Topological Logics over Euclidean Spaces

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joint work with

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RCC-8

(Egenhofer & Franzosa, 91): 9-intersections $\begin{pmatrix} A \cap B & A \cap \delta B & A \cap B' \\ \delta A \cap B & \delta A \cap \delta B & \delta A \cap B' \\ A' \cap B & A' \cap \delta B & A' \cap B' \end{pmatrix}$

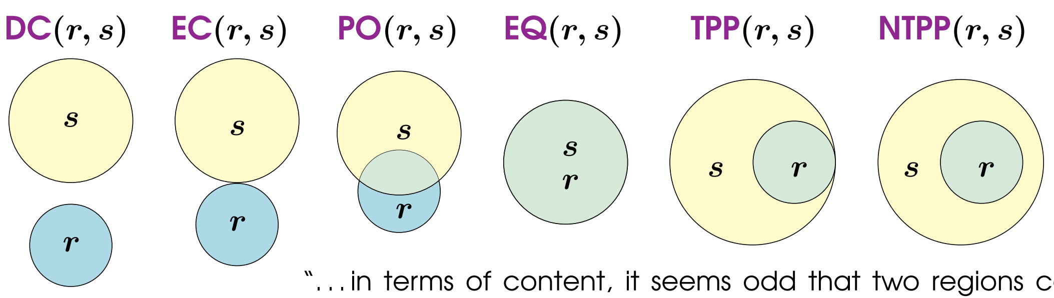
"The binary topological relation between two objects, A and B, in $\mathbb{R}^2$ is based upon the intersection of A's interior, boundary and exterior with B's interior, boundary and exterior."

regions = 'homogenously 2-dimensional objects with connected boundaries'

8 relations are possible between a pair of regions (out of 2^9)

(Randell, Cui & Cohn, 92): first-order theory of connection $C(x, y)$

(Whitehead, 1929)



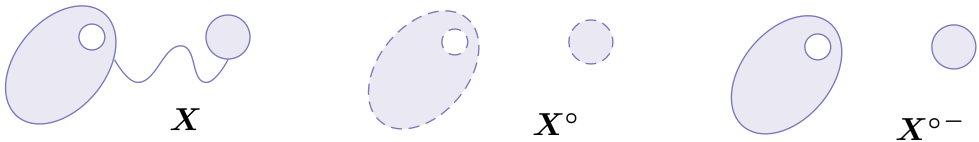
"...in terms of content, it seems odd that two regions can be distinct, but that each occupies the same amount of space..."

Regular closed sets

$X \subseteq T$ is **regular closed** if $X = X^{\circ -}$

(i.e., the set coincides with the closure of its interior)

RC(T) = sets of the form $X^{\circ -}$, for $X \subseteq T$



(Bennett 94): $\mathcal{RCC}$-8 is a **fragment** of $\mathcal{S}4_u$:

regions = variables, which are interpreted by regular closed sets

$\Box_u(p \leftrightarrow \Diamond\Box p)$

DC$(r, s) = \Box_u(r \wedge s \to \bot)$

TPP$(r, s) = \Box_u(r \to s) \wedge \neg\Box_u(s \to r) \wedge \Diamond_u(r \wedge \neg s)$

...

(Renz 98): Satisfiability of $\mathcal{RCC}$-8-formulas in the class of all topological spaces is **NP**-complete

Every consistent $\mathcal{RCC}$-8-formula is satisfied in a model over $\mathbb{R}^n$, $n \geq 3$, where all variables are interpreted as **internally-connected closed polyhedra**

Topological logics

RC(T) is a **Boolean algebra** $(\text{RC}(T), +, \cdot, -, \emptyset, T)$,
where $X + Y = X \cup Y$, $X \cdot Y = (X \cap Y)^{\circ -}$ and $-X = (\overline{X})^{-}$

topological model $\mathfrak{M} = (T, \cdot^{\mathfrak{M}})$
T a topological space
$\cdot^{\mathfrak{M}}$ a valuation

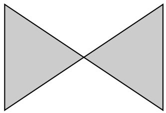
terms: **regular closed subsets of T**

$$\tau \quad ::= \quad r_i \quad | \quad \tau_1 + \tau_2 \quad | \quad \tau_1 \cdot \tau_2 \quad | \quad -\tau \quad | \quad \mathbf{0} \quad | \quad \mathbf{1}$$

formulas: **true** or **false**

$$\varphi \quad ::= \quad \tau_1 \subseteq \tau_2 \quad | \quad C(\tau_1, \tau_2) \quad | \quad c(\tau) \quad | \quad c^{\circ}(\tau) \quad | \quad \neg\varphi \quad | \quad \varphi_1 \wedge \varphi_2 \quad | \quad \dots$$

$\mathfrak{M} \models \tau_1 \subseteq \tau_2$ iff $\tau_1^{\mathfrak{M}} \subseteq \tau_2^{\mathfrak{M}}$
$\mathfrak{M} \models C(\tau_1, \tau_2)$ iff $\tau_1^{\mathfrak{M}} \cap \tau_2^{\mathfrak{M}} \neq \emptyset$
$\mathfrak{M} \models c(\tau)$ iff $\tau^{\mathfrak{M}}$ is connected
$\mathfrak{M} \models c^{\circ}(\tau)$ iff $(\tau^{\circ})^{\mathfrak{M}}$ is connected



NB. $\mathcal{RCC}$-8 is a topological logic:
DC$(r, s) = \neg C(r, s)$
TPP$(r, s) = (r \subseteq s) \wedge \neg(s \subseteq r) \wedge C(r, -s)$
...

$\mathcal{B}c^\circ$ over arbitrary topological spaces

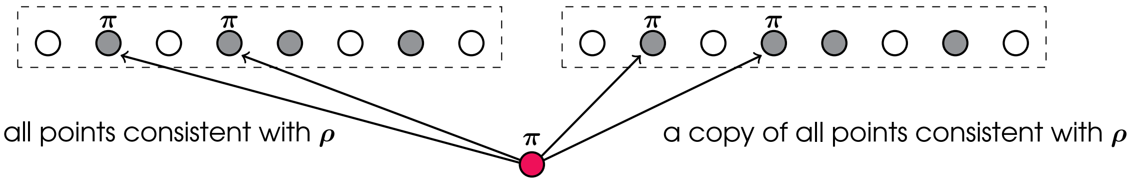
$\mathcal{B}c^\circ$ is the language with predicates $\subseteq$ and c° and full Boolean terms

Theorem. Satisfiability of $\mathcal{B}c^\circ$-formulas in the class of **all** topological spaces is **NP**-complete

Proof. Normal form:

$$\varphi \;=\; (\rho = \mathbf{0}) \;\wedge \bigwedge_{1\le j\le m} (\sigma_j \ne \mathbf{0}) \wedge \bigwedge_{1\le i\le n} \big(c^\circ(\pi_i) \wedge (\pi_i \ne \mathbf{0})\big) \;\wedge \bigwedge_{1\le k\le p} \neg c^\circ(\tau_k)$$

Step 1. If φ is satisfiable then it is satisfiable in a **saturated** Aleksandrov model:



Step 2. Select $m + 2p + 2n$ points and,

for each $1 \le k \le p$, select $\le n$ points $y_{\overline{\tau_k},\pi_i} \in \pi_i^{\mathfrak{A}} \cap (-\tau_k)^{\mathfrak{A}}$ (if the set is not empty)

polynomial finite model property

$\mathcal{B}c$ over arbitrary topological spaces

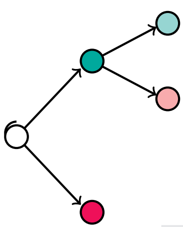
$\mathcal{B}c$ is the language with predicates $\subseteq$ and c and full Boolean terms

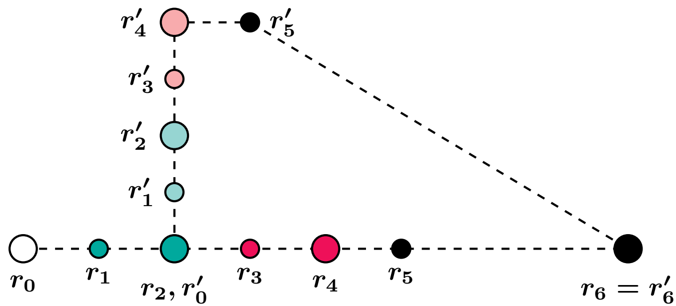
Theorem. Satisfiability of $\mathcal{B}c$-formulas in the class of **all** topological spaces is **ExpTime**-complete

Proof. (lower bound) Every satisfiable f-la is satisfied in a finite Aleskandrov space

> connectedness in an Aleksandrov space (W, R) =
> graph-theoretic connectedness of $(W, R \cup R^{-1})$

encoding of binary trees





$$(r_0 \neq \mathbf{0}) \wedge (r_6 \neq \mathbf{0}) \quad \wedge \quad c\left(\sum_{i=0}^{6} r_i\right) \quad \wedge \bigwedge_{|i-j|>1} \neg c(r_i + r_j)$$

$$(r_2 \subseteq r'_0)$$
$$(r_4 \subseteq r'_0)$$

Euclidean spaces

Theorem. Satisfiability of $\mathcal{B}c$- and $\mathcal{C}c^\circ$-formulas in **RC**$(\mathbb{R}^n)$, $n \geq 2$, is **ExpTime**-hard

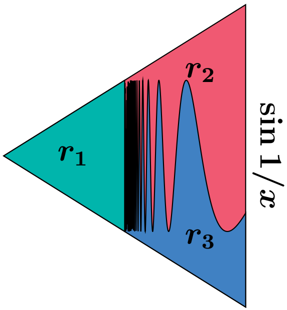
Proof. *a*) finite trees are enough to encode alternating TM with polynomial tape

b) $\neg c(\tau_1 + \tau_2) = \neg C(\tau_1, \tau_2)$, for internally-connected τ_1, τ_2

NB. This proof does not work for $\mathcal{B}c^\circ$ ($\neg c^\circ(\tau_1 + \tau_2)$ is too weak)

Polygons v Regular Closed Sets

$$\bigwedge_{i=1}^{3} c^\circ(r_i) \wedge c^\circ(r_1 + r_2 + r_3) \rightarrow \bigvee_{i=2}^{3} c^\circ(r_1 + r_i)$$



However, this $\mathcal{B}c^\circ$-formula is valid if the r_i are **semi-linear sets** (i.e., polygons)

RCP$(\mathbb{R}^n)$ is the class of models over $\mathbb{R}^n$ with valuations assigning n-dimensional polyhedra to variables

$\mathcal{B}c^\circ$ over RCP$(\mathbb{R}^n)$

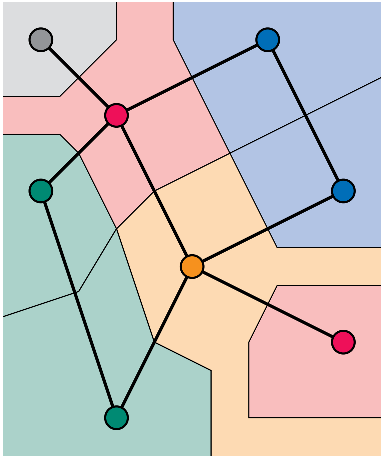
A **graph model** $\mathfrak{G} = (G, \cdot^{\mathfrak{G}})$:

$G = (V, E)$ is a (finite undirected simple) graph

$r_i^{\mathfrak{G}} \subseteq V$

$+, \cdot$ and $-$ are the union, intersection and complement

$\mathfrak{G} \models c(\tau)$ iff $\tau^{\mathfrak{G}}$ is connected



A **neighbourhood graph** of

an internally-connected partition $X_1, \dots, X_n$ is

$G = (V, E)$, where $G = \{1, \dots, n\}$

$E = \{(i, j) \mid (X_i + X_j)^\circ \text{ connected}\}$

A $\mathcal{B}c^\circ$-formula is satisfiable over **RCP**$(\mathbb{R}^n)$, $n \geq 3$, iff it has a graph model

EXPTIME-complete

is satisfiable over **RCP**$(\mathbb{R}^2)$ iff it has a **planar** graph model

EXPTIME-hard

NB. Upper complexity bound for **RCP**$(\mathbb{R}^2)$ is not known

Summary of results

lang.	$\mathbb{R}$			$\mathbb{R}^2$			$\mathbb{R}^3$				RC
	RCP($\mathbb{R}$)		RC($\mathbb{R}$)	RCP($\mathbb{R}^2$)		RC($\mathbb{R}^2$)	RCP($\mathbb{R}^3$)		RC($\mathbb{R}^3$)		
$\mathcal{RCC}$-8c°	NP	$\neq$	NP	NP			NP				
$\mathcal{RCC}$-8c				NP							
$\mathcal{B}c^\circ$	NP			$\geq$ EXP	$\neq$	?	EXP	$\neq$	?	?	NP
$\mathcal{B}c$				$\geq$ EXP	$\neq$	$\geq$ EXP	$\geq$ EXP	?	$\geq$ EXP	?	EXP
$\mathcal{C}c^\circ$	PSPACE	$\neq$	PSPACE	$\geq$ EXP	$\neq$	$\geq$ EXP	$\geq$ EXP	$\neq$	$\geq$ EXP	$\neq$	EXP
$\mathcal{C}c$				$\geq$ EXP	$\neq$	$\geq$ EXP	$\geq$ EXP	?	$\geq$ EXP	?	EXP

References

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