Proving Program Termination via Term Rewriting

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Advanced Course at the International School on Rewriting 2017

Eindhoven, The Netherlands

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Automated Termination Provers for Term Rewrite Systems

- AProVE (Aachen, ...)
- CiME3 (Paris)
- HOT (Cachan)
- Jambox (Amsterdam)
- Matchbox (Leipzig)
- Mu-Term (Valencia)
- MultumNonMulta (Kassel)
- NaTT (Innsbruck)
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- VMTL (Vienna)
- Wanda (Copenhagen)
- ...

- Powerful push-button termination analysis tools for Term Rewrite Systems (TRSs)
- Development spurred by annual International Termination Competition (termCOMP) since 2003
- termCOMP initially for term rewriting, now also C, Java, Haskell, Prolog
- **Can we use tools for TRSs also for programming languages?**

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Termination Analysis for Programs – Why?

- push-button analysis
- does not need separate specification (hard to get right)
- termination is in most cases a desirable property
- non-termination can be security issue (Denial of Service)
- in 2011: PHP and Java issues with floating-point number parser
  - `http://www.exploringbinary.com/php-hangs-on-numeric-value-2-2250738585072011e-308/`
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  → Microsoft's Terminator project

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How can Term Rewriting Contribute?

- term rewriting is Turing-complete
- can represent inductive data structures (trees) in a natural way

Idea 1: port techniques from TRSs to each programming language
$\rightarrow$ but: lots of repeated work

Idea 2: two-stage approach

- **front-end** for language-specific aspects, **extracts TRS** such that termination of TRS implies termination of the program
- **back-end**: reuse optimized off-the-shelf termination prover for TRSs

This course: How can we construct such a front-end?

- look at general principle
- look at two concrete programming languages as examples
  - Haskell (functional, lazy)
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$$\mathcal{R} = \begin{cases} \mathsf{minus}(x, \mathsf{0}) & \to & x \\ \mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{minus}(x, y) \\ \mathsf{quot}(\mathsf{0}, \mathsf{s}(y)) & \to & \mathsf{0} \\ \mathsf{quot}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{s}(\mathsf{quot}(\mathsf{minus}(x, y), \mathsf{s}(y))) \end{cases}$$

Term rewriting: Evaluate terms by applying rules from $\mathcal{R}$

$$\mathsf{minus}(\mathsf{s}(\mathsf{s}(\mathsf{0})), \mathsf{s}(\mathsf{0})) \to_{\mathcal{R}} \mathsf{minus}(\mathsf{s}(\mathsf{0}), \mathsf{0}) \to_{\mathcal{R}} \mathsf{s}(\mathsf{0})$$

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Termination: No infinite evaluation sequences $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} \dots$

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Show termination using Dependency Pairs

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  - delete $s \to t$ with $s \succ t$ from $\mathcal{P}$ ($\succ$ well founded, *not monotonic*)
- Find $(\succsim, \succ)$ automatically via SAT and SMT solving

Example (Constraints for Division)

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Use polynomial interpretation $[\,\cdot\,]$ over $\mathbb{N}$ [Lankford '75] with

$$
\begin{array}{rclcrcl}
[\mathsf{quot}^\sharp](x_1, x_2) & = & x_1 + x_2 & \qquad & [\mathsf{quot}](x_1, x_2) & = & x_1 + x_2 \\
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[0] & = & 0 & & [\mathsf{s}](x_1) & = & x_1 + 1
\end{array}
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$\curvearrowright$ $(\succsim, \succ)$ induced by $[\,\cdot\,]$ solves all term constraints

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$\curvearrowright$ termination of division algorithm proved $\square$

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- execute program **symbolically** from initial states of the program, handle language peculiarities here

```
f : if ...
        ...
    else
        ...
        g : while ...
                ...
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 & \downarrow & \\
 & \mathsf{f}(\dots) & \\
\swarrow & & \searrow \\
\dots & & \mathsf{g}(\vec{s})
\end{array}
$$

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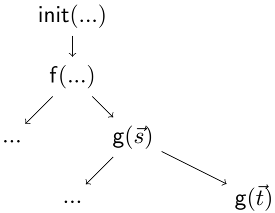

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- use **generalization** of program states, get over-approximation of all possible program runs (≈ control-flow graph with extra info)
- related: Abstract Interpretation [Cousot and Cousot, *POPL '77*]

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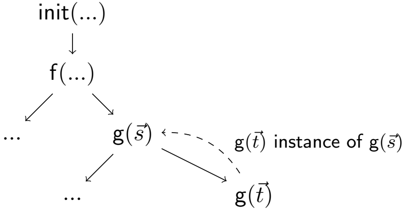

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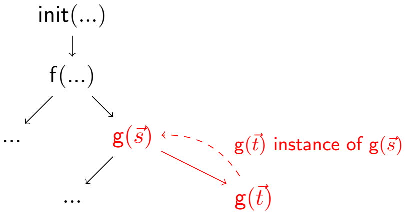

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- related: Abstract Interpretation [Cousot and Cousot, *POPL '77*]
- **extract TRS** from cycles in the representation
- if TRS terminates

  $\Rightarrow$ any **concrete program execution** can use cycles only finitely often

  $\Rightarrow$ the program **must terminate**

```
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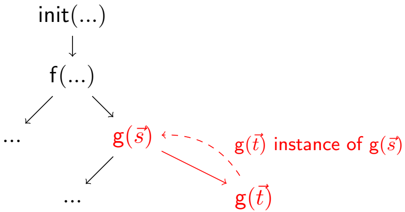
                 ...
```



Here: Haskell

Haskell 98

- Widely used functional programming language
- Goal: analyze termination, reuse techniques for term rewrite systems

Approach
[Giesl, Raffelsieper, Schneider-Kamp, Swiderski, Thiemann, *TOPLAS '11*]

- Translate from Haskell 98 to TRS
- Prove termination of the TRS using standard techniques for TRSs

$\Rightarrow$ Implies termination of the Haskell program!

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From Haskell to TRSs

Challenges

- higher-order: functional variables, λ-abstractions, ...
  But: Standard framework for TRSs works on first-order terms
- lazy evaluation
  But: Standard TRS techniques consider all evaluation strategies
- polymorphic types
  But: TRSs are untyped
- usually not all Haskell functions terminate $\rightarrow \infty$ data (streams)
  But: TRS techniques analyze termination of all terms

Data Structures

- data Nat = Z | S Nat
  - type constructor: Nat
  - data constructors: Z :: Nat, S :: Nat $\to$ Nat
- data List a = Nil | Cons a (List a)
  - type constructor: List of arity 1
  - data constructors: Nil :: List a, Cons :: $a \to$ (List a) $\to$ (List a)

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Terms (well-typed)

- Variables: x, y, $\ldots$
- Function Symbols: constructors (Z, S, Nil, Cons) & defined (from, take)
- Applications $(t_1\ t_2)$
  - S Z represents number 1
  - Cons x Nil $\equiv$ (Cons x) Nil represents $[x]$

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Types

- Type Variables: a, b, ...
- Applications of type constructors to types: List Nat, $a \to$ (List a), ...

S Z	has type	Nat
Cons x Nil	has type	List a

Syntax of Haskell

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Function Declarations (example)

$$\mathsf{from}\,x = \mathsf{Cons}\,x\,(\mathsf{from}\,(\mathsf{S}\,x))$$

$$\mathsf{from} :: \mathsf{Nat} \to \mathsf{List\,Nat}$$

$$\mathsf{from}\ x \ \equiv\ [x, x+1, x+2, \ldots]$$

$$\begin{aligned} \mathsf{take}\,\mathsf{Z}\,xs &= \mathsf{Nil} \\ \mathsf{take}\,n\,\mathsf{Nil} &= \mathsf{Nil} \\ \mathsf{take}\,(\mathsf{S}\,n)\,(\mathsf{Cons}\,x\,xs) &= \mathsf{Cons}\,x\,(\mathsf{take}\,n\,xs) \end{aligned}$$

$$\mathsf{take} :: \mathsf{Nat} \to (\mathsf{List}\,a) \to (\mathsf{List}\,a)$$

$$\mathsf{take}\ n\ [x_1, \ldots, x_n, \ldots] \ \equiv\ [x_1, \ldots, x_n]$$

Syntax of Haskell

Function Declarations (general)

$$f\ \ell_1 \ldots \ell_n = r$$

- f is **defined** function symbol
- n is **arity** of f
- r is arbitrary term
- $\ell_1 \ldots \ell_n$ are linear **patterns** (terms from constructors and variables)

Function Declarations (example)

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Syntax of Haskell

Approach also works with

- built-in data structures
- type classes

All other Haskell constructs are eliminated by automatic transformations!

- lambda abstractions
- Conditions
- Local Declarations
- ...

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replace	$\backslash\, u\ m \to \mathsf{take}\, u\, (\mathsf{from}\, m)$
by	f
where	$\mathsf{f}\ u\ m = \ \mathsf{take}\, u\, (\mathsf{from}\, m)$

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  replace $\backslash\, t_1 \ldots t_n \to t$ with free variables $x_1, \ldots, x_m$
  by $\mathsf{f}\, x_1 \ldots x_m$
  where $\mathsf{f}\, x_1 \ldots x_m\, t_1 \ldots t_n = t$

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Semantics of Haskell

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\end{aligned}$$

- **Evaluation Relation** $\rightarrow_{\mathsf{H}}$

$$\mathsf{from}\ \mathsf{Z}$$

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from Z

evaluation position

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$$\begin{aligned} & \mathsf{from}\, m \\ \to_{\mathsf{H}}\ & \mathsf{Cons}\, m\, (\mathsf{from}\, (\mathsf{S}\, m)) \\ \to_{\mathsf{H}}\ & \mathsf{Cons}\, m\, (\mathsf{Cons}\, (\mathsf{S}\, m)\, (\mathsf{from}\, (\mathsf{S}\, (\mathsf{S}\, m)))) \\ \to_{\mathsf{H}}\ & \ldots \end{aligned}$$

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$$\mathsf{take}\,(\mathsf{S}\,\mathsf{Z})\,(\mathsf{from}\, m)$$

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$\mathsf{take}\ (\mathsf{S}\ \mathsf{Z})\ (\mathsf{from}\ m)$ *evaluation position*

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H-Termination

Analyze H-**termination**:

If I only plug terminating arguments into my initial term, will the Haskell interpreter always give me an answer?

Formally:

- **H-Termination** of ground term t if
  - t does not start infinite evaluation $t \to_{\mathsf{H}} \ldots$
  - if $t \to_{\mathsf{H}}^{*} (f\ t_1 \ldots t_n)$, f defined, $n < \mathrm{arity}(f)$, then $(f\ t_1 \ldots t_n\ t')$ is also H-terminating if t' is H-terminating
  - if $t \to_{\mathsf{H}}^{*} (c\ t_1 \ldots t_n)$, c constructor, then $t_1, \ldots, t_n$ are also H-terminating.
- **H-Termination** of arbitrary term t if $t\sigma$ H-terminates for all substitutions σ with H-terminating terms.

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- x

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- x is H-terminating

H-Termination

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- x is H-terminating
  from

H-Termination

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- **H-Termination** of arbitrary term t if
  $t\sigma$ H-terminates for all substitutions σ with H-terminating terms.
- x is H-terminating
  from is not H-terminating (from Z has infinite evaluation)

H-Termination

Analyze H-**termination**:

If I only plug terminating arguments into my initial term,
will the Haskell interpreter always give me an answer?

Formally:

- **H-Termination** of ground term t if
  - t does not start infinite evaluation $t \to_{\mathsf{H}} \ldots$
  - if $t \to_{\mathsf{H}}^{*} (f\ t_1 \ldots t_n)$, f defined, $n < \mathrm{arity}(f)$,
    then $(f\ t_1 \ldots t_n\ t')$ is also H-terminating if t' is H-terminating
  - if $t \to_{\mathsf{H}}^{*} (c\ t_1 \ldots t_n)$, c constructor,
    then $t_1, \ldots, t_n$ are also H-terminating.
- **H-Termination** of arbitrary term t if
  $t\sigma$ H-terminates for all substitutions σ with H-terminating terms.
- x is H-terminating
  from is not H-terminating (from Z has infinite evaluation)
  take u (from m)

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  take u (from m) is H-terminating
  Cons u (from m) is not H-terminating

From Haskell to Termination Graphs

$\mathsf{from}\,x = \mathsf{Cons}\,x\,(\mathsf{from}\,(\mathsf{S}\,x))$

$\mathsf{take}\,\mathsf{Z}\,xs = \mathsf{Nil}$
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Goal: Prove (H-)termination of initial term $\mathsf{take}\,u\,(\mathsf{from}\,m)$

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Naive approach

- Use defining equations directly
- fails, since from is not terminating
- disregards Haskell's evaluation strategy

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Our approach [Giesl et al, *TOPLAS '11*]

- evaluate **initial term** a few steps
  $\Rightarrow$ termination graph ($\approx$ abstract interpretation)
- our "abstract domain" for Haskell program states: *a single term*
- do not transform Haskell into TRS directly,
  but transform termination graph into TRS

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$\boxed{\mathsf{take}\, u\, (\mathsf{from}\, m)}$

- begin with node marked with initial term
- 4 expansion rules to add children to leaves (more in paper)
- expansion rules try to *evaluate* terms

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$\boxed{\mathsf{take}\, u\, (\mathsf{from}\, m)}$

- ***Case* rule**:
  - evaluation has to continue with variable u
  - instantiate u by all possible constructor terms of correct type

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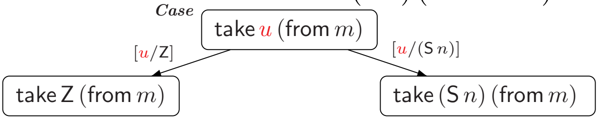

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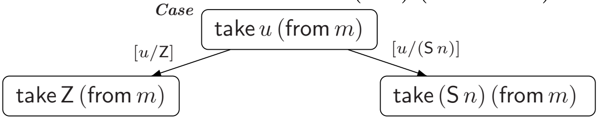

- **Main Property of Termination Graphs**:

  A node is H-terminating if all its children are H-terminating.

From Haskell to Termination Graphs

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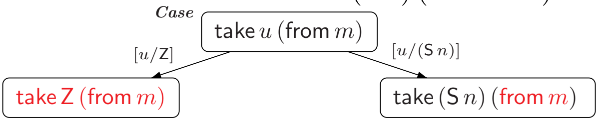

- ***Eval*** rule:

  performs one evaluation step with $\to_{\mathsf{H}}$

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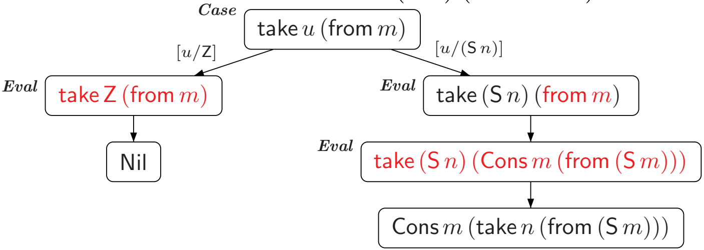

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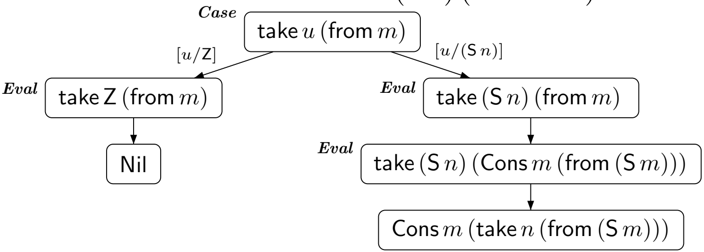

Case and ***Eval*** rule perform *narrowing*
w.r.t. Haskell's evaluation strategy and types

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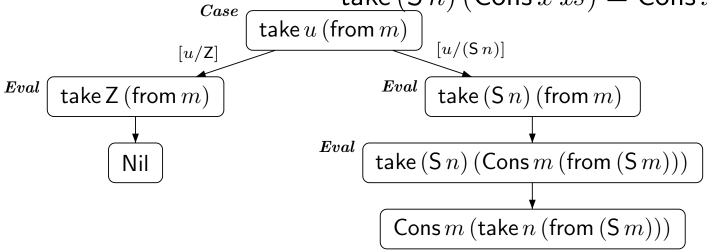

- ***ParSplit*** rule:

  if head of term is a constructor like Cons or a variable,
  then continue with the parameters

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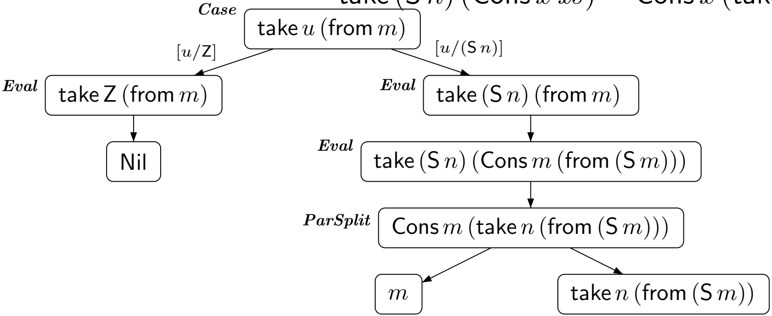

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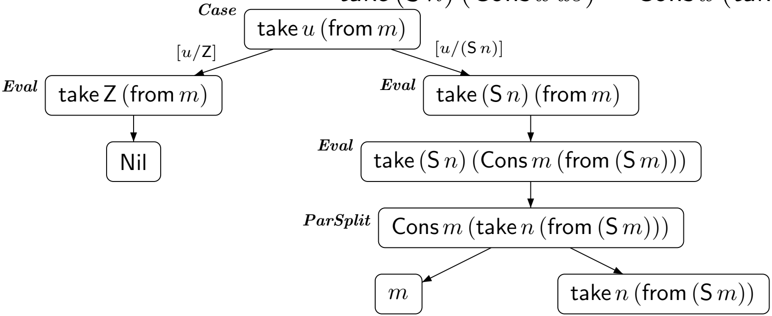

- one could continue with ***Case***, ***Eval***, ***ParSplit***
  $\Rightarrow$ infinite tree
- Instead: ***Ins*** rule to obtain finite graphs

From Haskell to Termination Graphs

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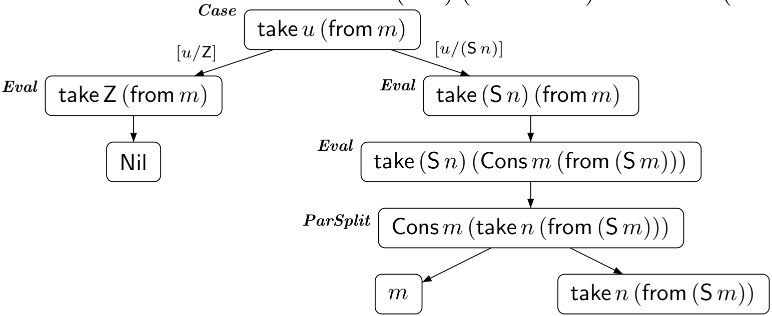

- ***Ins* rule:**
  - if leaf t is instance of t', then add instantiation edge from t to t'
  - one may re-use an existing node for t', if possible

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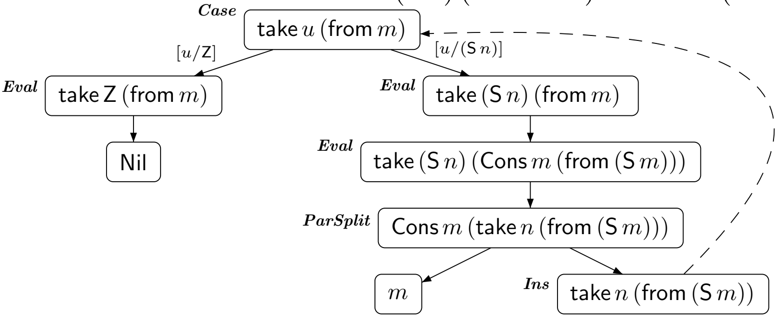

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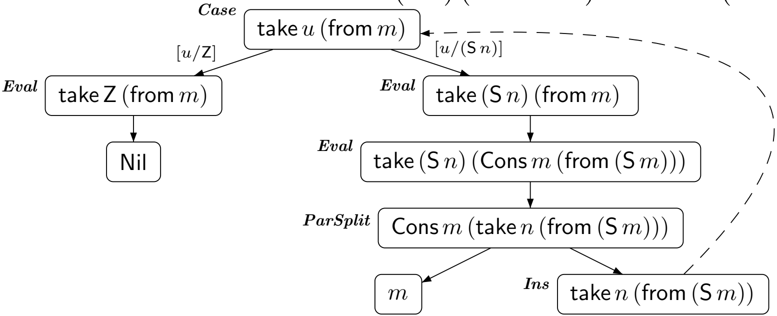

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  - since instantiation is $[u/n,\ m/(\text{S}\ m)]$, add child nodes n and $(\text{S}\ m)$

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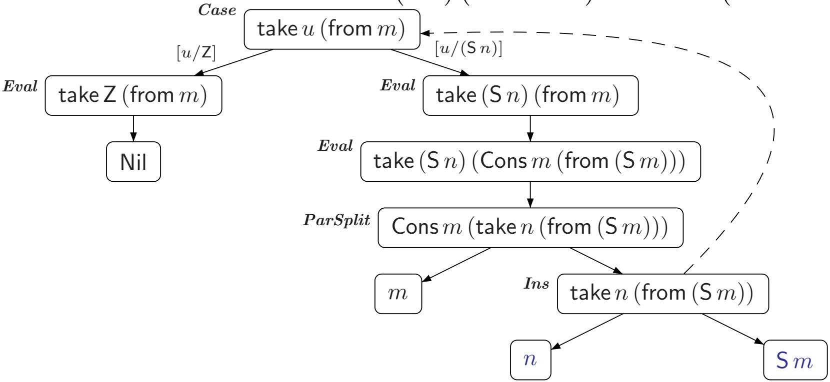

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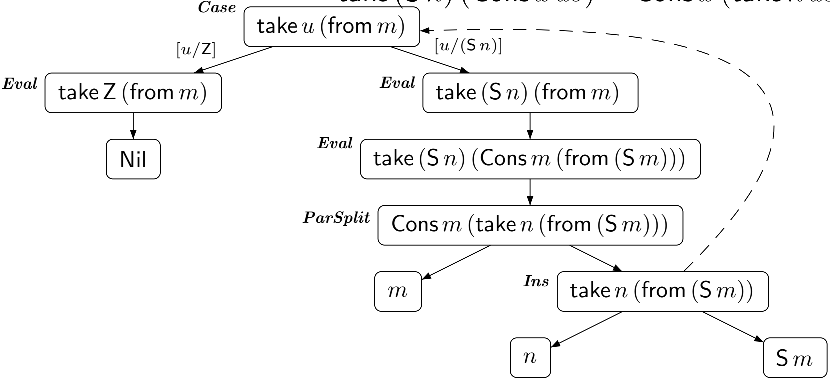

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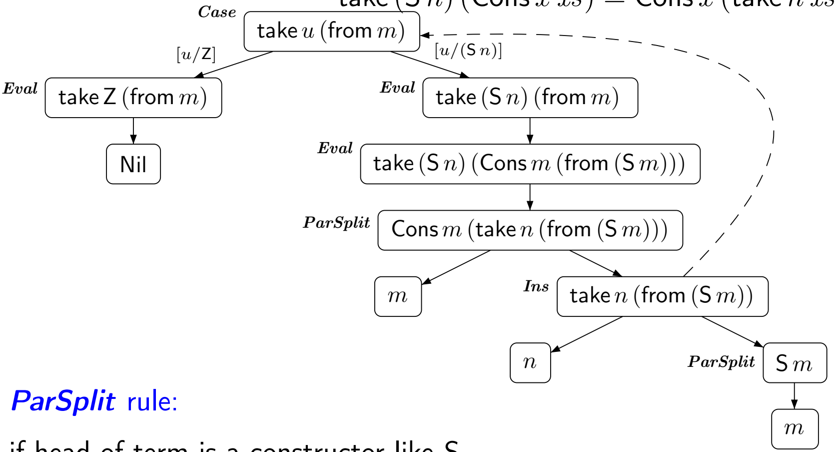

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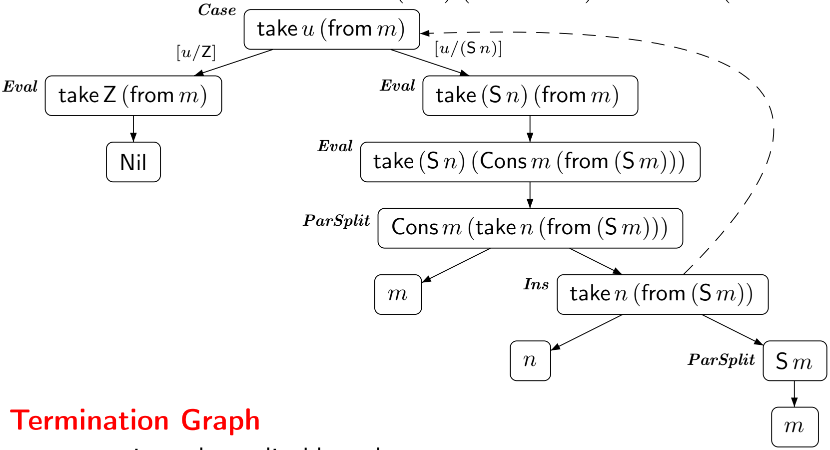

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From Haskell to Termination Graphs

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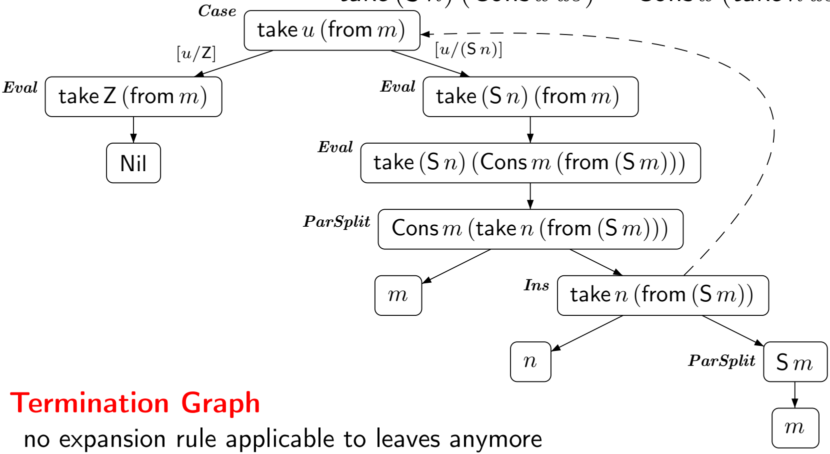

- **Termination Graph**
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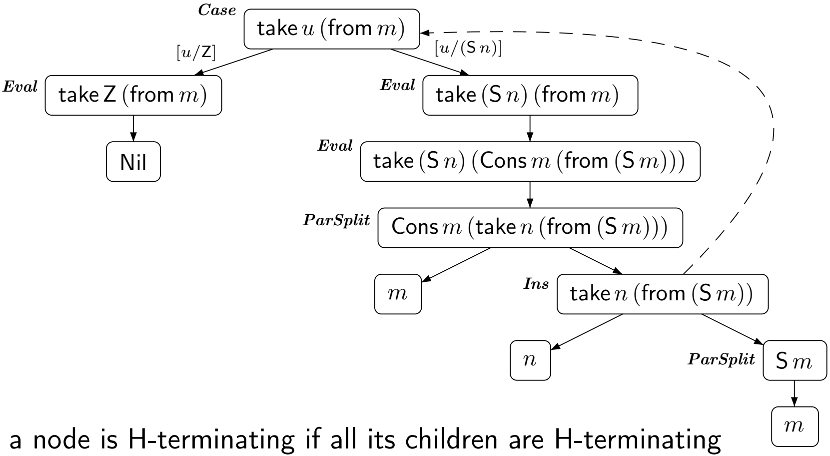
- **Goal:** Prove H-termination of all terms in termination graph

From Termination Graphs to DP Problems

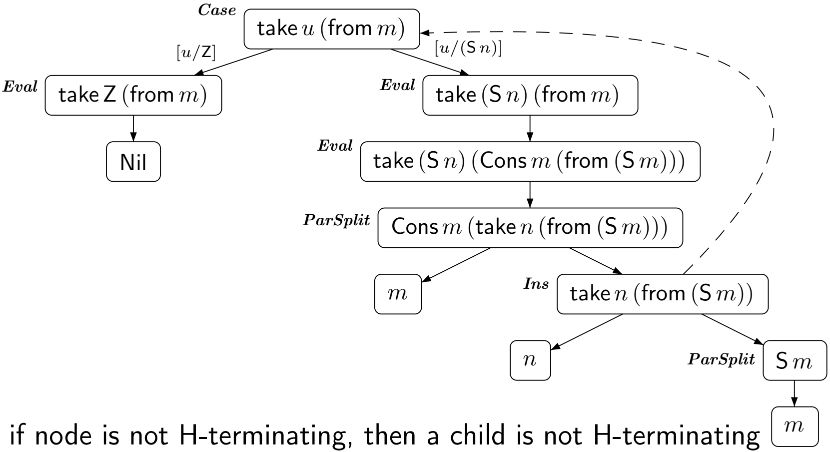
- Prove H-termination of all terms in termination graph



- a node is H-terminating if all its children are H-terminating

From Termination Graphs to DP Problems

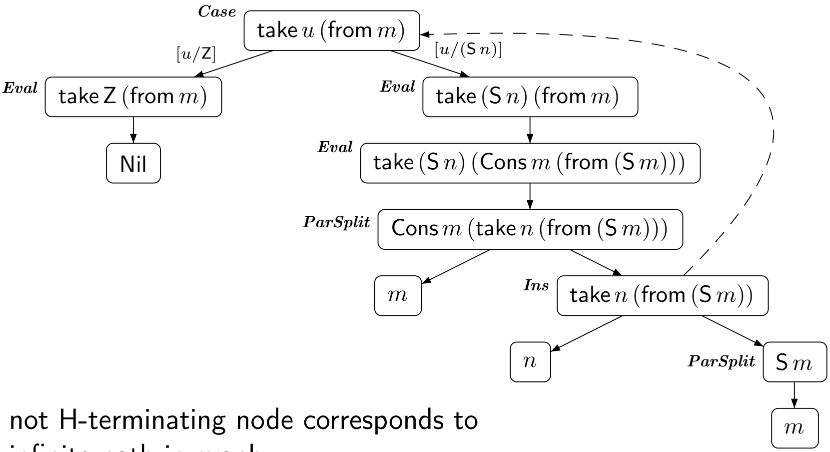
- Prove H-termination of all terms in termination graph



- if node is not H-terminating, then a child is not H-terminating

From Termination Graphs to DP Problems

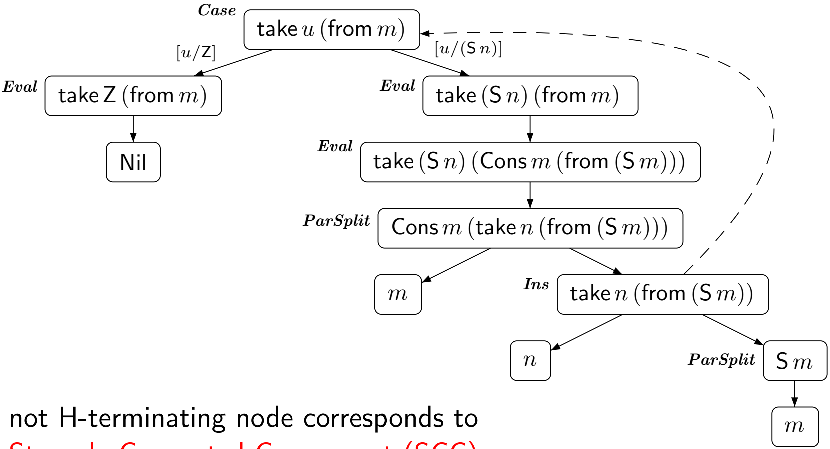
- Prove H-termination of all terms in termination graph



- not H-terminating node corresponds to infinite path in graph

From Termination Graphs to DP Problems

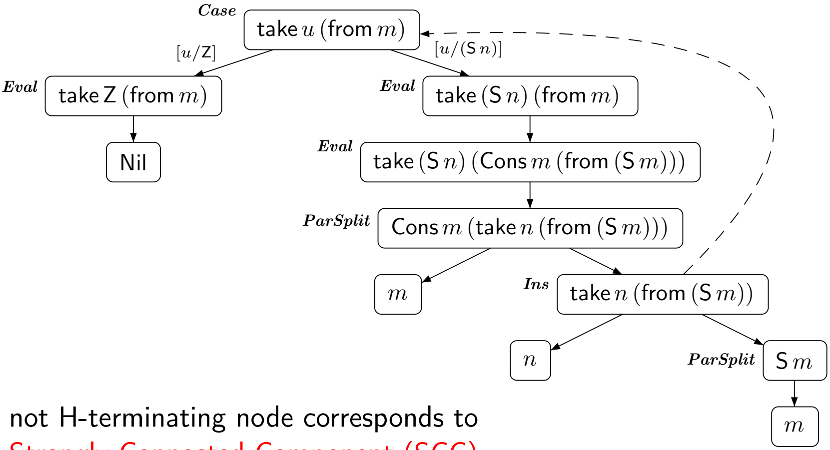
- Prove H-termination of all terms in termination graph



- not H-terminating node corresponds to Strongly Connected Component (SCC)

From Termination Graphs to DP Problems

- Prove H-termination of all terms for each SCC



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From Termination Graphs to DP Problems

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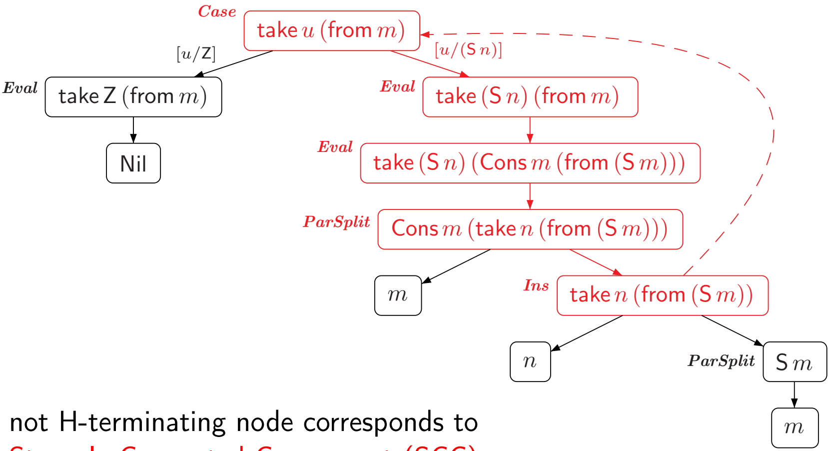

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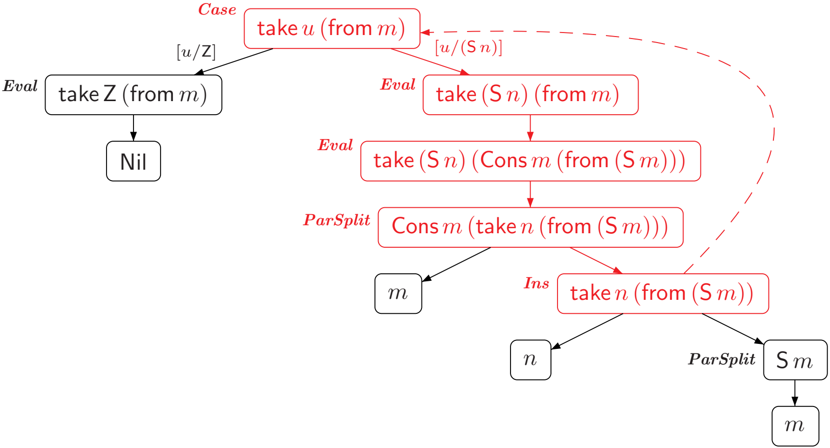
From Termination Graphs to DP Problems

- Every infinite path traverses an instantiation edge infinitely often



From Termination Graphs to DP Problems

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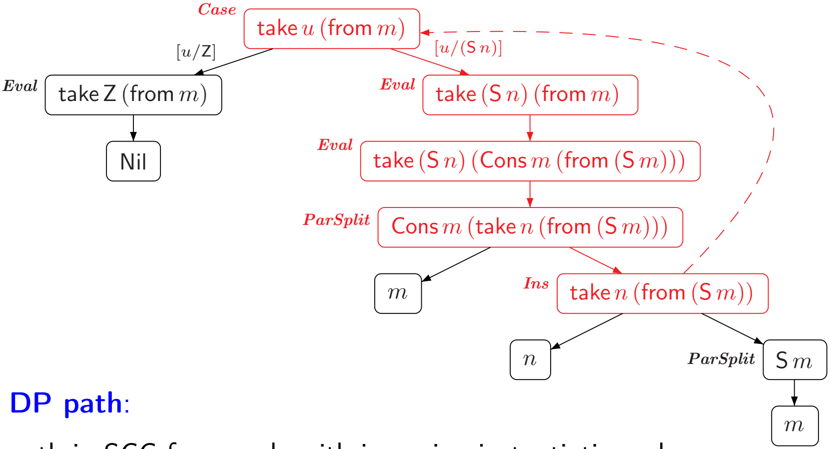
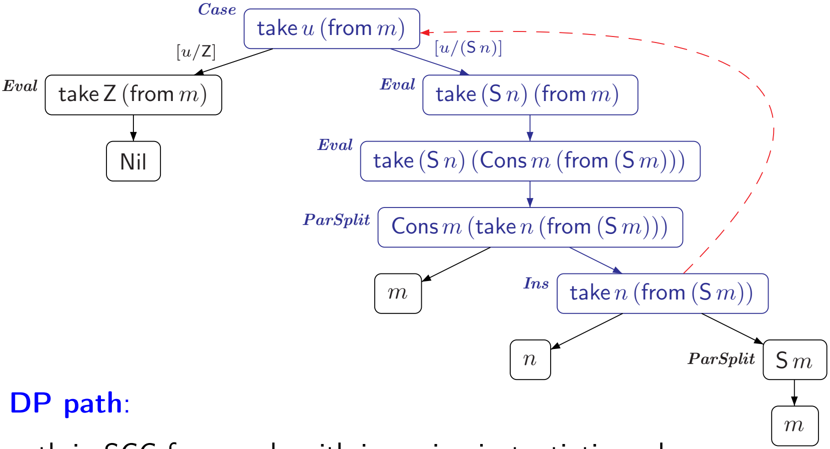


- **DP path**:

  path in SCC from node with incoming instantiation edge
  to node with outgoing instantiation edge

From Termination Graphs to DP Problems

- Every infinite path traverses a DP path infinitely often



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From Termination Graphs to DP Problems

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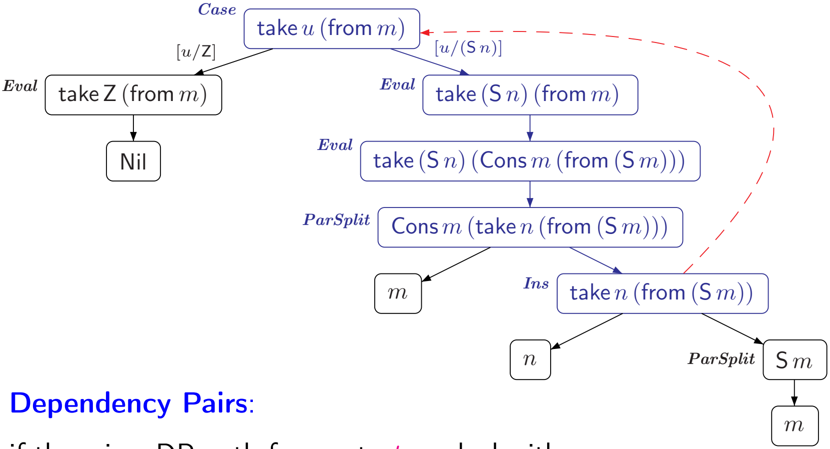
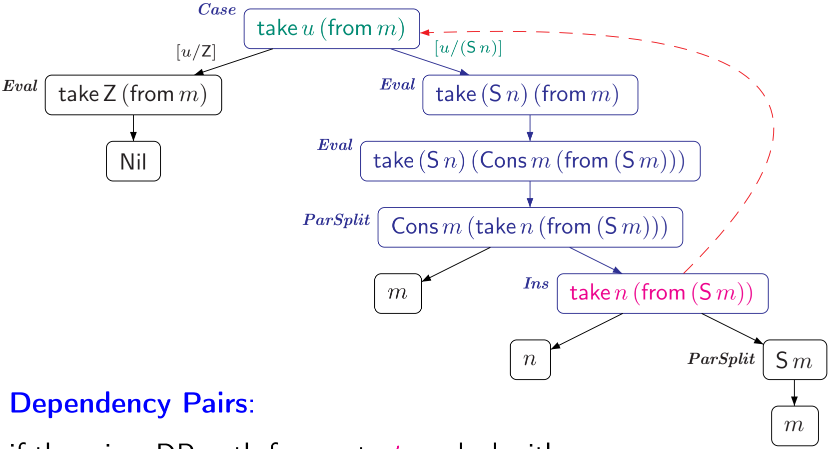


- **Dependency Pairs**:

  if there is a DP path from s to t marked with μ,
  then generate the dependency pair $s\,\mu \to t$

From Termination Graphs to DP Problems

- Every infinite path traverses a DP path infinitely often
  $\Rightarrow$ generate a dependency pair for every DP path



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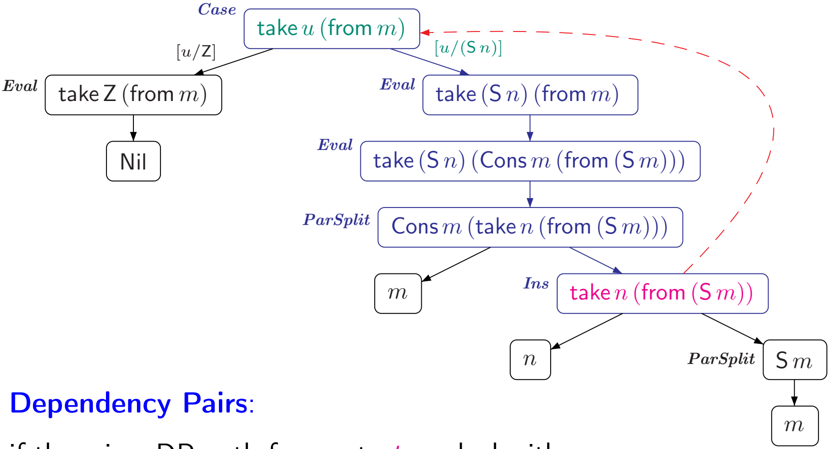
From Termination Graphs to DP Problems

- Dependency Pair $\mathcal{P}$: $\mathsf{take}(\mathsf{S}(n), \mathsf{from}(m)) \to \mathsf{take}(n, \mathsf{from}(\mathsf{S}(m)))$
  Rules $\mathcal{R}$: $\emptyset$



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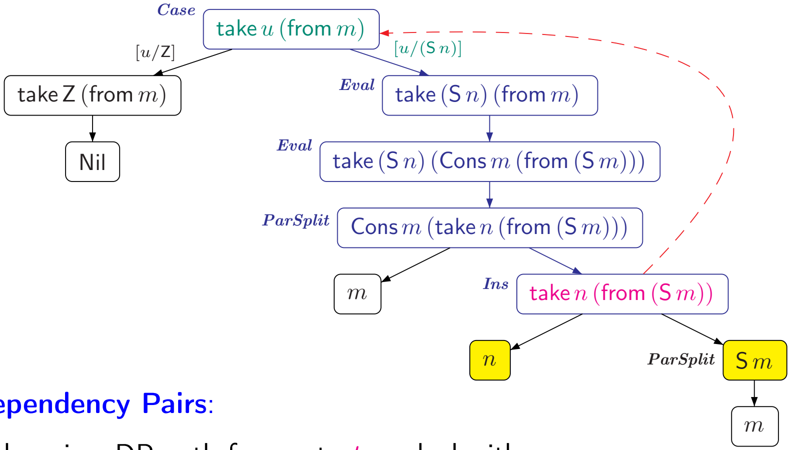

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  if there is a DP path from s to t marked with μ,
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From Termination Graphs to DP Problems

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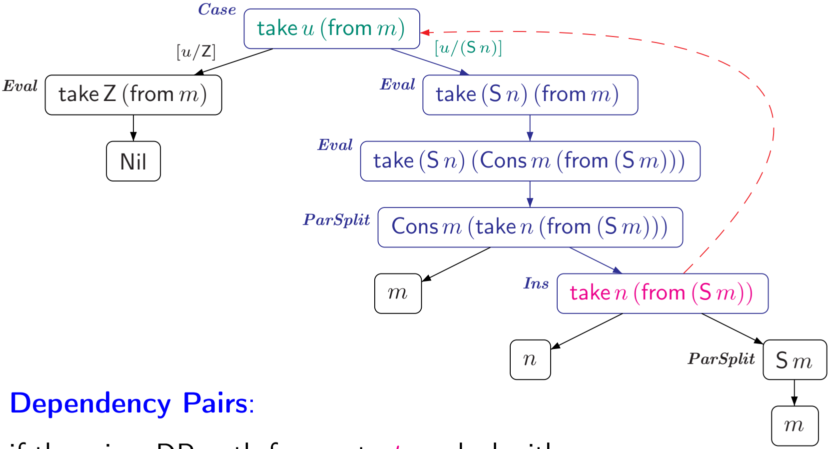

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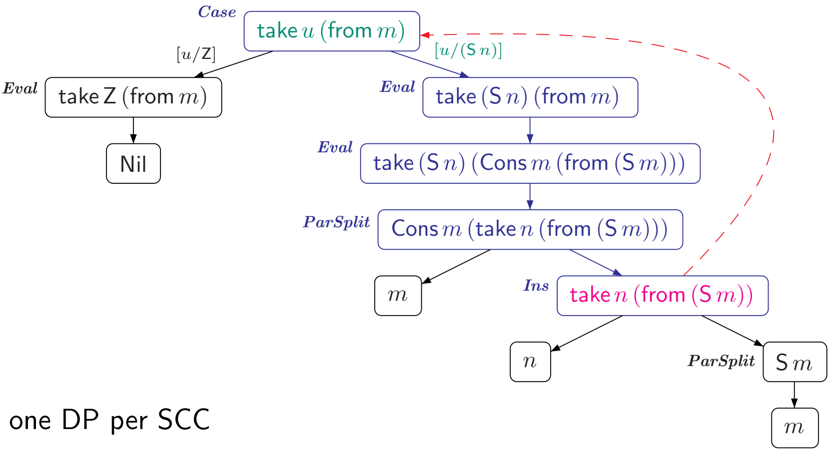

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- no rules in $\mathcal{R}$ if no defined symbols to evaluate in rhs of DP

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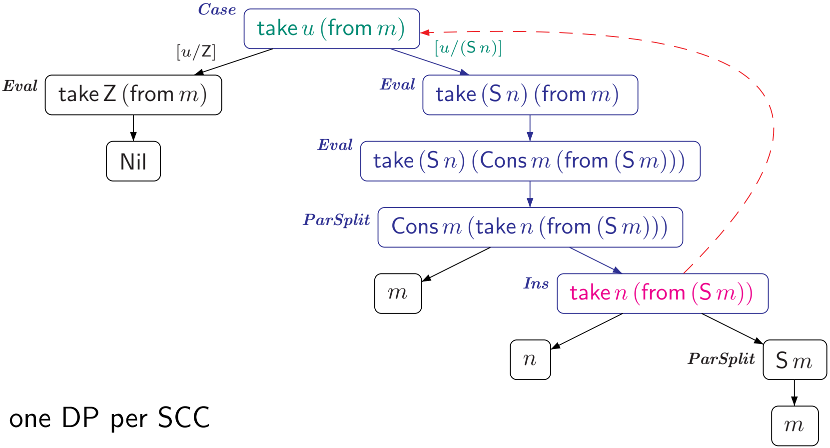

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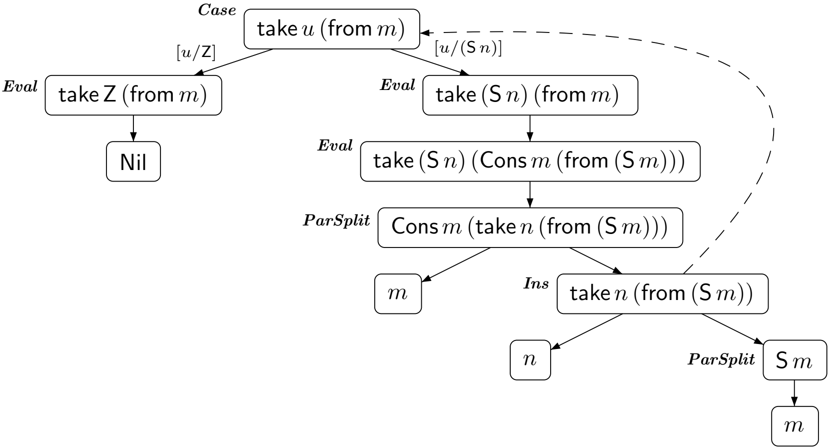

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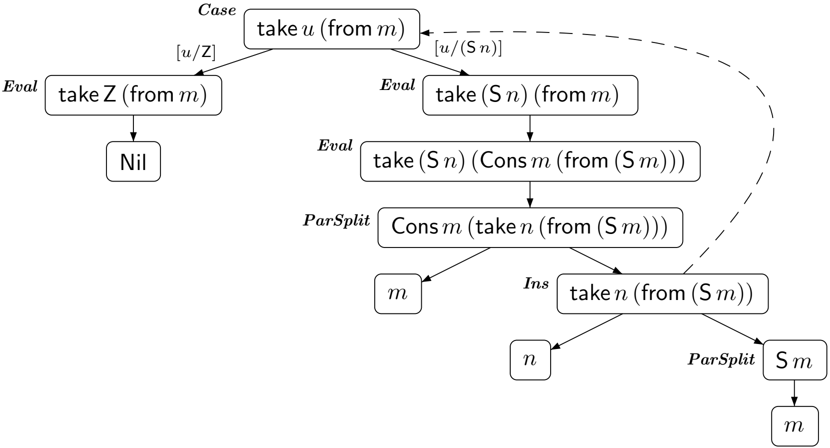

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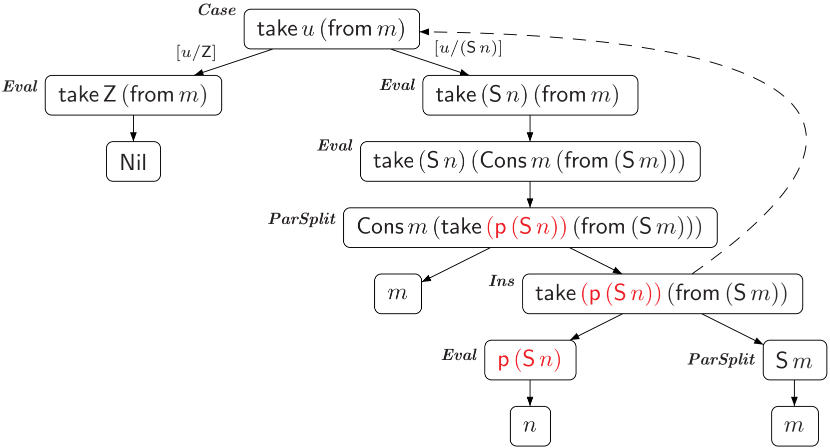

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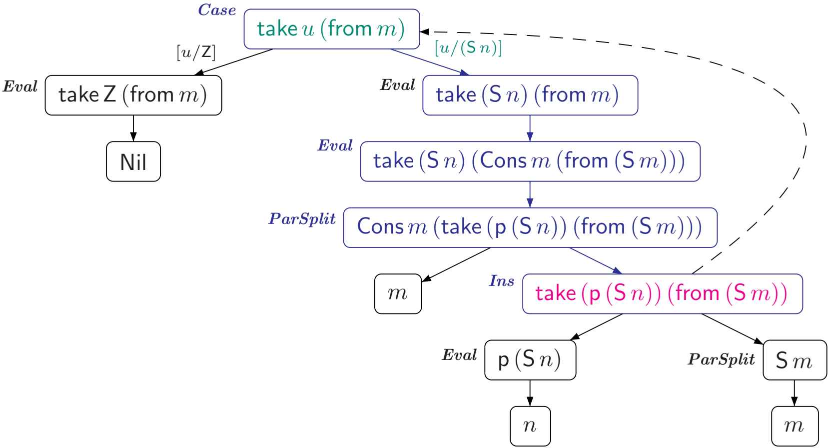

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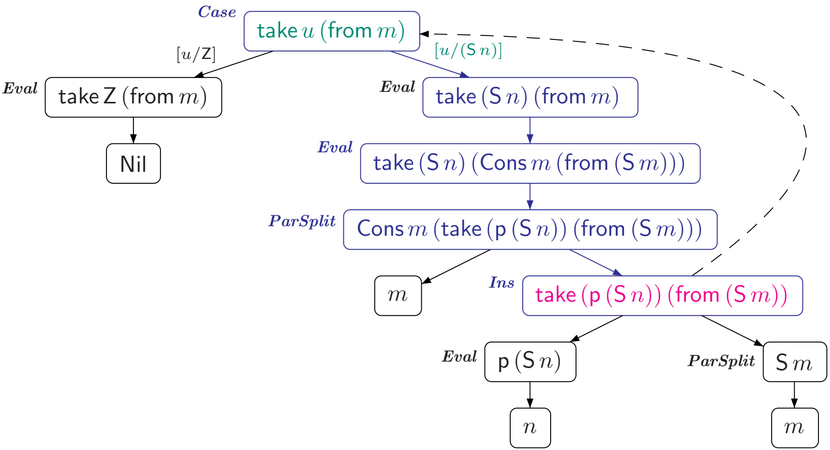

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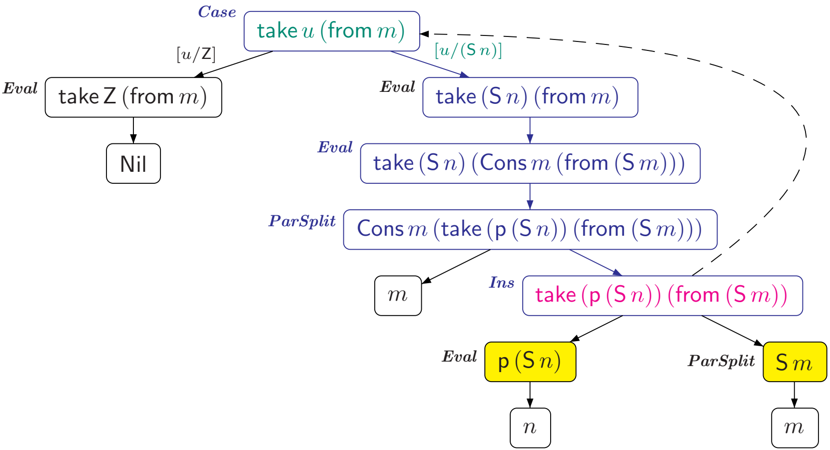

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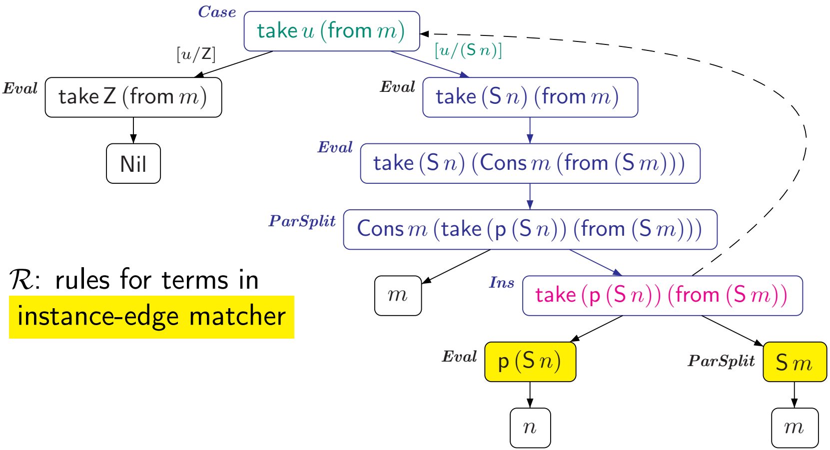

- $\mathcal{R}$: rules for terms in instance-edge matcher

From Termination Graphs to DP Problems

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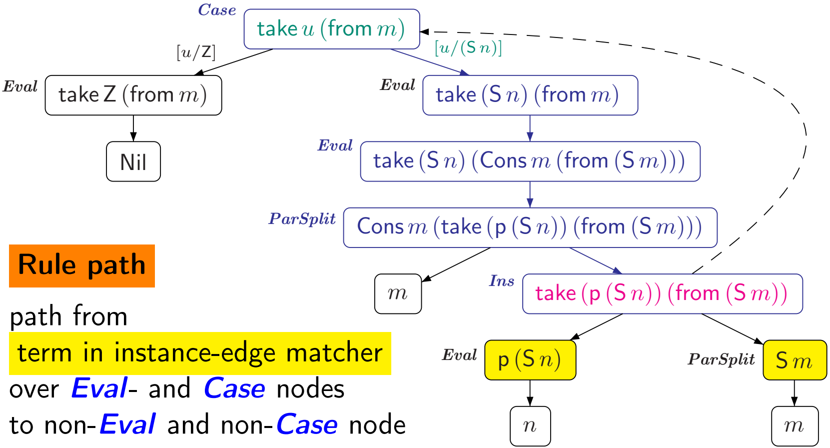

- **Rule path**

  path from
  term in instance-edge matcher
  over ***Eval***- and ***Case*** nodes
  to non-***Eval*** and non-***Case*** node

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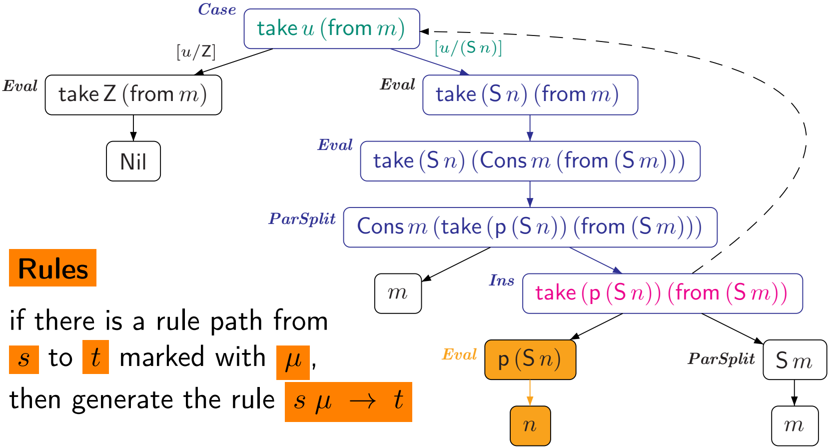

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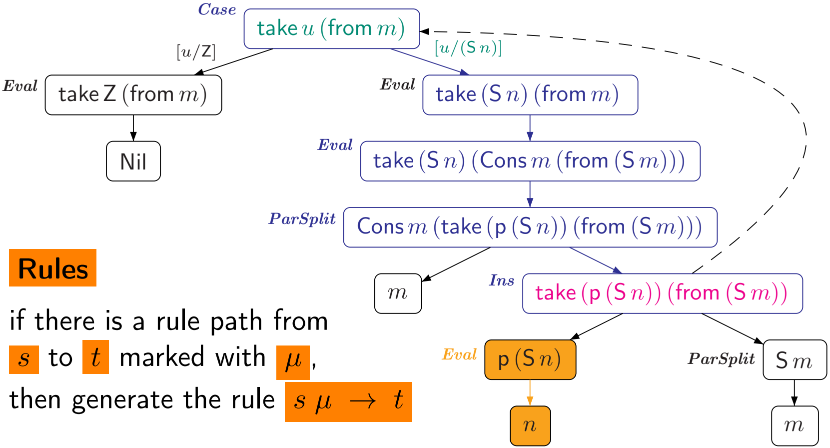

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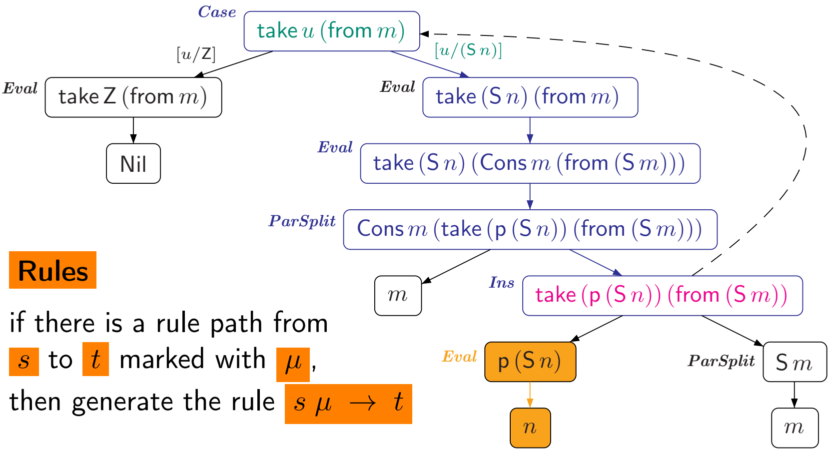

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Experiments

Implementation in termination prover AProVE
(uses improved step termination graph $\to$ DP problem)

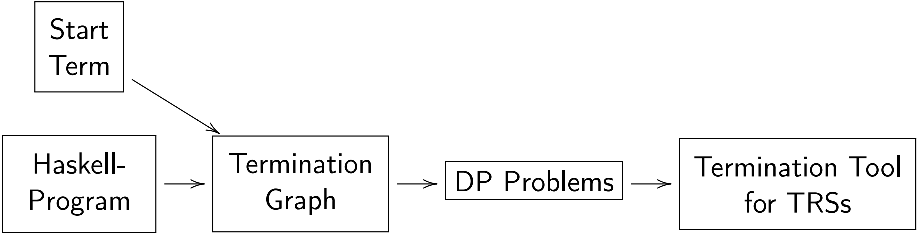
http://aprove.informatik.rwth-aachen.de/

Experiments on Haskell libraries

- `FiniteMap`, `List`, `Monad`, `Prelude`, `Queue`
- 300 seconds timeout
- AProVE shows H-Termination for 999 out of 1272 functions

The Tool Chain for Haskell

- Termination analysis of Haskell 98 via transformation to TRSs



- Language specifics are handled in transformation front-end

$\Rightarrow$ Apply TRS analysis back-end for **several** programming languages!

- Successful evaluation on Haskell 98 standard libraries

Details: [Giesl et al., *TOPLAS '11*]

http://aprove.informatik.rwth-aachen.de/eval/Haskell

Conclusion of Part I

Analyze program termination in 2 steps:

- Program → term rewrite system
- Term rewrite system → termination proof

Termination analysis for languages other than Haskell:

- Logic programming: Prolog
  [van Raamsdonk, *ICLP '97*; Schneider-Kamp et al, *TOCL '09*; Giesl et al, *PPDP '12*]
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Haskell Exercises I

Consider the following Haskell program.

```
data Nat = Z | S Nat
data List a = Nil | Cons a (List a)

mylength Nil = Z
mylength (Cons x xs) = S (mylength xs)

mysum Nil = Z
mysum (Cons x xs) = plus x (mysum xs)

plus Z y = y
plus (S x) y = S (plus x y)
```

Question 1

Consider the start term `mylength x`.

(a) Is this start term H-terminating?

(b) Construct a termination graph for this start term.

(c) Extract a DP problem from the termination graph from part (b).

(d) Prove that this DP problem is "terminating", i.e., that no infinite call sequences are possible.

(e) Check your solutions with the web interface (or a local installation) of the termination prover AProVE:
http://aprove.informatik.rwth-aachen.de/
(note that AProVE preprocesses the termination graph before the step to DP problems so that the output will look slightly differently).

Haskell Exercises III

Question 2 (slightly harder/more interesting)

Consider the start term `mysum x`. Proceed with it as in Question 1.

Hint: One can draw instantiation edges also to nodes that are not yet present in the termination graph.

Question 3

What strengths and limitations do you expect this approach to termination proving of Haskell programs to have?

Recap: from Haskell to Term Rewriting for Termination

Recipe for proving program termination by reusing TRS termination provers

- Decide on suitable symbolic representation of abstract program states (**abstract domain**)
  $\rightarrow$ what data objects can we represent as terms?
- Execute program **symbolically** from its initial states
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- Extract **rewrite rules** that "over-approximate" program runs in strongly-connected components of graph
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  $\Rightarrow$ implies termination of program from initial states

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Beyond Classic TRSs for Programs

Rewrite rules for Haskell programs in **standard** term rewriting
$\rightarrow$ no predefined rules for addition, multiplication, etc.

Drawbacks:

- throws away domain knowledge about built-in data types like integers
- need to analyze recursive rules for plus, times, ... over and over
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Constrained Term Rewriting, what's that?

Term rewriting "with batteries included"

- first-order
- no fixed evaluation strategy
- no fixed order of rules to apply
- typed
- with pre-defined data structures (integers, arrays, bitvectors, ...), usually from SMT-LIB theories
- rewrite rules with SMT constraints

$\Rightarrow$ Term rewriting + SMT solving for automated reasoning

- General forms available, e.g., Logically Constrained TRSs [Kop, Nishida, *FroCoS '13*]
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Constrained Rewriting by Example

Consider a variation of the take-from program ...

Example (Constrained Rewrite System)

$$
\begin{array}{rcll}
\ell_0(n, r) & \to & \ell_1(n, r, \mathsf{Nil}) & \\
\ell_1(n, r, xs) & \to & \ell_1(n-1, r+1, \mathsf{Cons}(r, xs)) & [n > 0] \\
\ell_1(n, r, xs) & \to & \ell_2(xs) & [n = 0]
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Possible rewrite sequence:

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$$\begin{array}{rcll} \ell_0(n, r) & \to & \ell_1(n, r, \mathsf{Nil}) & \\ \ell_1(n, r, xs) & \to & \ell_1(n-1, r+1, \mathsf{Cons}(r, xs)) & [n > 0] \\ \ell_1(n, r, xs) & \to & \ell_2(xs) & [n = 0] \end{array}$$

Possible rewrite sequence:

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Here 7, 8, $\ldots$ are predefined constants.

Papers on termination of imperative programs often about **integers** as data

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Example (Imperative Program)

$$\textbf{if } (\mathsf{x} \geq 0)$$
$$\quad \textbf{while } (\mathsf{x} \neq 0)$$
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Does this program terminate?

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Papers on termination of imperative programs often about **integers** as data

Example (Imperative Program)

ℓ_0: **if** (x ≥ 0)
ℓ_1: **while** (x $\neq 0$)
ℓ_2: x $=$ x $- 1$;

Does this program terminate?

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Java Challenges

Java: object-oriented imperative language

- sharing and aliasing (several references to the same object)
- side effects
- cyclic data objects (e.g., `list.next == list`)
- object-orientation with inheritance
- . . .

Java Example

```
public class MyInt {

  // only wrap a primitive int
  private int val;

  // count "num" up to the value in "limit"
  public static void count(MyInt num, MyInt limit) {
    if (num == null || limit == null) {
      return;
    }
    // introduce sharing
    MyInt copy = num;
    while (num.val < limit.val) {
      copy.val++;
    }
  }
}
```

Does `count` terminate for all inputs? Why (not)?

(You may assume that `num` and `limit` are not references to the same object.)

Approach to Termination Analysis of Java

Tailor two-stage approach from Haskell analysis to Java [Otto et al, *RTA '10*]

Back-end: From rewrite system to termination proof

- Constrained term rewriting with integers [Giesl et al, *JAR '17*]
- Termination techniques for rewriting and for integers can be integrated

Front-end: From Java to constrained rewrite system

- Build **termination graph** that over-approximates all runs of Java program (abstract interpretation)
- Termination graph has **invariants** for integers and heap object shape (trees?)
- Extract rewrite system from termination graph

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Java: Source Code vs Bytecode

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```
00: aload_0
01: ifnull 8
04: aload_1
05: ifnonnull 9
08: return
09: aload_0
10: astore_2
11: aload_0
12: getfield val
15: aload_1
16: getfield val
19: if_icmpge 35
22: aload_2
23: aload_2
24: getfield val
27: iconst_1
28: iadd
29: putfield val
32: goto 11
35: return
```

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Here: **Java source code**

Ingredients for the Abstract Domain

1. program counter value (line number)
2. values of variables (treating `int` as $\mathbb{Z}$)
3. over-approximating info on possible variable values
   - integers: use intervals, e.g. $\mathtt{x} \in [4,\ 7]$ or $\mathtt{y} \in [0,\ \infty)$
   - heap memory with objects, **no sharing** unless stated otherwise
   - `MyInt`(?): maybe `null`, maybe a `MyInt` object

Heap predicates:

- Two references may be equal: $o_1 =^? o_2$
- Two references may share: $o_1 \searrow\!\!\swarrow o_2$
- Reference may have cycles: $o_1\,!$

03	$\mathtt{num} : o_1, \mathtt{limit} : o_2$
$o_1 : \mathtt{MyInt}(?)$ $o_2 : \mathtt{MyInt}(\mathtt{val} = i_1)$ $i_1 : [4, 80]$	

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   public class MyInt {
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1:      if (num == null
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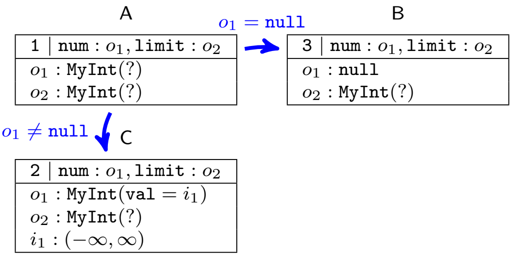

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means: refine X with $cond$, then evaluate to Y; here combined for brevity (narrowing; Haskell: $\boldsymbol{Case} + \boldsymbol{Eval}$)

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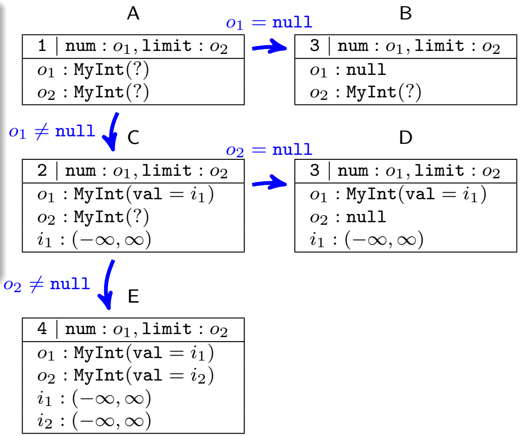

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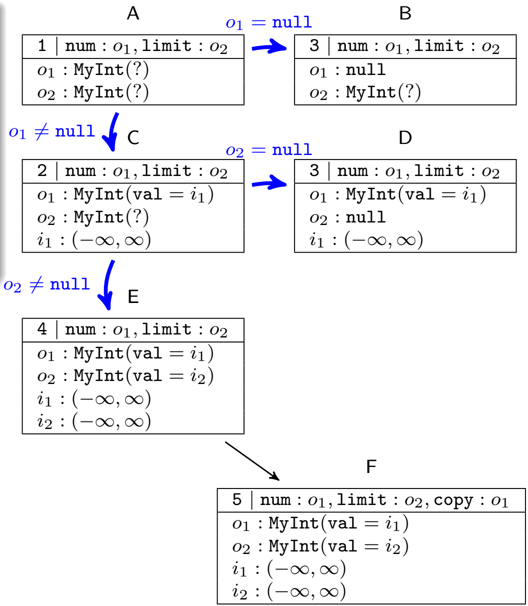

$X \longrightarrow Y$

means: evaluate X to Y
(Haskell: ***Eval***)

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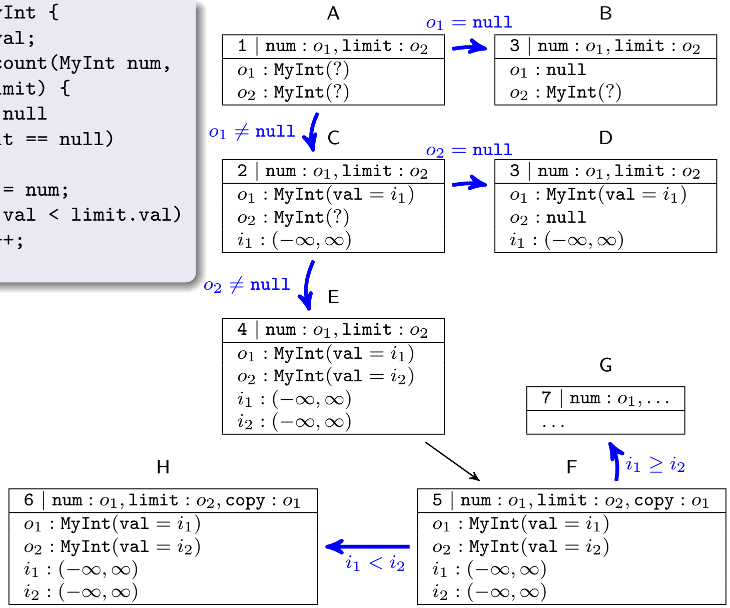

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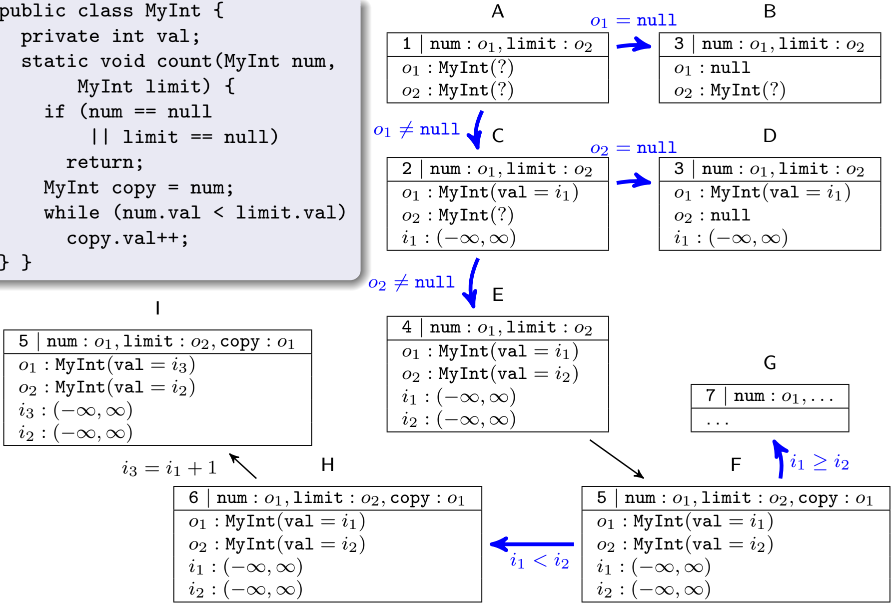

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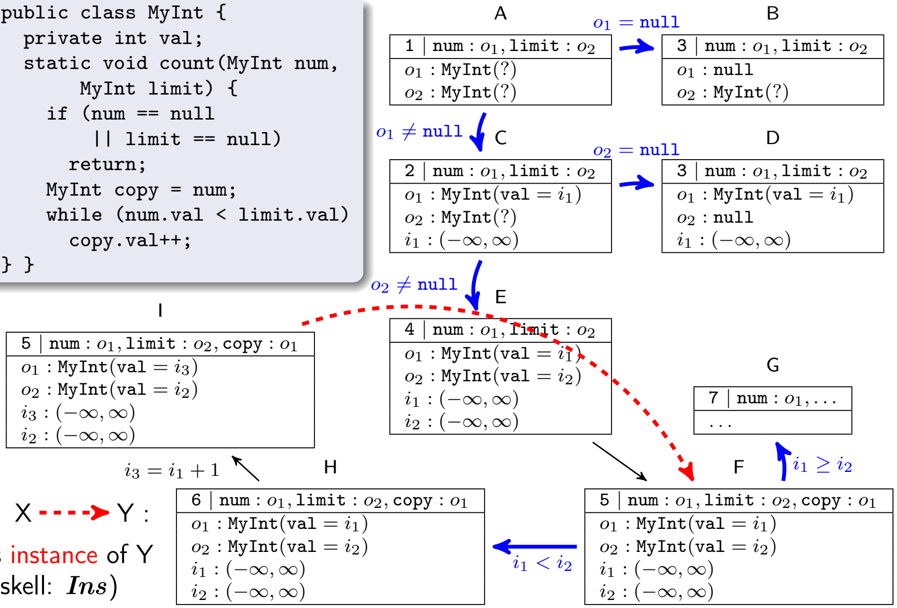

X $\dashrightarrow$ Y :

X is instance of Y

(Haskell: *Ins*)

From Java to Termination Graphs

Termination Graphs

- symbolic over-approximation of all computations (abstract interpretation)
- expand nodes until all leaves correspond to program ends
- by suitable generalization steps, one can always get a **finite** termination graph
- state s_1 is instance of state s_2 if all concrete states described by s_1 are also described by s_2

Using Termination Graphs for Termination Proofs

- every concrete Java computation corresponds to a **computation path** in the termination graph (related: DP paths for Haskell as suffixes of *non-(H-)terminating* computations)
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Q

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$o_1 : \texttt{MyInt}(?)$
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$o_3 : \texttt{null}$
$o_4 : \texttt{MyList}(?)$
$o_4\,!$
$i_1 : [7, \infty)$
$i_2 : (-\infty, \infty)$

For every class `C` with n fields, introduce an n-ary function symbol C

- term for o_1: o_1
- term for o_2: MyInt(i_2)
- term for o_3: null
- term for o_4: x (new variable)
- term for i_1: i_1 with **side constraint** $i_1 \geq 7$
  (invariant $i_1 \geq 7$ to be added to constrained rewrite rules for state Q)

Transformation of Objects to Terms

```
public class A {
  int a;
}

public class B extends A {
  int b;
}

...
A x = new A();
x.a = 1;

B y = new B();
y.a = 2;
y.b = 3;
```

- for every class `C` with n fields, introduce $(n+1)$-ary function symbol C
- first argument: part of the object corresponding to subclasses of `C`
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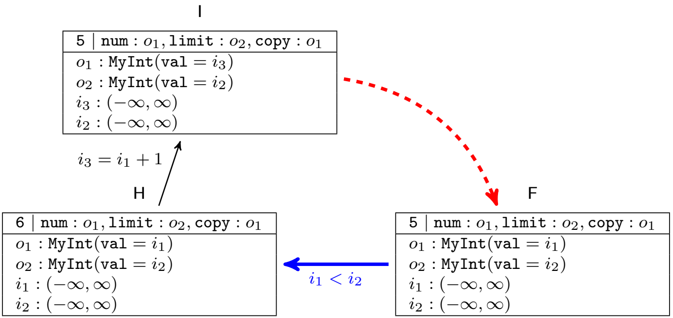
  $\to$ eoc for end of class
- term for `y`: $\mathsf{jlO}(\mathsf{A}(\mathsf{B}(\mathsf{eoc}, 3), 2))$
- every class extends `Object`!
  ($\to$ jlO $\equiv$ `java.lang.Object`)



- State F: $\ell_{\mathsf{F}}(\ \mathsf{jlO}(\mathsf{MyInt}(\mathsf{eoc}, i_1)),\ \mathsf{jlO}(\mathsf{MyInt}(\mathsf{eoc}, i_2))\ )$

  State H: $\ell_{\mathsf{H}}(\ \mathsf{jlO}(\mathsf{MyInt}(\mathsf{eoc}, i_1)),\ \mathsf{jlO}(\mathsf{MyInt}(\mathsf{eoc}, i_2))\ )$
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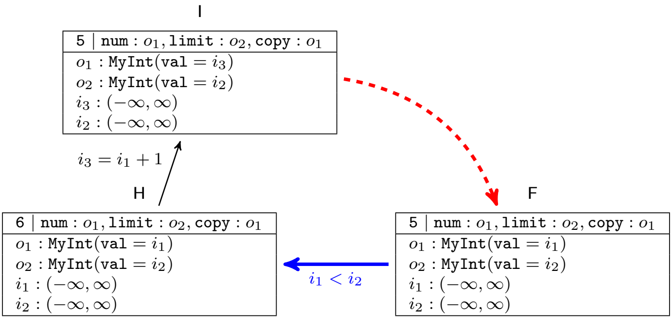
- Termination easy to show (intuitively: $i_2 - i_1$ decreases against bound 0)

From the Termination Graph to Terms and Rules



- State F: $\ell_{\mathsf{F}}(\ \mathsf{jIO}(\mathsf{MyInt}(\mathsf{eoc}, i_1)),\ \ \mathsf{jIO}(\mathsf{MyInt}(\mathsf{eoc}, i_2))\ \ )$

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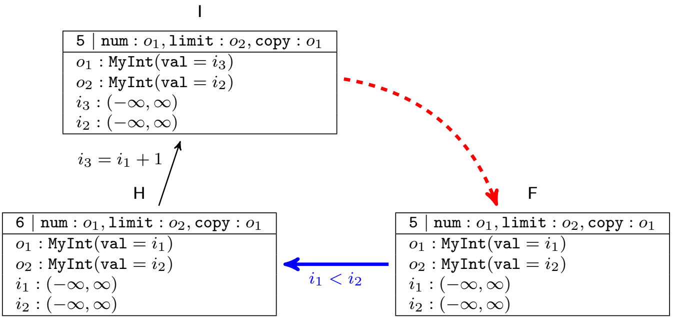

- State F: $\ell_{\mathsf{F}}($ jlO(MyInt(eoc, i_1)), jlO(MyInt(eoc, i_2)) $)$

  $\rightarrow$

  State H: $\ell_{\mathsf{H}}($ jlO(MyInt(eoc, i_1)), jlO(MyInt(eoc, i_2)) $)$ $[i_1 < i_2]$
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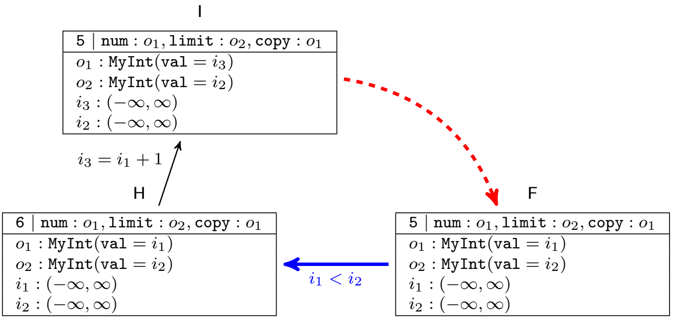

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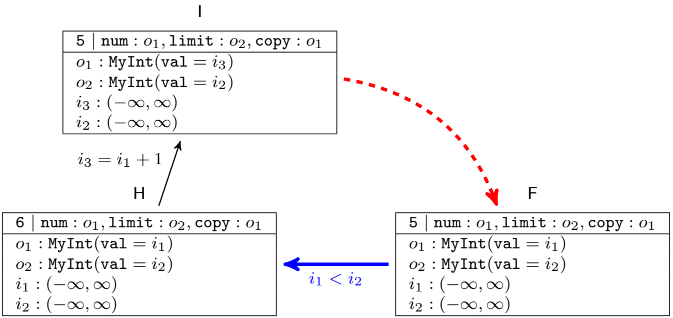

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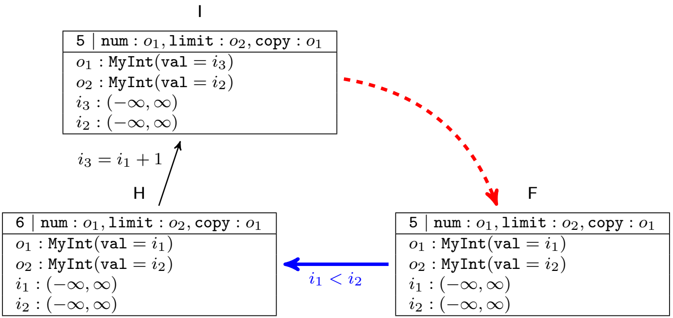

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Extensions

- **modular** termination proofs and **recursion** [Brockschmidt et al, *RTA '11*]
- proving **reachability** and **non-termination** (uses only termination graph) [Brockschmidt et al, *FoVeOOS '11*]
- proving termination with **cyclic data objects** (preprocessing in termination graph) [Brockschmidt et al, *CAV '12*]
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Conclusion Part II

Java:

- Successful empirical evaluation of Java approach on Termination Problems Database, including Java classes (e.g., `LinkedList`)
- Approach also successful at Termination Competition (other tools like COSTA, Julia abstract data structures to numbers instead of terms)

Overall:

- Common theme for program analysis by rewriting:
  - handle language specifics in **front-end**
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- Haskell: single term as abstract domain to represent program state
- Java: more complex abstract domain, use constrained rewriting

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  - handle language specifics in **front-end**
  - transitions between program states become rewrite rules for **TRS termination back-end**
- Haskell: single term as abstract domain to represent program state
- Java: more complex abstract domain, use constrained rewriting

Question 4

Recall the imperative program fragment on slide 34 (the `while` loop counting down).

In the lecture we added the invariant $x \geq 0$ to the constrained rewrite system. Construct a termination graph for the program that also finds this invariant and extract a constrained rewrite system with the invariant from this termination graph.

Java Exercises II

Question 5

```
public class List {
  private List next;

  public static int length(List xs) {
    int res = 0;                            // 1
    while (xs != null) {                    // 2
      xs = xs.next;                         // 3
      res++;                                // 4
    }
    return res;                             // 5
  }
}
```

Construct a termination graph for `length`, then extract the corresponding constrained rewrite system for the SCCs in the graph.
Can you prove termination of the resulting constrained rewrite system?

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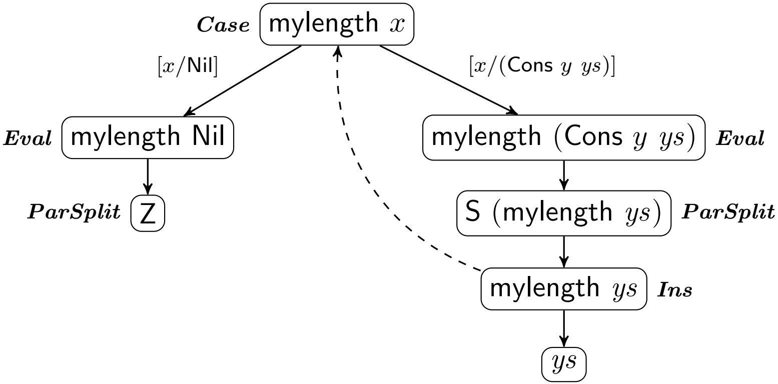
Solution for Question 1 (page 1)

(a) Yes. Intuition: The recursive call to `mylength` is on a list that is shorter than the original (H-terminating!) list. Thus, the end of the list will eventually be reached, and the recursion ends in its base case.

(b)



(c) $\mathcal{P}$: mylength(Cons(y, ys)) $\to$ mylength(ys)
$\mathcal{R}$: $\emptyset$

Solution for Question 1 (page 2)

(d) We can prove termination via a linear polynomial interpretation $[\,\cdot\,]$ of the function symbols to $\mathbb{N}$, such as:

$$[\text{mylength}](x_1) \;=\; x_1 \qquad\qquad [\text{Cons}](x_1, x_2) \;=\; x_1 + x_2 + 1$$

Alternatively, we could also use the embedding order or any path order.

(e) AProVE uses an adaption of the size-change termination principle [Lee, Jones, Ben-Amram, *POPL '01*] to term rewriting and dependency pairs [Thiemann, Giesl, *AAECC '05*] for the termination proof:

`http://www.dcs.bbk.ac.uk/~carsten/isr2017/Ex1.html`

Note that AProVE uses a slightly improved version of the step from termination graphs to DP problems. This can lead to simpler outputs than our translation from the lecture, in particular if Haskell terms with higher-order symbols are involved.

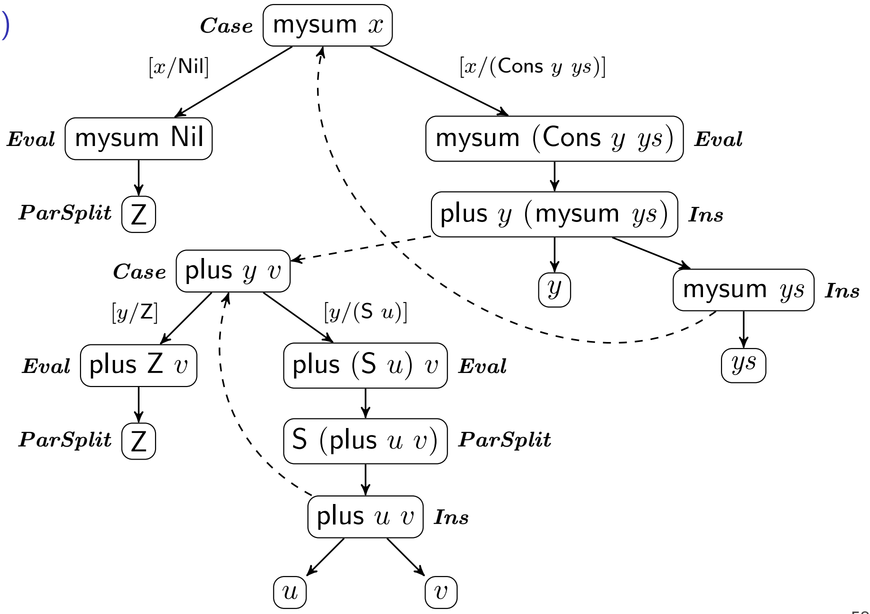
Further details: [Giesl et al, *TOPLAS '11*]

Solution for Question 2 (page 1)

(a) Yes. The reason is that both `mysum` and `plus` are terminating since their recursive calls are on arguments that get smaller and smaller.

Solution for Question 2 (page 2)

(b)



Solution for Question 2 (page 3)

(c) $\mathcal{P}$: $\mathsf{mysum}(\mathsf{Cons}(y, ys)) \to \mathsf{mysum}(ys)$
$\mathsf{plus}(\mathsf{S}(u), v) \to \mathsf{plus}(u, v)$
$\mathcal{R}$: $\emptyset$

(d) We can prove termination via a linear polynomial interpretation $[\,\cdot\,]$ of the function symbols to $\mathbb{N}$, such as:

$$[\mathsf{mysum}](x_1) = x_1 \qquad [\mathsf{Cons}](x_1, x_2) = x_1 + x_2 + 1$$
$$[\mathsf{plus}](x_1, x_2) = x_1 \qquad [\mathsf{S}](x_1) = x_1 + 1$$

Alternatively, we could also use the embedding order or any path order. (In general, more powerful techniques can be required for a successful termination proof of a Haskell program. However, the examples that we have considered in the exercises terminate for relatively straightforward reasons.)

(e) AProVE again uses the size-change termination principle for both DPs in $\mathcal{P}$ to prove termination:

http://www.dcs.bbk.ac.uk/~carsten/isr2017/Ex2.html

Solution for Question 3

Some strengths that one might expect (non-exhaustive list):

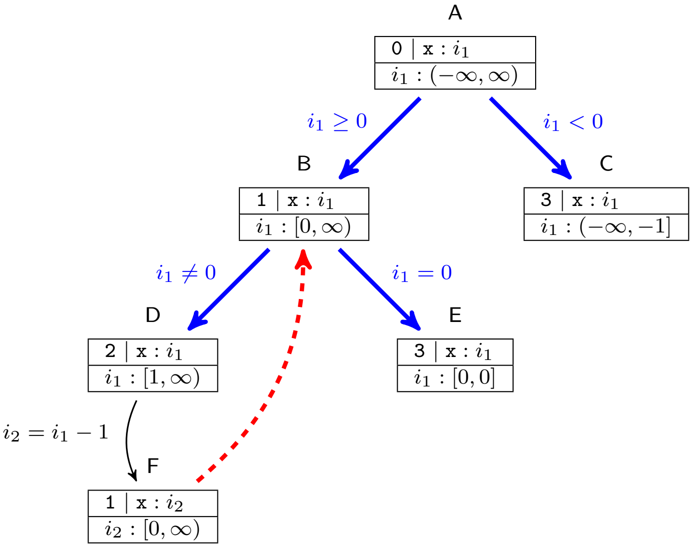
- support of user-defined data structures by representation as terms
- as termination tools for TRSs improve over time thanks to on-going development, so does this overall approach

Some weaknesses that one might expect (non-exhaustive list):

- support of built-in data structures (e.g., `Integer`) and their operations (e.g., `+`, `*`, ...) by terms over a finite signature and recursive rewrite rules on them is cumbersome; does not benefit from specialized program analysis techniques for built-in data structures, e.g., invariant synthesis (but: could improve using constrained rewriting with built-in data structures as translation target)
- termination back-end must prove termination of *all* terms; start term information is "lost in translation" (but: could include the path from initial node to SCCs in translated system and prove termination from only the start terms for the *initial* node in the resulting problem; would need TRS termination tools that benefit from this information)

Solution for Question 4 (page 1)

We get the following termination graph for the program:



Solution for Question 4 (page 2)

If we translate the whole termination graph, we get the following constrained rewrite rules:

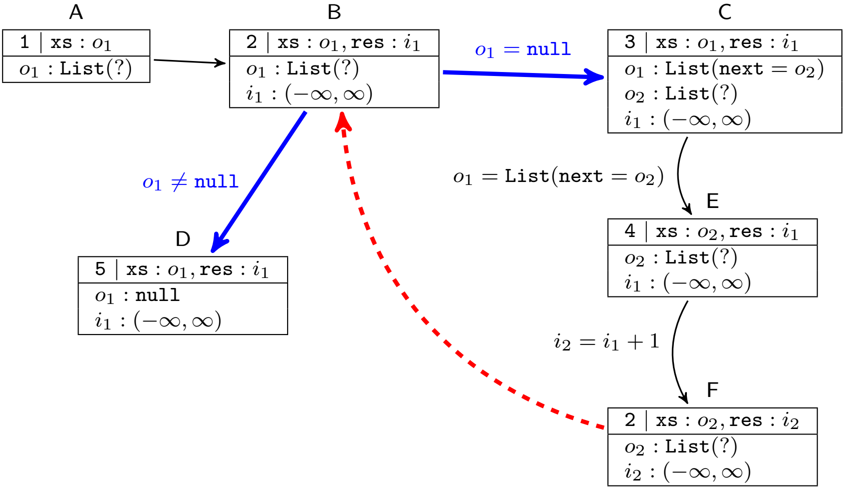
$$\begin{array}{lcll}
\ell_{\mathsf{A}}(i_1) & \to & \ell_{\mathsf{B}}(i_1) & [i_1 \geq 0] \\
\ell_{\mathsf{A}}(i_1) & \to & \ell_{\mathsf{C}}(i_1) & [i_1 < 0] \\
\ell_{\mathsf{B}}(i_1) & \to & \ell_{\mathsf{D}}(i_1) & [i_1 \neq 0 \wedge i_1 \geq 0] \\
\ell_{\mathsf{D}}(i_1) & \to & \ell_{\mathsf{B}}(i_1 - 1) & [i_1 \geq 1] \\
\ell_{\mathsf{B}}(i_1) & \to & \ell_{\mathsf{E}}(i_1) & [i_1 = 0]
\end{array}$$

Apart from the different names for function symbols and variables, we get essentially the same result as on slide 34, with two differences:

- Instead of ℓ_3, we now have the two different end-of-program symbols ℓ_{C} and ℓ_{E}.
- The second-to-last rule has $i_1 \geq 1$ as its condition, which is stronger than $i_1 \geq 0$ (i.e., $i_1 \geq 1$ implies $i_1 \geq 0$).

Solution for Question 5 (page 1)

We get the following termination graph for the program:



Solution for Question 5 (page 2)

The SCC of the graph gives rise to the following constrained rewrite rules:

$$\begin{array}{rcl} \ell_{\mathsf{B}}(\mathsf{jlO}(\mathsf{List}(\mathsf{eoc}, o_2)), i_1) & \to & \ell_{\mathsf{C}}(\mathsf{jlO}(\mathsf{List}(\mathsf{eoc}, o_2)), i_1) \\ \ell_{\mathsf{C}}(\mathsf{jlO}(\mathsf{List}(\mathsf{eoc}, o_2)), i_1) & \to & \ell_{\mathsf{E}}(o_2, i_1) \\ \ell_{\mathsf{E}}(o_2, i_1) & \to & \ell_{\mathsf{B}}(o_2, i_1 + 1) \end{array}$$

The dependency pairs for these rules are identical to the rules (except that we may rename the defined function symbols). The following polynomial interpretation $[\,\cdot\,]$ to $\mathbb{N}$ lets us conclude the termination proof:

$$\begin{array}{rclcrcl} [\ell_{\mathsf{B}}](o, i) & = & o + 1 & \qquad & [\ell_{\mathsf{C}}](o, i) & = & o \\ [\ell_{\mathsf{E}}](o, i) & = & o + 2 & & [\mathsf{List}](c, o) & = & o + 3 \\ [\mathsf{jlO}](c) & = & c & & [\mathsf{eoc}] & = & 0 \end{array}$$

In practice, tools like AProVE will first apply techniques to simplify and combine the obtained rewrite rules. Here we may obtain this single rule:

$$\ell(\mathsf{jlO}(\mathsf{List}(\mathsf{eoc}, o_2)), i_1) \quad \to \quad \ell(o_2, i_1 + 1)$$

Details: [Giesl et al, *JAR '17*]