xs', ys) & \xrightarrow{1} & \mathsf{isort}(xs, \mathsf{insert}(x, ys)) & \mid \quad xs' = 1 + x + xs \\
\mathsf{insert}(x, \mathsf{nil}) & \to & \mathsf{cons}(x, \mathsf{nil}) & \\
\mathsf{insert}(x, \mathsf{cons}(y, ys)) & \to & \mathsf{if}(\mathsf{gt}(x, y), x, \mathsf{cons}(y, ys)) & \\
\mathsf{if}(\mathsf{true}, x, \mathsf{cons}(y, ys)) & \to & \mathsf{cons}(y, \mathsf{insert}(x, ys)) & \\
\mathsf{if}(\mathsf{false}, x, \mathsf{cons}(y, ys)) & \to & \mathsf{cons}(x, \mathsf{cons}(y, ys)) & \\
\mathsf{gt}(0, y) & \xrightarrow{=} & \mathsf{false} & \\
\mathsf{gt}(\mathsf{s}(x), 0) & \xrightarrow{=} & \mathsf{true} & \\
\mathsf{gt}(\mathsf{s}(x), \mathsf{s}(y)) & \xrightarrow{=} & \mathsf{gt}(x, y) &
\end{array}$$

1. abstract terms to integers

Bird's Eye View of the Transformation

Example

$$
\begin{array}{rcll}
\mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & \mid \quad xs' = 1 \\
\mathsf{isort}(xs', ys) & \xrightarrow{1} & \mathsf{isort}(xs, \mathsf{insert}(x, ys)) & \mid \quad xs' = 1 + x + xs \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & 2 + x & \mid \quad ys' = 1 \\
\mathsf{insert}(x, \mathsf{cons}(y, ys)) & \to & \mathsf{if}(\mathsf{gt}(x, y), x, \mathsf{cons}(y, ys)) & \\
\mathsf{if}(\mathsf{true}, x, \mathsf{cons}(y, ys)) & \to & \mathsf{cons}(y, \mathsf{insert}(x, ys)) & \\
\mathsf{if}(\mathsf{false}, x, \mathsf{cons}(y, ys)) & \to & \mathsf{cons}(x, \mathsf{cons}(y, ys)) & \\
\mathsf{gt}(\mathsf{0}, y) & \xrightarrow{=} & \mathsf{false} & \\
\mathsf{gt}(\mathsf{s}(x), \mathsf{0}) & \xrightarrow{=} & \mathsf{true} & \\
\mathsf{gt}(\mathsf{s}(x), \mathsf{s}(y)) & \xrightarrow{=} & \mathsf{gt}(x, y) &
\end{array}
$$

1. abstract terms to integers

Bird's Eye View of the Transformation

Example

$$
\begin{array}{rcl|l}
\mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & xs' = 1 \\
\mathsf{isort}(xs', ys) & \xrightarrow{1} & \mathsf{isort}(xs, \mathsf{insert}(x, ys)) & xs' = 1 + x + xs \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & 2 + x & ys' = 1 \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & \mathsf{if}(\mathsf{gt}(x, y), x, ys') & ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + y + \mathsf{insert}(x, ys) & b = 1 \wedge ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + ys' & b = 1 \wedge ys' = 1 + y + ys \\
\mathsf{gt}(x', y') & \xrightarrow{0} & 1 & x' = 1 \\
\mathsf{gt}(x', y') & \xrightarrow{0} & 1 & x' = 1 + x \wedge y' = 1 \\
\mathsf{gt}(x', y') & \xrightarrow{0} & \mathsf{gt}(x, y) & x' = 1 + x \wedge y' = 1 + y
\end{array}
$$

1. abstract terms to integers

Bird's Eye View of the Transformation

Example

$$\begin{array}{rcll}
\mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & \mid \quad xs' = 1 \\
\mathsf{isort}(xs', ys) & \xrightarrow{1} & \mathsf{isort}(xs, \mathsf{insert}(x, ys)) & \mid \quad xs' = 1 + x + xs \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & 2 + x & \mid \quad ys' = 1 \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & \mathsf{if}(\mathsf{gt}(x, y), x, ys') & \mid \quad ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + y + \mathsf{insert}(x, ys) & \mid \quad b = 1 \wedge ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + ys' & \mid \quad b = 1 \wedge ys' = 1 + y + ys \\
\mathsf{gt}(x', y') & \xrightarrow{0} & 1 & \mid \quad x' = 1 \\
\mathsf{gt}(x', y') & \xrightarrow{0} & 1 & \mid \quad x' = 1 + x \wedge y' = 1 \\
\mathsf{gt}(x', y') & \xrightarrow{0} & \mathsf{gt}(x, y) & \mid \quad x' = 1 + x \wedge y' = 1 + y
\end{array}$$

1. abstract terms to integers
    - $[c](x_1, \dots, x_n) = 1 + x_1 + \cdots + x_n$ for constructors c
    - note: variables range over $\mathbb{N}$
    - just $+$ and $\cdot$

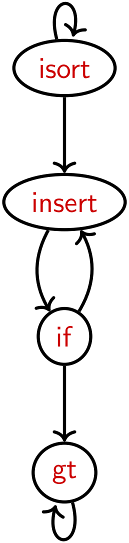
Bird's Eye View of the Transformation

Example

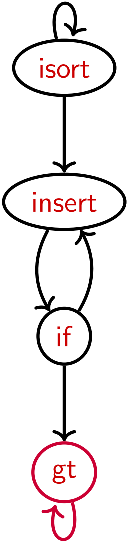
$$\begin{array}{rcll}
\mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & \mid \quad xs' = 1 \\
\mathsf{isort}(xs', ys) & \xrightarrow{1} & \mathsf{isort}(xs, \mathsf{insert}(x, ys)) & \mid \quad xs' = 1 + x + xs \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & 2 + x & \mid \quad ys' = 1 \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & \mathsf{if}(\mathsf{gt}(x, y), x, ys') & \mid \quad ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + y + \mathsf{insert}(x, ys) & \mid \quad b = 1 \land ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + ys' & \mid \quad b = 1 \land ys' = 1 + y + ys \\
\mathsf{gt}(x', y') & \xrightarrow{0} & 1 & \mid \quad x' = 1 \\
\mathsf{gt}(x', y') & \xrightarrow{0} & 1 & \mid \quad x' = 1 + x \land y' = 1 \\
\mathsf{gt}(x', y') & \xrightarrow{0} & \mathsf{gt}(x, y) & \mid \quad x' = 1 + x \land y' = 1 + y
\end{array}$$

1. abstract terms to integers
    - $[c](x_1, \dots, x_n) = 1 + x_1 + \cdots + x_n$ for constructors c
    - note: variables range over $\mathbb{N}$
    - just $+$ and $\cdot$
2. analyse result size for bottom-SCC (Strongly Connected Component) of call graph using standard ITS tools

Call Graph & Bottom SCCs



Call Graph & Bottom SCCs



Bird's Eye View

Example

$$\begin{array}{rcl|l}
\mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & xs' = 1 \\
\mathsf{isort}(xs', ys) & \xrightarrow{1} & \mathsf{isort}(xs, \mathsf{insert}(x, ys)) & xs' = 1 + x + xs \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & 2 + x & ys' = 1 \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & \mathsf{if}(\mathsf{gt}(x, y), x, ys') & ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + y + \mathsf{insert}(x, ys) & b = 1 \wedge ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + ys' & b = 1 \wedge ys' = 1 + y + ys \\
\mathsf{gt}(x', y') & \xrightarrow{0} & 1 & x' = 1 \\
\mathsf{gt}(x', y') & \xrightarrow{0} & 1 & x' = 1 + x \wedge y' = 1 \\
\mathsf{gt}(x', y') & \xrightarrow{0} & \mathsf{gt}(x, y) & x' = 1 + x \wedge y' = 1 + y
\end{array}$$

1. abstract terms to integers
    - $[c](x_1, \dots, x_n) = 1 + x_1 + \cdots + x_n$ for constructors c
    - note: variables range over $\mathbb{N}$
    - just $+$ and $\cdot$
2. analyse result size for bottom-SCC using standard ITS tools

Bird’s Eye View

Example

$$\begin{array}{rclcl}
\mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & \mid & xs' = 1 \\
\mathsf{isort}(xs', ys) & \xrightarrow{1} & \mathsf{isort}(xs, \mathsf{insert}(x, ys)) & \mid & xs' = 1 + x + xs \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & 2 + x & \mid & ys' = 1 \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & \mathsf{if}(\mathsf{gt}(x, y), x, ys') & \mid & ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + y + \mathsf{insert}(x, ys) & \mid & b = 1 \wedge ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + ys' & \mid & b = 1 \wedge ys' = 1 + y + ys \\
\mathsf{gt}(x', y') & \xrightarrow{0} & 1 & \mid & x' = 1 \\
\mathsf{gt}(x', y') & \xrightarrow{0} & 1 & \mid & x' = 1 + x \wedge y' = 1 \\
\mathsf{gt}(x', y') & \xrightarrow{0} & \mathsf{gt}(x, y) & \mid & x' = 1 + x \wedge y' = 1 + y
\end{array}$$

1. abstract terms to integers
    - $[c](x_1, \ldots, x_n) = 1 + x_1 + \cdots + x_n$ for constructors c
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Bird's Eye View

Example

$$
\begin{array}{rcl|l}
\mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & xs' = 1 \\
\mathsf{isort}(xs', ys) & \xrightarrow{1} & \mathsf{isort}(xs, \mathsf{insert}(x, ys)) & xs' = 1 + x + xs \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & 2 + x & ys' = 1 \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & \mathsf{if}(\mathsf{gt}(x, y), x, ys') & ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + y + \mathsf{insert}(x, ys) & b = 1 \wedge ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + ys' & b = 1 \wedge ys' = 1 + y + ys \\
\mathsf{gt}(x', y') & \xrightarrow{0} & 1 & x' = 1 \\
\mathsf{gt}(x', y') & \xrightarrow{0} & 1 & x' = 1 + x \wedge y' = 1 \\
\mathsf{gt}(x', y') & \xrightarrow{0} & \mathsf{gt}(x, y) & x' = 1 + x \wedge y' = 1 + y
\end{array}
$$

1. abstract terms to integers
    - $[c](x_1, \ldots, x_n) = 1 + x_1 + \cdots + x_n$ for constructors c
    - note: variables range over $\mathbb{N}$
    - just $+$ and $\cdot$
2. analyse result size for bottom-SCC using standard ITS tools
3. analyse runtime of bottom-SCC using standard ITS tools

Bird's Eye View

Example

$$\begin{array}{rcll}
\mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & \mid \quad xs' = 1 \\
\mathsf{isort}(xs', ys) & \xrightarrow{1} & \mathsf{isort}(xs, \mathsf{insert}(x, ys)) & \mid \quad xs' = 1 + x + xs \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & 2 + x & \mid \quad ys' = 1 \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & \mathsf{if}(b, x, ys') & \mid \quad ys' = 1 + y + ys \land b \leq 1 \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + y + \mathsf{insert}(x, ys) & \mid \quad b = 1 \land ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + ys' & \mid \quad b = 1 \land ys' = 1 + y + ys
\end{array}$$

1. abstract terms to integers
   - $[c](x_1, \ldots, x_n) = 1 + x_1 + \cdots + x_n$ for constructors c
   - note: variables range over $\mathbb{N}$
   - just $+$ and $\cdot$
2. analyse result size for bottom-SCC using standard ITS tools
3. analyse runtime of bottom-SCC using standard ITS tools

Abstracting Terms to Integers: Pitfalls

Terminating Variants

Term Rewriting	**Integer Transition Systems**
start terms may have variables	ground start terms only

Example

$\mathsf{h}(x) \to \mathsf{f}(\mathsf{g}(x)) \qquad \mathsf{f}(x) \to \mathsf{f}(x) \qquad \mathsf{g}(\mathsf{a}) \xrightarrow{=} \mathsf{g}(\mathsf{a})$

Terminating Variants

Term Rewriting	**Integer Transition Systems**
start terms may have variables	ground start terms only

Example

$$\mathsf{h}(x) \longrightarrow \mathsf{f}(\mathsf{g}(x)) \qquad \mathsf{f}(x) \longrightarrow \mathsf{f}(x) \qquad \mathsf{g}(\mathsf{a}) \xrightarrow{=} \mathsf{g}(\mathsf{a})$$

innermost rewriting: $\mathsf{h}(x) \longrightarrow \mathsf{f}(\mathsf{g}(x)) \longrightarrow \mathsf{f}(\mathsf{g}(x)) \longrightarrow \ldots$

Terminating Variants

Term Rewriting	**Integer Transition Systems**
start terms may have variables	ground start terms only

Example

$\mathsf{h}(x) \longrightarrow \mathsf{f}(\mathsf{g}(x)) \qquad \mathsf{f}(x) \longrightarrow \mathsf{f}(x) \qquad \mathsf{g}(\mathsf{a}) \overset{=}{\longrightarrow} \mathsf{g}(\mathsf{a})$

innermost rewriting: $\mathsf{h}(x) \longrightarrow \mathsf{f}(\mathsf{g}(x)) \longrightarrow \mathsf{f}(\mathsf{g}(x)) \longrightarrow \ldots$ $\mathcal{O}(\infty)$

Terminating Variants

Term Rewriting	**Integer Transition Systems**
start terms may have variables	ground start terms only

Example

$$\mathsf{h}(x) \longrightarrow \mathsf{f}(\mathsf{g}(x)) \qquad \mathsf{f}(x) \longrightarrow \mathsf{f}(x) \qquad \mathsf{g}(\mathsf{a}) \xrightarrow{=} \mathsf{g}(\mathsf{a})$$

innermost rewriting: $\mathsf{h}(x) \longrightarrow \mathsf{f}(\mathsf{g}(x)) \longrightarrow \mathsf{f}(\mathsf{g}(x)) \longrightarrow \dots$ $\mathcal{O}(\infty)$

- Just ground rewriting?

Terminating Variants

Term Rewriting	**Integer Transition Systems**
start terms may have variables	ground start terms only

Example

$$\mathsf{h}(x) \longrightarrow \mathsf{f}(\mathsf{g}(x)) \qquad \mathsf{f}(x) \longrightarrow \mathsf{f}(x) \qquad \mathsf{g}(\mathsf{a}) \xrightarrow{=} \mathsf{g}(\mathsf{a})$$

innermost rewriting: $\mathsf{h}(x) \longrightarrow \mathsf{f}(\mathsf{g}(x)) \longrightarrow \mathsf{f}(\mathsf{g}(x)) \longrightarrow \ldots$ $\mathcal{O}(\infty)$

ground rewriting: $\mathsf{h}(\mathsf{a}) \longrightarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \xrightarrow{=} \mathsf{f}(\mathsf{g}(\mathsf{a})) \xrightarrow{=} \ldots$

- Just ground rewriting?

Terminating Variants

Term Rewriting	**Integer Transition Systems**
start terms may have variables	ground start terms only

Example

$$\mathsf{h}(x) \longrightarrow \mathsf{f}(\mathsf{g}(x)) \qquad \mathsf{f}(x) \longrightarrow \mathsf{f}(x) \qquad \mathsf{g}(\mathsf{a}) \xrightarrow{=} \mathsf{g}(\mathsf{a})$$

innermost rewriting: $\mathsf{h}(x) \longrightarrow \mathsf{f}(\mathsf{g}(x)) \longrightarrow \mathsf{f}(\mathsf{g}(x)) \longrightarrow \dots$ $\mathcal{O}(\infty)$

ground rewriting: $\mathsf{h}(\mathsf{a}) \longrightarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \xrightarrow{=} \mathsf{f}(\mathsf{g}(\mathsf{a})) \xrightarrow{=} \dots$ $\mathcal{O}(1)$

- Just ground rewriting?

Terminating Variants

Term Rewriting	**Integer Transition Systems**
start terms may have variables	ground start terms only

Example

$\mathsf{h}(x) \to \mathsf{f}(\mathsf{g}(x)) \qquad \mathsf{f}(x) \to \mathsf{f}(x) \qquad \mathsf{g}(\mathsf{a}) \xrightarrow{=} \mathsf{g}(\mathsf{a})$

innermost rewriting: $\mathsf{h}(x) \to \mathsf{f}(\mathsf{g}(x)) \to \mathsf{f}(\mathsf{g}(x)) \to \dots$ $\mathcal{O}(\infty)$

ground rewriting: $\mathsf{h}(\mathsf{a}) \to \mathsf{f}(\mathsf{g}(\mathsf{a})) \xrightarrow{=} \mathsf{f}(\mathsf{g}(\mathsf{a})) \xrightarrow{=} \dots$ $\mathcal{O}(1)$

- Just ground rewriting?
- Add terminating variant of relative rules!

Terminating Variants

Term Rewriting	**Integer Transition Systems**
start terms may have variables	ground start terms only

Example

$$\mathsf{h}(x) \to \mathsf{f}(\mathsf{g}(x)) \qquad \mathsf{f}(x) \to \mathsf{f}(x) \qquad \mathsf{g}(\mathsf{a}) \xrightarrow{=} \mathsf{g}(\mathsf{a})$$

innermost rewriting: $\mathsf{h}(x) \to \mathsf{f}(\mathsf{g}(x)) \to \mathsf{f}(\mathsf{g}(x)) \to \dots$ $\mathcal{O}(\infty)$

ground rewriting: $\mathsf{h}(\mathsf{a}) \to \mathsf{f}(\mathsf{g}(\mathsf{a})) \xrightarrow{=} \mathsf{f}(\mathsf{g}(\mathsf{a})) \xrightarrow{=} \dots$ $\mathcal{O}(1)$

- Just ground rewriting?
- Add terminating variant of relative rules!

Definition

$\mathcal{N}$ is a terminating variant of $\mathcal{S}$ iff $\mathcal{N}$ terminates and every $\mathcal{N}$-normal form is an $\mathcal{S}$-normal form.

Terminating Variants

Term Rewriting	**Integer Transition Systems**
start terms may have variables	ground start terms only

Example

$$\mathsf{h}(x) \to \mathsf{f}(\mathsf{g}(x)) \qquad \mathsf{f}(x) \to \mathsf{f}(x) \qquad \mathsf{g}(\mathsf{a}) \xrightarrow{=} \mathsf{g}(\mathsf{a}) \qquad \mathsf{g}(\mathsf{a}) \xrightarrow{=} \mathsf{a}$$

innermost rewriting: $\mathsf{h}(x) \to \mathsf{f}(\mathsf{g}(x)) \to \mathsf{f}(\mathsf{g}(x)) \to \dots$ $\mathcal{O}(\infty)$

ground rewriting: $\mathsf{h}(\mathsf{a}) \to \mathsf{f}(\mathsf{g}(\mathsf{a})) \xrightarrow{=} \mathsf{f}(\mathsf{g}(\mathsf{a})) \xrightarrow{=} \dots$ $\mathcal{O}(1)$

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Terminating Variants

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innermost rewriting: $\mathsf{h}(x) \to \mathsf{f}(\mathsf{g}(x)) \to \mathsf{f}(\mathsf{g}(x)) \to \dots$ $\qquad \mathcal{O}(\infty)$

ground rewriting: $\mathsf{h}(\mathsf{a}) \to \mathsf{f}(\mathsf{g}(\mathsf{a})) \xrightarrow{=} \mathsf{f}(\mathsf{g}(\mathsf{a})) \xrightarrow{=} \dots$ $\qquad \mathcal{O}(1)$

with terminating variant: $\mathsf{h}(\mathsf{a}) \to \mathsf{f}(\mathsf{g}(\mathsf{a})) \xrightarrow{=} \mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{a}) \to \dots$

- Just ground rewriting?
- Add terminating variant of relative rules!

Definition

$\mathcal{N}$ is a terminating variant of $\mathcal{S}$ iff $\mathcal{N}$ terminates and every $\mathcal{N}$-normal form is an $\mathcal{S}$-normal form.

Terminating Variants

Term Rewriting	**Integer Transition Systems**
start terms may have variables	ground start terms only

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$\mathsf{h}(x) \to \mathsf{f}(\mathsf{g}(x)) \qquad \mathsf{f}(x) \to \mathsf{f}(x) \qquad \mathsf{g}(\mathsf{a}) \xrightarrow{=} \mathsf{g}(\mathsf{a}) \qquad \mathsf{g}(\mathsf{a}) \xrightarrow{=} \mathsf{a}$

innermost rewriting: $\mathsf{h}(x) \to \mathsf{f}(\mathsf{g}(x)) \to \mathsf{f}(\mathsf{g}(x)) \to \ldots$ $\mathcal{O}(\infty)$

ground rewriting: $\mathsf{h}(\mathsf{a}) \to \mathsf{f}(\mathsf{g}(\mathsf{a})) \xrightarrow{=} \mathsf{f}(\mathsf{g}(\mathsf{a})) \xrightarrow{=} \ldots$ $\mathcal{O}(1)$

with terminating variant: $\mathsf{h}(\mathsf{a}) \to \mathsf{f}(\mathsf{g}(\mathsf{a})) \xrightarrow{=} \mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{a}) \to \ldots$ $\mathcal{O}(\infty)$

- Just ground rewriting?
- Add terminating variant of relative rules!

Definition

$\mathcal{N}$ is a terminating variant of $\mathcal{S}$ iff $\mathcal{N}$ terminates and every $\mathcal{N}$-normal form is an $\mathcal{S}$-normal form.

Ensuring Complete Definedness

Term Rewriting	**Integer Transition Systems**
arbitrary matchers	integer substitutions only

Example

$\mathsf{f}(x) \longrightarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \qquad \mathsf{g}(\mathsf{b}(\mathsf{a})) \longrightarrow \mathsf{a}$

Ensuring Complete Definedness

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Example

$$\mathsf{f}(x) \longrightarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \qquad \mathsf{g}(\mathsf{b}(\mathsf{a})) \longrightarrow \mathsf{a}$$

original TRS: $\mathsf{f}(\mathsf{a}) \longrightarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \longrightarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \longrightarrow \dots$

Ensuring Complete Definedness

Term Rewriting	**Integer Transition Systems**
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$$\mathsf{f}(x) \longrightarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \qquad \mathsf{g}(\mathsf{b}(\mathsf{a})) \longrightarrow \mathsf{a}$$

original TRS: $\mathsf{f}(\mathsf{a}) \longrightarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \longrightarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \longrightarrow \dots$ $\mathcal{O}(\infty)$

Ensuring Complete Definedness

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resulting ITS: $\mathsf{f}(1) \xrightarrow{1} \mathsf{f}(\mathsf{g}(1))$

Ensuring Complete Definedness

Term Rewriting	**Integer Transition Systems**
arbitrary matchers	integer substitutions only

Example

$$\mathsf{f}(x) \longrightarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \qquad \mathsf{g}(\mathsf{b}(\mathsf{a})) \longrightarrow \mathsf{a}$$

original TRS: $\mathsf{f}(\mathsf{a}) \longrightarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \longrightarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \longrightarrow \dots$ $\mathcal{O}(\infty)$

resulting ITS: $\mathsf{f}(1) \xrightarrow{1} \mathsf{f}(\mathsf{g}(1))$ $\mathcal{O}(1)$

Ensuring Complete Definedness

Term Rewriting	**Integer Transition Systems**
arbitrary matchers	integer substitutions only

Example

$\mathsf{f}(x) \longrightarrow \mathsf{f}(\mathsf{g}(\mathsf{a}))$ $\qquad$ $\mathsf{g}(\mathsf{b}(\mathsf{a})) \longrightarrow \mathsf{a}$

original TRS: $\mathsf{f}(\mathsf{a}) \longrightarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \longrightarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \longrightarrow \dots$ $\qquad$ $\mathcal{O}(\infty)$

resulting ITS: $\mathsf{f}(1) \xrightarrow{1} \mathsf{f}(\mathsf{g}(1))$ $\qquad$ $\mathcal{O}(1)$

Definition

A TRS is completely defined iff its ground normal forms do not contain defined symbols.

Ensuring Complete Definedness

Term Rewriting	**Integer Transition Systems**
arbitrary matchers	integer substitutions only

Example

$$\mathsf{f}(x) \to \mathsf{f}(\mathsf{g}(\mathsf{a})) \qquad \mathsf{g}(\mathsf{b}(\mathsf{a})) \to \mathsf{a} \qquad \mathsf{g}(x) \xrightarrow{=} \mathsf{a}$$

original TRS: $\mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{g}(\mathsf{a})) \to \mathsf{f}(\mathsf{g}(\mathsf{a})) \to \dots$ $\mathcal{O}(\infty)$

resulting ITS: $\mathsf{f}(1) \xrightarrow{1} \mathsf{f}(\mathsf{g}(1))$ $\mathcal{O}(1)$

Definition

A TRS is completely defined iff its ground normal forms do not contain defined symbols.

TRS not completely defined? $\curvearrowright$ Add suitable terminating variant!

Ensuring Complete Definedness

Term Rewriting	**Integer Transition Systems**
arbitrary matchers	integer substitutions only

Example

$$\mathsf{f}(x) \to \mathsf{f}(\mathsf{g}(\mathsf{a})) \qquad \mathsf{g}(\mathsf{b}(\mathsf{a})) \to \mathsf{a} \qquad \mathsf{g}(x) \xrightarrow{=} \mathsf{a}$$

original TRS: $\mathsf{f}(\mathsf{a}) \to \mathsf{f}(\mathsf{g}(\mathsf{a})) \to \mathsf{f}(\mathsf{g}(\mathsf{a})) \to \dots$ $\mathcal{O}(\infty)$

resulting ITS: $\mathsf{f}(1) \xrightarrow{1} \mathsf{f}(\mathsf{g}(1))$ $\mathcal{O}(1)$

ITS after completion: $\mathsf{f}(1) \xrightarrow{1} \mathsf{f}(\mathsf{g}(1)) \xrightarrow{0} \mathsf{f}(1) \xrightarrow{1} \mathsf{f}(\mathsf{g}(1)) \xrightarrow{0} \dots$

Definition

A TRS is completely defined iff its ground normal forms do not contain defined symbols.

TRS not completely defined? $\curvearrowright$ Add suitable terminating variant!

Ensuring Complete Definedness

Term Rewriting	**Integer Transition Systems**
arbitrary matchers	integer substitutions only

Example

$$\mathsf{f}(x) \longrightarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \qquad \mathsf{g}(\mathsf{b}(\mathsf{a})) \longrightarrow \mathsf{a} \qquad \mathsf{g}(x) \xrightarrow{=} \mathsf{a}$$

original TRS: $\mathsf{f}(\mathsf{a}) \longrightarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \longrightarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \longrightarrow \dots$ $\mathcal{O}(\infty)$

resulting ITS: $\mathsf{f}(1) \xrightarrow{1} \mathsf{f}(\mathsf{g}(1))$ $\mathcal{O}(1)$

ITS after completion: $\mathsf{f}(1) \xrightarrow{1} \mathsf{f}(\mathsf{g}(1)) \xrightarrow{0} \mathsf{f}(1) \xrightarrow{1} \mathsf{f}(\mathsf{g}(1)) \xrightarrow{0} \dots$ $\mathcal{O}(\infty)$

Definition

A TRS is completely defined iff its ground normal forms do not contain defined symbols.

TRS not completely defined? $\curvearrowright$ Add suitable terminating variant!

Ensuring Complete Definedness

Term Rewriting	**Integer Transition Systems**
arbitrary matchers	integer substitutions only

Example

$$\mathsf{f}(x) \longrightarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \qquad \mathsf{g}(\mathsf{b}(\mathsf{a})) \longrightarrow \mathsf{a} \qquad \mathsf{g}(x) \xrightarrow{=} \mathsf{a}$$

original TRS: $\mathsf{f}(\mathsf{a}) \longrightarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \longrightarrow \mathsf{f}(\mathsf{g}(\mathsf{a})) \longrightarrow \ldots$ $\mathcal{O}(\infty)$

resulting ITS: $\mathsf{f}(1) \xrightarrow{1} \mathsf{f}(\mathsf{g}(1))$ $\mathcal{O}(1)$

ITS after completion: $\mathsf{f}(1) \xrightarrow{1} \mathsf{f}(\mathsf{g}(1)) \xrightarrow{0} \mathsf{f}(1) \xrightarrow{1} \mathsf{f}(\mathsf{g}(1)) \xrightarrow{0} \ldots$ $\mathcal{O}(\infty)$

Definition

A TRS is completely defined iff its well-typed ground normal forms do not contain defined symbols.

TRS not completely defined? $\curvearrowright$ Add suitable terminating variant!

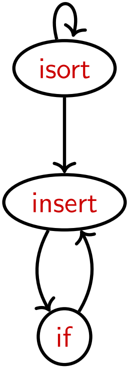
Bird's Eye View

Example

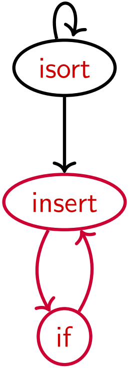
$$\begin{array}{rcl|l}
\mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & xs' = 1 \\
\mathsf{isort}(xs', ys) & \xrightarrow{1} & \mathsf{isort}(xs, \mathsf{insert}(x, ys)) & xs' = 1 + x + xs \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & 2 + x & ys' = 1 \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & \mathsf{if}(b, x, ys') & ys' = 1 + y + ys \wedge b \leq 1 \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + y + \mathsf{insert}(x, ys) & b = 1 \wedge ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + ys' & b = 1 \wedge ys' = 1 + y + ys
\end{array}$$

1. abstract terms to integers
2. analyse result size for bottom-SCC using standard ITS tools
3. analyse runtime of bottom-SCC using standard ITS tools

Call Graph & Bottom SCCs



Call Graph & Bottom SCCs



Bird's Eye View

Example

$$\begin{array}{rcll}
\mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & \mid \quad xs' = 1 \\
\mathsf{isort}(xs', ys) & \xrightarrow{1} & \mathsf{isort}(xs, \mathsf{insert}(x, ys)) & \mid \quad xs' = 1 + x + xs \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & 2 + x & \mid \quad ys' = 1 \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & \mathsf{if}(b, x, ys') & \mid \quad ys' = 1 + y + ys \wedge b \leq 1 \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + y + \mathsf{insert}(x, ys) & \mid \quad b = 1 \wedge ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + ys' & \mid \quad b = 1 \wedge ys' = 1 + y + ys
\end{array}$$

1. abstract terms to integers
2. analyse result size for bottom-SCC using standard ITS tools
3. analyse runtime of bottom-SCC using standard ITS tools

Bird's Eye View

Example

$$
\begin{array}{rcll}
\mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & \mid \quad xs' = 1 \\
\mathsf{isort}(xs', ys) & \xrightarrow{1} & \mathsf{isort}(xs, \mathsf{insert}(x, ys)) & \mid \quad xs' = 1 + x + xs \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & 2 + x & \mid \quad ys' = 1 \\
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\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + y + \mathsf{insert}(x, ys) & \mid \quad b = 1 \wedge ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + ys' & \mid \quad b = 1 \wedge ys' = 1 + y + ys
\end{array}
$$

1. abstract terms to integers
2. analyse result size for bottom-SCC using standard ITS tools
3. analyse runtime of bottom-SCC using standard ITS tools

Analyse Size Using Standard ITS Tools

Using Runtime Analysis to Compute Size Bounds

Idea: time bound for insert in transformed rules gives size bound for insert in original rules

Example

$$\begin{array}{llll}
\mathsf{insert}(x, ys') & \xrightarrow{1} & 2 + x & \mid \quad ys' = 1 \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & \mathsf{if}(b, x, ys') & \mid \quad ys' = 1 + y + ys \land b \leq 1 \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + y + \mathsf{insert}(x, ys) & \mid \quad b = 1 \land ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + ys' & \mid \quad b = 1 \land ys' = 1 + y + ys
\end{array}$$

Using Runtime Analysis to Compute Size Bounds

Idea: time bound for insert in transformed rules gives size bound for insert in original rules

Example

$$\begin{array}{llll}
\mathsf{insert}(x, ys') & \xrightarrow{1} & 2 + x & \mid\quad ys' = 1 \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & \mathsf{if}(b, x, ys') & \mid\quad ys' = 1 + y + ys \wedge b \leq 1 \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + y + \mathsf{insert}(x, ys) & \mid\quad b = 1 \wedge ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + ys' & \mid\quad b = 1 \wedge ys' = 1 + y + ys
\end{array}$$

Idea: move "integer context" to weights

Using Runtime Analysis to Compute Size Bounds

Idea: time bound for insert in transformed rules gives size bound for insert in original rules

Example

$$\begin{array}{rclcl}
\mathsf{insert}(x, ys') & \xrightarrow{2+x} & 2 + x & | & ys' = 1 \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & \mathsf{if}(b, x, ys') & | & ys' = 1 + y + ys \wedge b \leq 1 \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + y + \mathsf{insert}(x, ys) & | & b = 1 \wedge ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + ys' & | & b = 1 \wedge ys' = 1 + y + ys
\end{array}$$

Idea: move "integer context" to weights

Using Runtime Analysis to Compute Size Bounds

Idea: time bound for insert in transformed rules gives size bound for insert in original rules

Example

$$\begin{array}{rclcl}
\mathsf{insert}(x, ys') & \xrightarrow{2+x} & 2 + x & \mid & ys' = 1 \\
\mathsf{insert}(x, ys') & \xrightarrow{0} & \mathsf{if}(b, x, ys') & \mid & ys' = 1 + y + ys \land b \leq 1 \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + y + \mathsf{insert}(x, ys) & \mid & b = 1 \land ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + ys' & \mid & b = 1 \land ys' = 1 + y + ys
\end{array}$$

Idea: move "integer context" to weights

Using Runtime Analysis to Compute Size Bounds

Idea: time bound for insert in transformed rules gives size bound for insert in original rules

Example

$$\begin{array}{rclcl}
\mathsf{insert}(x, ys') & \xrightarrow{2+x} & 2 + x & \mid & ys' = 1 \\
\mathsf{insert}(x, ys') & \xrightarrow{0} & \mathsf{if}(b, x, ys') & \mid & ys' = 1 + y + ys \wedge b \leq 1 \\
\mathsf{if}(b, x, ys') & \xrightarrow{1+y} & 1 + y + \mathsf{insert}(x, ys) & \mid & b = 1 \wedge ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + ys' & \mid & b = 1 \wedge ys' = 1 + y + ys
\end{array}$$

Idea: move "integer context" to weights

Using Runtime Analysis to Compute Size Bounds

Idea: time bound for insert in transformed rules gives size bound for insert in original rules

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\mathsf{insert}(x, ys') & \xrightarrow{2+x} & 2 + x & \mid \quad ys' = 1 \\
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\mathsf{if}(b, x, ys') & \xrightarrow{1+y} & 1 + y + \mathsf{insert}(x, ys) & \mid \quad b = 1 \wedge ys' = 1 + y + ys \\
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Idea: move "integer context" to weights

Using Runtime Analysis to Compute Size Bounds

Idea: time bound for insert in transformed rules gives size bound for insert in original rules

Example

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\mathsf{insert}(x, ys') & \xrightarrow{2+x} & 2 + x & \mid \quad ys' = 1 \\
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\mathsf{if}(b, x, ys') & \xrightarrow{1+ys'} & 1 + ys' & \mid \quad b = 1 \wedge ys' = 1 + y + ys
\end{array}$$

Idea: move "integer context" to weights $\curvearrowright$ $\mathsf{sz}(\mathsf{insert}(x, ys')) \leq 1 + x + ys'$

Using Runtime Analysis to Compute Size Bounds

Idea: time bound for insert in transformed rules gives size bound for insert in original rules

Example

$$\begin{array}{rcll} \mathsf{insert}(x, ys') & \xrightarrow{2+x} & 2 + x & \mid \quad ys' = 1 \\ \mathsf{insert}(x, ys') & \xrightarrow{0} & \mathsf{if}(b, x, ys') & \mid \quad ys' = 1 + y + ys \wedge b \leq 1 \\ \mathsf{if}(b, x, ys') & \xrightarrow{1+y} & 1 + y + \mathsf{insert}(x, ys) & \mid \quad b = 1 \wedge ys' = 1 + y + ys \\ \mathsf{if}(b, x, ys') & \xrightarrow{1+ys'} & 1 + ys' & \mid \quad b = 1 \wedge ys' = 1 + y + ys \end{array}$$

Idea: move "integer context" to weights $\curvearrowright$ $\mathsf{sz}(\mathsf{insert}(x, ys')) \leq 1 + x + ys'$

Example

$$\mathsf{f}(x) \xrightarrow{1} 2 + x \cdot \mathsf{f}(x-1) \quad \mid \quad x > 0$$

Using Runtime Analysis to Compute Size Bounds

Idea: time bound for insert in transformed rules gives size bound for insert in original rules

Example

$$\begin{array}{rcll}
\mathsf{insert}(x, ys') & \xrightarrow{2+x} & 2 + x & \mid \quad ys' = 1 \\
\mathsf{insert}(x, ys') & \xrightarrow{0} & \mathsf{if}(b, x, ys') & \mid \quad ys' = 1 + y + ys \wedge b \leq 1 \\
\mathsf{if}(b, x, ys') & \xrightarrow{1+y} & 1 + y + \mathsf{insert}(x, ys) & \mid \quad b = 1 \wedge ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1+ys'} & 1 + ys' & \mid \quad b = 1 \wedge ys' = 1 + y + ys
\end{array}$$

Idea: move "integer context" to weights $\curvearrowright$ $\mathsf{sz}(\mathsf{insert}(x, ys')) \leq 1 + x + ys'$

Example

$$\mathsf{f}(x) \xrightarrow{1} 2 + x \cdot \mathsf{f}(x - 1) \quad \mid \quad x > 0$$

Idea: use accumulator

Using Runtime Analysis to Compute Size Bounds

Idea: time bound for insert in transformed rules gives size bound for insert in original rules

Example

$$\begin{array}{rcll}
\mathsf{insert}(x, ys') & \xrightarrow{2+x} & 2 + x & \mid\quad ys' = 1 \\
\mathsf{insert}(x, ys') & \xrightarrow{0} & \mathsf{if}(b, x, ys') & \mid\quad ys' = 1 + y + ys \wedge b \leq 1 \\
\mathsf{if}(b, x, ys') & \xrightarrow{1+y} & 1 + y + \mathsf{insert}(x, ys) & \mid\quad b = 1 \wedge ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1+ys'} & 1 + ys' & \mid\quad b = 1 \wedge ys' = 1 + y + ys
\end{array}$$

Idea: move "integer context" to weights $\curvearrowright$ $\mathsf{sz}(\mathsf{insert}(x, ys')) \leq 1 + x + ys'$

Example

$$\begin{array}{rcll}
\mathsf{f}(x) & \xrightarrow{1} & 2 + x \cdot \mathsf{f}(x - 1) & \mid\quad x > 0 \\
\mathsf{f}(x, acc) & \xrightarrow{acc \cdot 2} & 2 + x \cdot \mathsf{f}(x - 1, acc \cdot x) & \mid\quad x > 0
\end{array}$$

Idea: use accumulator

Bird's Eye View

Example

$$
\begin{array}{rcl|l}
\mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & xs' = 1 \\
\mathsf{isort}(xs', ys) & \xrightarrow{1} & \mathsf{isort}(xs, \mathsf{insert}(x, ys)) & xs' = 1 + x + xs \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & 2 + x & ys' = 1 \\
\mathsf{insert}(x, ys') & \xrightarrow{1} & \mathsf{if}(b, x, ys') & ys' = 1 + y + ys \land b \leq 1 \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + y + \mathsf{insert}(x, ys) & b = 1 \land ys' = 1 + y + ys \\
\mathsf{if}(b, x, ys') & \xrightarrow{1} & 1 + ys' & b = 1 \land ys' = 1 + y + ys
\end{array}
$$

1. abstract terms to integers
2. analyse result size for bottom-SCC using standard ITS tools
3. analyse runtime of bottom-SCC using standard ITS tools

Bird's Eye View

Example

$$\begin{array}{lcllcl} \mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & \mid & xs' = 1 \\ \mathsf{isort}(xs', ys) & \xrightarrow{1} & \mathsf{isort}(xs, \mathsf{insert}(x, ys)) & \mid & xs' = 1 + x + xs \end{array}$$

1. abstract terms to integers
2. analyse result size for bottom-SCC using standard ITS tools
3. analyse runtime of bottom-SCC using standard ITS tools

Analyse Runtime Using Standard Tools

Removing Nested Function Calls

Example

$$\begin{array}{lcll} \mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & \mid \quad xs' = 1 \\ \mathsf{isort}(xs', ys) & \xrightarrow{1} & \mathsf{isort}(xs, \mathsf{insert}(x, ys)) & \mid \quad xs' = 1{+}x{+}xs \end{array}$$

- $\mathsf{sz}(\mathsf{insert}(x, ys)) \leq 1 + x + ys$
- $\mathsf{rt}(\mathsf{insert}(x, ys)) \leq 2 \cdot ys$

Removing Nested Function Calls

Example

$$\begin{array}{lcll} \mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & \mid \quad xs' = 1 \\ \mathsf{isort}(xs', ys) & \xrightarrow{1} & \mathsf{isort}(xs, \mathsf{insert}(x, ys)) & \mid \quad xs' = 1{+}x{+}xs \end{array}$$

- $\mathsf{sz}(\mathsf{insert}(x, ys)) \leq 1 + x + ys$
- $\mathsf{rt}(\mathsf{insert}(x, ys)) \leq 2 \cdot ys$
- add costs of nested function call

Removing Nested Function Calls

Example

$$\begin{array}{lcl|l} \mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & xs' = 1 \\ \mathsf{isort}(xs', ys) & \xrightarrow{1+2\cdot ys} & \mathsf{isort}(xs, \mathsf{insert}(x, ys)) & xs' = 1{+}x{+}xs \end{array}$$

- $\mathsf{sz}(\mathsf{insert}(x, ys)) \leq 1 + x + ys$
- $\mathsf{rt}(\mathsf{insert}(x, ys)) \leq 2 \cdot ys$
- add costs of nested function call

Removing Nested Function Calls

Example

$$\begin{array}{lcllc}
\mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & \mid & xs' = 1 \\
\mathsf{isort}(xs', ys) & \xrightarrow{1+2\cdot ys} & \mathsf{isort}(xs, \mathsf{insert}(x, ys)) & \mid & xs' = 1+x+xs
\end{array}$$

- $\mathsf{sz}(\mathsf{insert}(x, ys)) \leq 1 + x + ys$
- $\mathsf{rt}(\mathsf{insert}(x, ys)) \leq 2 \cdot ys$
- add costs of nested function call
- replace nested function call by fresh variable x_f

Removing Nested Function Calls

Example

$$\begin{array}{llll} \mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & \mid \quad xs' = 1 \\ \mathsf{isort}(xs', ys) & \xrightarrow{1+2\cdot ys} & \mathsf{isort}(xs, x_f) & \mid \quad xs' = 1{+}x{+}xs \end{array}$$

- $\mathsf{sz}(\mathsf{insert}(x, ys)) \leq 1 + x + ys$
- $\mathsf{rt}(\mathsf{insert}(x, ys)) \leq 2 \cdot ys$
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- replace nested function call by fresh variable x_f

Removing Nested Function Calls

Example

$$\begin{array}{lcllcl} \mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & & \mid & xs' = 1 \\ \mathsf{isort}(xs', ys) & \xrightarrow{1+2\cdot ys} & \mathsf{isort}(xs, x_f) & & \mid & xs' = 1{+}x{+}xs \end{array}$$

- $\mathsf{sz}(\mathsf{insert}(x, ys)) \leq 1 + x + ys$
- $\mathsf{rt}(\mathsf{insert}(x, ys)) \leq 2 \cdot ys$
- add costs of nested function call
- replace nested function call by fresh variable x_f
- add constraint "$x_f \leq$ size bound"

Removing Nested Function Calls

Example

$$\begin{array}{lcl|l} \mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & xs' = 1 \\ \mathsf{isort}(xs', ys) & \xrightarrow{1+2\cdot ys} & \mathsf{isort}(xs, x_f) & xs' = 1+x+xs \wedge x_f \leq 1+x+ys \end{array}$$

- $\mathsf{sz}(\mathsf{insert}(x, ys)) \leq 1 + x + ys$
- $\mathsf{rt}(\mathsf{insert}(x, ys)) \leq 2 \cdot ys$
- add costs of nested function call
- replace nested function call by fresh variable x_f
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Removing Nested Function Calls

Example

$$\begin{array}{lcl|l} \mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & xs' = 1 \\ \mathsf{isort}(xs', ys) & \xrightarrow{1+2\cdot ys} & \mathsf{isort}(xs, x_f) & xs' = 1{+}x{+}xs \land x_f \leq 1{+}x{+}ys \end{array}$$

- $\mathsf{sz}(\mathsf{insert}(x, ys)) \leq 1 + x + ys$
- $\mathsf{rt}(\mathsf{insert}(x, ys)) \leq 2 \cdot ys$
- add costs of nested function call
- replace nested function call by fresh variable x_f
- add constraint "$x_f \leq$ size bound"

↷ $\mathsf{rt}(\mathsf{isort}(xs', ys)) \leq \mathcal{O}(xs'^2 + xs' \cdot ys)$

Removing Nested Function Calls

Example

$$\begin{array}{lcll} \mathsf{isort}(xs', ys) & \xrightarrow{1} & ys & \mid \quad xs' = 1 \\ \mathsf{isort}(xs', ys) & \xrightarrow{1+2\cdot ys} & \mathsf{isort}(xs, x_f) & \mid \quad xs' = 1 + x + xs \wedge x_f \leq 1 + x + ys \end{array}$$

- $\mathsf{sz}(\mathsf{insert}(x, ys)) \leq 1 + x + ys$
- $\mathsf{rt}(\mathsf{insert}(x, ys)) \leq 2 \cdot ys$
- add costs of nested function call
- replace nested function call by fresh variable x_f
- add constraint "$x_f \leq$ size bound"

↷ $\mathsf{rt}(\mathsf{isort}(xs', ys)) \leq \mathcal{O}(xs'^2 + xs' \cdot ys)$

- similar techniques to eliminate *outer* function calls

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Experiments

ITS tools CoFloCo, KoAT, and PUBS used as backends.

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Results on the TPDB (922 examples):

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ITS tools CoFloCo, KoAT, and PUBS used as backends.

Results on the TPDB (922 examples):

- AProVE + ITS backend finds better bounds than AProVE & TcT for 127 TRSs
- transformation a useful additional inference technique for upper bounds

From irc of TRSs to Integer Transition Systems: Summary

- Abstraction from terms to integers
- Modular bottom-up approach using standard ITS tools
- Approach complements and improves state of the art
- Note: abstraction **hard-coded** to term size

⇒ Future work: more flexible approach?

Derivational_Complexity_Full_Rewriting/AG01/#3.12, TPDB

$$\begin{array}{rcl|rcl}
\mathsf{app}(\mathsf{nil}, y) & \rightarrow & y & \mathsf{app}(\mathsf{add}(n, x), y) & \rightarrow & \mathsf{add}(n, \mathsf{app}(x, y)) \\
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5. Upper bound $\mathcal{O}(n^4)$ for RITS complexity carries over to $\mathrm{dc}_{\mathcal{R}}$ of input!

AProVE finds lower bound $\Omega(n^3)$ for $\mathrm{dc}_{\mathcal{R}}$ using induction technique.

Input for Automated Tools (1/4)

Automated tools at the Termination and Complexity Competition 2021:

- AProVE: `https://aprove.informatik.rwth-aachen.de/`
- TcT: `https://tcs-informatik.uibk.ac.at/tools/tct/`

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Input format for runtime complexity:[41]

```
(VAR x y)
(GOAL COMPLEXITY)
(STARTTERM CONSTRUCTOR-BASED)
(RULES
  plus(0, y) -> y
  plus(s(x), y) -> s(plus(x, y))
)
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Input for Automated Tools (2/4)

Innermost runtime complexity:

```
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(STRATEGY INNERMOST)
(RULES
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)
```

Input for Automated Tools (3/4)

Derivational complexity:

```
(VAR x y)
(GOAL COMPLEXITY)
(STARTTERM UNRESTRICTED)
(RULES
  plus(0, y) -> y
  plus(s(x), y) -> s(plus(x, y))
)
```

Input for Automated Tools (4/4)

Innermost derivational complexity:

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What if Complexity Analysis Tools have Bugs?

Problem noted in the early Termination Competitions:

- Tools may give contradictory answers on some (few) inputs.

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Solution for termination and complexity of TRSs:

- Proof output by TRS tools in a standard (XML) format
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- Example for TRS complexity: IsaFoR with certifier CeTA[42]

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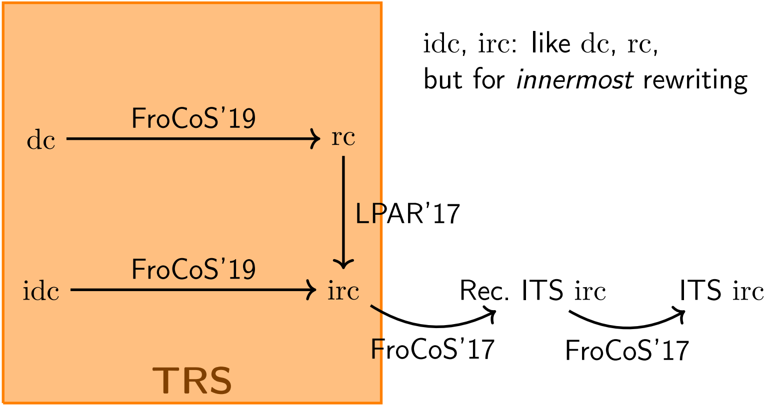
[42] R. Thiemann, C. Sternagel: *Certification of Termination Proofs Using CeTA*, TPHOLs 2009, `http://cl-informatik.uibk.ac.at/software/ceta/`

A Landscape of Complexity Properties and Transformations

idc, irc: like dc, rc,
but for *innermost* rewriting



$$\begin{tikzcd}
\mathrm{dc} \arrow[r, "\text{FroCoS'19}"] & \mathrm{rc} \arrow[d, "\text{LPAR'17}"] & & \\
\mathrm{idc} \arrow[r, "\text{FroCoS'19}"] & \mathrm{irc} \arrow[r, bend right, "\text{FroCoS'17}"'] & \text{Rec. ITS } \mathrm{irc} \arrow[r, bend right, "\text{FroCoS'17}"'] & \text{ITS } \mathrm{irc}
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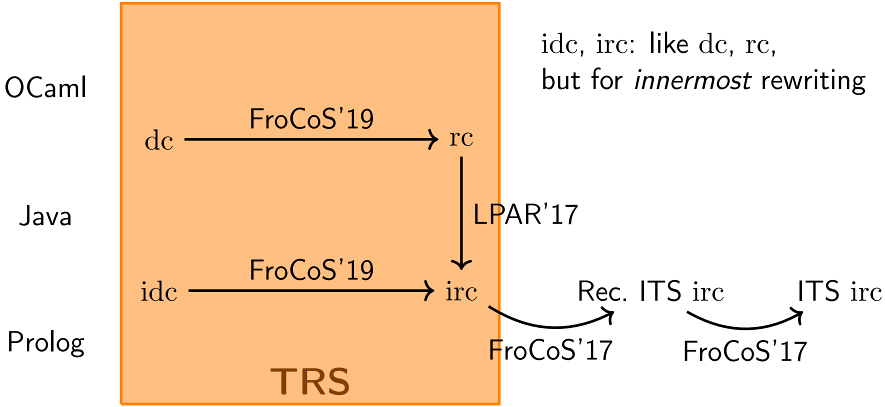

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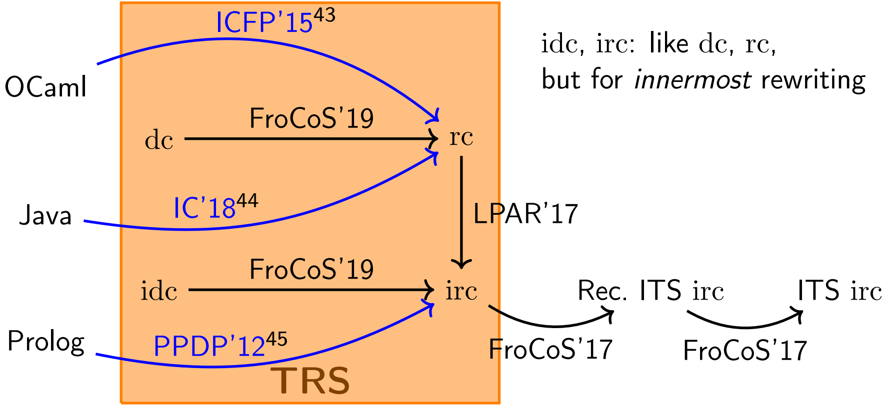

OCaml $\xrightarrow{\text{ICFP'15[43]}}$ rc

dc $\xrightarrow{\text{FroCoS'19}}$ rc

Java $\xrightarrow{\text{IC'18[44]}}$ rc

rc $\xrightarrow{\text{LPAR'17}}$ irc

idc $\xrightarrow{\text{FroCoS'19}}$ irc

Prolog $\xrightarrow{\text{PPDP'12[45]}}$ irc

irc $\xrightarrow{\text{FroCoS'17}}$ Rec. ITS irc $\xrightarrow{\text{FroCoS'17}}$ ITS irc

TRS

idc, irc: like dc, rc,
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---

[43] M. Avanzini, U. Dal Lago, G. Moser: *Analysing the Complexity of Functional Programs: Higher-Order Meets First-Order*, ICFP '15

[44] G. Moser, M. Schaper: *From Jinja bytecode to term rewriting: A complexity reflecting transformation*, IC '18

[45] J. Giesl, T. Ströder, P. Schneider-Kamp, F. Emmes, C. Fuhs: *Symbolic evaluation graphs and term rewriting: A general methodology for analyzing logic programs*, PPDP '12

Program Complexity Analysis via Term Rewriting: OCaml

Complexity analysis for functional programs (OCaml) by translation to term rewriting

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Challenge for translation to TRS: OCaml is higher-order – functions can take functions as arguments: $\mathsf{map}(F, xs)$

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Complexity analysis for functional programs (OCaml) by translation to term rewriting

Challenge for translation to TRS: OCaml is higher-order – functions can take functions as arguments: $\mathsf{map}(F, xs)$

Solution:

- Defunctionalisation to: $\mathsf{a}(\mathsf{a}(\mathsf{map}, F), xs)$
- Analyse start term with non-functional parameter types, then partially evaluate functions to instantiate higher-order variables
- Further program transformations

⇒ First-order TRS $\mathcal{R}$ with $\mathrm{rc}_{\mathcal{R}}(n)$ an upper bound for the complexity of the OCaml program

Program Complexity Analysis via Term Rewriting: Prolog and Java

Complexity analysis for Prolog programs and for Java programs by translation to term rewriting

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Program Complexity Analysis via Term Rewriting: Prolog and Java

Complexity analysis for Prolog programs and for Java programs by translation to term rewriting

Common ideas:

- Analyse program via symbolic execution and generalisation (a form of abstract interpretation[46])
- Deal with language specifics in program analysis
- Extract TRS $\mathcal{R}$ such that $\mathrm{rc}_{\mathcal{R}}(n)$ is provably at least as high as runtime of program on input of size n
- Can represent tree structures of program as terms in TRS!

---

[46] P. Cousot, R. Cousot: *Abstract interpretation: a unified lattice model for static analysis of programs by construction or approximation of fixpoints*, POPL '77

Current Developments

- **amortised** complexity analysis for term rewriting[47]

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[47] G. Moser, M. Schneckenreither: *Automated amortised resource analysis for term rewrite systems*, SCP '20

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- analysis of **parallel**-innermost runtime complexity[51]

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[50] C. Kop, D. Vale: *Tuple interpretations for higher-order rewriting*, FSCD '21

[51] T. Baudon, C. Fuhs, L. Gonnord: *Parallel complexity of term rewriting systems*, WST '21

Conclusion

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Thanks a lot for your attention!

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