Chapter 1

SPATIAL LOGIC + TEMPORAL LOGIC =?

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1. Introduction

As follows from the title of this chapter, our primary aim is to analyse possible solutions to the equation

$$\boxed{\textit{Spatial logic} \ + \ \textit{Temporal logic} \ = \ x} \tag{1.1}$$

where the items on the left-hand side are some standard *spatial* and *temporal logics*, and + is some 'operator' combining these two logics into a single one. The question we are concerned with is how the computational complexity and the expressive power of the component logics are related to the complexity and expressiveness of the resulting *spatio-temporal logic* x under *various combination operators* +.

To convey the flavour of the problems we are facing when attempting to answer this question, let us consider two standard spatial and temporal logics and try to combine them.

Recall from Ch. **??** of this Handbook that one of the basic and natural logics for reasoning about space is the 'modal' logic $\mathcal{S}4_u$ equipped with the Boolean operators over subsets of a topological space and the 'modal' operators $\mathbf{I}$ and $\mathbf{C}$ interpreted as the topological interior and closure, respectively. In this language we can say, for example, that two spatial objects X and Y are externally connected, $\mathsf{EC}(X, Y)$ in symbols, in the sense that X and Y share some points but none of them belongs to the interior of X or Y. This can be expressed, e.g., by means of the following constraints:

$$X \cap Y \neq \emptyset \quad \text{and} \quad \mathbf{I}X \cap \mathbf{I}Y = \emptyset.$$

Reasoning in $\mathcal{S}4_u$ is perfectly well understood; it is known to be PSPACE-complete, and various reasonably effective reasoning systems are available.

For the temporal component we take the standard linear temporal logic $\mathcal{LTL}$ which extends propositional logic with the temporal operators $\bigcirc$ ('tomorrow'), $\Diamond_F$ (eventually), and $\Box_F$ (always in the future). $\mathcal{LTL}$ is interpreted over the flow of time consisting of the natural numbers $(\mathbb{N}, <)$. For example, the following formula says that a day is Saturday if, and only if, the next day is Sunday:

$$\Box_F(\mathit{Saturday} \leftrightarrow \bigcirc \mathit{Sunday}).$$

Reasoning in $\mathcal{LTL}$ is also thoroughly investigated; it is PSPACE-complete as well, and a number of temporal reasoning systems have been implemented.

Now our aim is to construct a combination of $\mathcal{S}4_u$ and $\mathcal{LTL}$ where we could express, for example, that today spatial objects X and Y are not externally connected, but tomorrow they are:

$$\neg\mathsf{EC}(X, Y) \ \wedge\ \bigcirc\mathsf{EC}(X, Y),$$

or that the spatial object X today is externally connected with the space $\bigcirc X$ it will be occupying tomorrow:

$$\mathsf{EC}(X, \bigcirc X),$$

or that, starting from some future moment, X will never change its position:

$$\Diamond_F \Box_F (X = \bigcirc X).$$

Having efficient spatial and temporal reasoners S and T at our disposal (for $\mathcal{S}4_u$ and $\mathcal{LTL}$, respectively), the quickest way of constructing a combined spatio-temporal reasoning system is to organise their joint work in a modular way: first, say, S treats the input constraints regarding formulas that start with temporal operators as atomic, then T deals with them regarding formulas with spatial operators as atomic, etc. Clearly, the resulting system works in PSPACE. But unfortunately, such a reasoner does not take into account any interaction between the spatial and temporal operators: the problem is that a spatio-temporal formula is recognised as valid by this reasoner only if it is valid in *arbitrary fusions of topological models with* (*possibly many*) *isomorphic copies of the flow of time* $(\mathbb{N}, <)$. In such models, spatial objects are not moving in *the same space* over *the same flow of time* because the topological space at moment n may have absolutely nothing to do with the space at moment $n+1$, or, dually, every point of space has its own history. In particular, one could expect the constraint

$$\bigcirc \mathsf{EC}(X, Y) \;\leftrightarrow\; \mathsf{EC}(\bigcirc X, \bigcirc Y)$$

to be a valid principle of spatial-temporal logics—yet, our reasoner would not confirm this: it would claim that the negation of this formula is satisfiable.

Of course, from a purely semantical perspective, this problem can easily be overcome by restricting the class of *intended models* to those where the same topological space is kept along the whole time line. In other words, we can assume that the underlying topological space does not change in time; what changes is the position, shape, size, etc. of spatial objects. Mathematically this means that the intended spatio-temporal models for combinations of $\mathcal{S}4_u$ and $\mathcal{LTL}$ are the Cartesian products of topological spaces and $(\mathbb{N}, <)$.

Such models provide a natural interpretation for the formulas considered above, with the last one being valid in all of them. But on the other hand, in order to deal with them we need a new, perhaps more sophisticated reasoning system. Is it, at least in principle, possible to design an effective complete and sound system of this kind?

A moment's reflection about the possible computational behaviour of such a system brings to memory another model, which logicians and computer scientists know all too well. We mean Turing's model of computation. The tape of a Turing machine can be regarded as a somewhat simplified model of space where a 'spatial object' is the collection of cells containing a certain symbol from the alphabet. Putting the problem in this perspective, one can immediately start suspecting that perhaps even a modestly expressive spatio-temporal language could be able to describe the change of spatial objects over time which corresponds to the computation of a Turing machine. And if this is indeed the case then, using the operator $\Diamond_F$ for 'eventually' it appears almost trivial to state that the Turing machine eventually reaches a halting state on a given input, which would mean that reasoning in the hybrid language cannot be decidable (or, even worse, that the set of valid spatio-temporal formulas is not recursively enumerable).

Now, obviously, the topological language $\mathcal{S}4_u$ and many other languages to be considered in this chapter are not designed to represent knowledge about the tape of a Turing machine (to begin with, there is no obvious topology on such a tape). Some much smarter 'encoding techniques' may be needed to prove that combinations of $\mathcal{S}4_u$ and $\mathcal{LTL}$ (and similar logics) are undecidable. Yet, the *first major result* of this chapter shows that the intuition behind the simulation of Turing machines discussed above is correct: naïve and straightforward combinations of spatial and temporal logics (interpreted in Cartesian products of time and space) almost invariably lead to undecidable hybrids.

The *second major result*, however, is that by closely inspecting the expressive means required to simulate Turing machines one can still find hierarchies of useful and expressive hybrids of $\mathcal{S}4_u$ and $\mathcal{LTL}$, their fragments, and some related logics which are decidable and of reasonably low complexity.

The structure of this chapter is as follows. In the next section, we discuss in more detail, but still on a rather abstract level, our main paradigm of 'snapshot spatio-temporal models' and most important reasoning problems relevant to these models.

Then, in Sec. 3 and 4, we discuss in detail the ingredients of the spatio-temporal logics to be constructed and investigated in this chapter. We consider two families of spatial logics. The first one is comprised of formalisms designed for reasoning about topological relations among spatial objects and ranging from $\mathcal{RCC}$-8 to $\mathcal{S}4_u$, possibly with component counting. A remarkable feature of these logics is their 'computational robustness' in the sense that the complexity of reasoning gradually increases from NP for $\mathcal{RCC}$-8 to PSpace and NExpTime for $\mathcal{S}4_u$ without and with component counting, respectively. Moreover, each complexity 'jump' in this hierarchy is clearly connected to the corresponding increase in the logic's expressiveness. Our second family of spatial logics consists of formalisms that are capable of reasoning about distances in metric spaces. Some of these logics will contain $\mathcal{S}4_u$ and, therefore, combine topological reasoning with reasoning about distances. These logics are also computationally robust, with the typical complexity being ExpTime.

The introduction to temporal logic systems in Sec. 4 is much shorter, as we only consider two approaches to logic modelling of time: time as a linear discrete sequence of time points or snapshots, and time as a tree-like structure of such snapshots representing some aspects of non-determinism. Other flows of time, say, continuous time, are not discussed, but pointers to the literature are provided.

Having introduced the logical systems for space and time, in Sec. 5 we discuss general combination principles—requirements and constraints for the operator + in (1.1)—which will guide us when designing combined spatio-temporal systems. Then, in Sec. 6 and 7, we use these principles to construct spatio-temporal logics out of the components introduced in Sec. 3 and 4. As before, the emphasis of this investigation is on the trade-off between the expressive power and the complexity of reasoning. We shall discover, in particular, that unlike the 'robust' component logics, their spatio-temporal hybrids turn out to be much more sensitive to seemingly minor changes in expressiveness.

In Sec. 8, we consider a somewhat different paradigm of spatio-temporal models and languages for reasoning about them: here we formalise spatio-temporal reasoning within the framework of dynamical systems based on topological and metric spaces with continuous and isometric functions, respectively. As logics for dynamical systems are discussed in detail elsewhere in this Handbook (see Ch. **??**), we concentrate here on the connection between the spatio-temporal systems introduced before and the dynamical systems perspective. It will turn out that in some cases the connections between the two approaches are so strong that results can be mutually imported from one area to the other.

Finally, in Sec. 9, we briefly discuss the relation between spatio-temporal logics and other temporalised formalisms, for example first-order temporal logics and temporal epistemic logics.

The reader who considers computational complexity less important and is interested in logic modelling of (relativistic) space-time using classical *first-order* logic is referred to Ch. **??**.

2. Static and changing spatial models

The intended models of standard spatial logics are usually based on 'mathematical spaces' such as (variations of) topological or metric spaces and their relational or algebraic representations or abstractions. We will consider many examples of such models and spaces in Sec. 3.1; more can be found elsewhere in this Handbook. Meanwhile, in order to discuss basic principles of introducing a temporal dimension into otherwise static spatial models, we neglect the concrete structure of these 'mathematical spaces' and concentrate on the generic properties of the models.

To represent spatial entities in models we require a countably infinite supply of *spatial variables* (that is, unary predicates) $p_0, p_1, \dots$. Thus, a generic spatial model can be thought of as a structure of the form

$$\mathfrak{M} = (\mathfrak{S}, p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}, \dots), \tag{1.2}$$

where $\mathfrak{S}$ is the underlying 'mathematical space' (say, a metric or topological space, or a structured collection of polygons on the Euclidean plane) and the $p_i^{\mathfrak{M}}$ are interpretations of the spatial variables as subsets of the domain of $\mathfrak{S}$.

Depending on the underlying spatial ontology, one can distinguish between two types of models:

- *point-based* models, where spatial objects are (explicitly or implicitly) thought of as consisting of sets of points, and
- models with *extended spatial entities* as basic elements (say, regions or intervals) together with certain relations between them.

Point-based spatial models. In a point-based model of the form (1.2), the underlying 'mathematical space' $\mathfrak{S}$ is a collection of points equipped with 'point-wise defined' operators (like a metric or topological space). Interpretations $p_i^{\mathfrak{M}}$ of spatial variables p_i (that is, subsets of the domain of $\mathfrak{S}$) represent spatial objects. Thus, a spatial object is identified with the set of points it occupies. By imposing various constraints on these interpretations—say, by allowing only polygons, circles or regular closed connected sets—we can reflect the desired requirements on the form of spatial objects.

Region-based spatial models. In a region-based model of the form (1.2), spatial objects are represented as (unstructured) *elements* of the underlying 'space' $\mathfrak{S}$. We may consider as the domain of $\mathfrak{S}$, for instance, the collection of polygons on the Euclidean plane and completely forget about the plane itself. The 'structure' of spatial objects is reflected then by certain relations among them (say, polygon x has a common edge with polygon y) which should be specified in $\mathfrak{S}$ (for details and further references see Ch. **??**, **??** and **??** of this Handbook). Spatial variables are again interpreted as sets of elements of the domain of $\mathfrak{S}$, for instance as a set of polygons approximating the map of the U.K. (including the Isle of Wight, the Hebrides, and other islands), or the singleton set containing (the polygonal approximation of) the Isle of Man.

In this chapter we only consider point-based spatial models, although some results and constructions can be generalised to region-based ones.

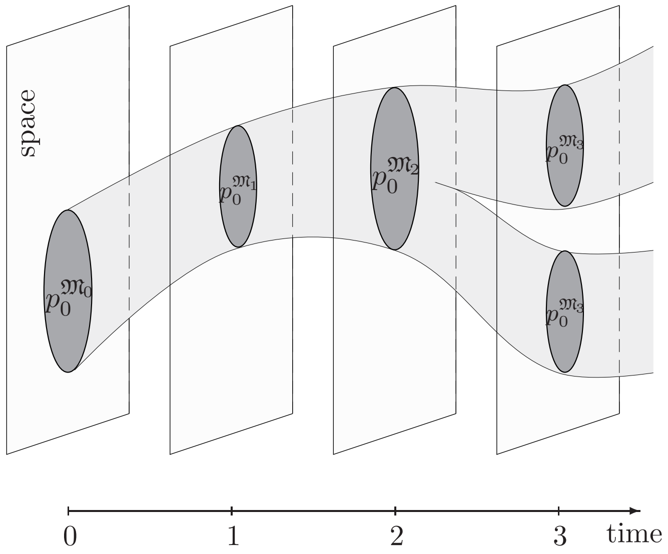


Figure 1.1. Linear snapshot model with a moving spatial object.

Snapshot spatio-temporal models. The intended models of temporal logics are supposed to represent the *change of states*—which, in our case, should be spatial models (1.2)—over time, under actions, etc. In most cases it makes sense to assume that space always remains the same. Moreover, one can usually simulate expanding, shrinking or varying space in some 'sufficiently large' constant space; see, e.g., (Gabbay et al., 2003). The motion of spatial objects can therefore be modelled by changing the interpretations $p_i^{\mathfrak{M}}$ of spatial objects from one state to another. (A different approach to modelling motion was taken by Muller (1998), who considered a moving object as a single spatio-temporal entity.)

There are many different time paradigms developed in philosophy, mathematics, physics, computer science and other disciplines: linear and branching, discrete and dense, point-based and interval-based, etc. (see, e.g., Gabbay et al., 1994; Gabbay et al., 2000; Fisher et al., 2005). In this chapter we mainly focus on the flow of time that can be represented by the natural numbers $(\mathbb{N}, <)$, where $<$ is the temporal precedence relation between time points. In this case our generic *snapshot spatio-temporal model* is simply an infinite sequence

$$\mathfrak{M}_0 = (\mathfrak{S}, p_0^{\mathfrak{M}_0}, p_1^{\mathfrak{M}_0}, \dots), \quad \mathfrak{M}_1 = (\mathfrak{S}, p_0^{\mathfrak{M}_1}, p_1^{\mathfrak{M}_1}, \dots), \quad \dots \qquad (1.3)$$

of spatial models of the form (1.2) with the same space $\mathfrak{S}$; see Fig. 1.1. In Sec. 4.2 and 6.2 we will briefly consider temporal and spatio-temporal models with branching (tree-like discrete) time that can capture some aspects of non-determinism. In either of these time paradigms the points of time can be taken as primitive temporal entities, assumed to be generated by state transition systems (a standard computer science approach), or by dynamical systems (a usual way in mathematics).

As an illustration let us consider the following example.

Spatial transition systems. Our main example running throughout the chapter is a spatial transition system which describes the changing geography of the Earth as we see it every day in BBC's weather forecasts, say, in Ten O'Clock News. Every day the state of the map is represented by a spatial model

$$\mathfrak{M} = (\mathfrak{E}, p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}, \dots),$$

where $\mathfrak{E}$ is a suitable mathematical model of the Earth surface and each $p_i^{\mathfrak{M}}$ is the space occupied by the geographical object modelled by p_i (either static as a town, a county or dynamic as a night frost or rainfall area, etc.) on that day. Starting from a certain day in the past, we can trace then the day-after-day changes that have happened till the present moment. Depending on our philosophical, religious, etc. views we can regard the future to be deterministic or not. In particular, we can imagine that today's state may evolve in many different ways.

In computer science, such scenarios are often described in terms of state transition systems—*spatial transition systems* in our context—which, in general, are tuples of the form

$$(S, \rightarrowtail, \mu, \mathfrak{S}, s_0), \tag{1.4}$$

where S is a nonempty set of *states*, $\rightarrowtail$ is a binary *transition* relation on S without dead-ends (states without outgoing $\rightarrowtail$), μ is a function associating with each state $s \in S$ a spatial model $\mu(s)$ of the form (1.2) based on the same space $\mathfrak{S}$, and s_0 is the *initial state*. Possible *evolutions* (or *transformations*) of this initial state are sequences

$$s_0 \rightarrowtail s_1 \rightarrowtail s_2 \rightarrowtail \dots, \tag{1.5}$$

where $s_i \in S$ for all $i \in \mathbb{N}$. Each of these evolutions obviously generates a linear snapshot model

$$\mu(s_0),\ \mu(s_1),\ \mu(s_2),\ \dots$$

of the form (1.3). In the deterministic case (as in the second example below) we have a single evolution. In general, however, the transition relation $\rightarrowtail$ of a spatial transition system can represent non-deterministic rules. Then it generates a discrete tree of evolutions (1.5).

What precisely can be told about these models depends of course on the concrete spatial and temporal logics we use. Here we give a few examples of English statements about our 'geographical transition system' that will be represented as spatio-temporal formulas in Sec. 5 and 6.1.

(A) If two clouds are disconnected now, then at the next moment they either remain disconnected or become externally connected.

(B) Kaliningrad is disconnected from the EU until the moment when Poland becomes a tangential proper part of the EU, after which Kaliningrad and the EU are externally connected forever.

(C) The current position of a hurricane overlaps its position in an hour.

(D) If tomorrow object X is at the place where object Y is today, then Y will have to move by tomorrow.

(E) The space occupied by Europe never changes.

(F) In two years the EU will be extended with Romania and Bulgaria.

(G) It will be raining over every part of England ever and ever again.

(H) If the Earth consists of water and land, and the space occupied by water expands, then the space occupied by land shrinks.

(I) Two deserts that expand by at least a mile in all directions every year must eventually intersect.

Reasoning tasks. We have not introduced yet any formal languages capable of talking and reasoning about spatio-temporal models—they depend on the concrete spatial and temporal logics we combine as well as the combination principles to be discussed later on in Sec. 5. Nonetheless, it does make sense to consider on this abstract level the main *reasoning problems* one might be interested in for some fictional language $\mathcal{L}$.

The most general and important problem we are going to consider is

- *satisfiability of spatio-temporal constraints.*

Suppose that we have formulated a finite set Γ of $\mathcal{L}$-formulas representing constraints on possible spatio-temporal scenarios. Then we are facing the following questions. Is this set Γ satisfiable (or consistent)? In other words, does there exist a spatio-temporal model realising these constraints? And if so, how such a model may look like? For example, can it be given by a finite transition system? Can it be based on a finite space?

Of particular interest to us will be algorithmic properties of the satisfiability problem. Is this problem *decidable*? That is, does there exist an algorithm which is capable of deciding, given an arbitrary finite set Γ of constraints, whether Γ is satisfiable? Are finite sets of satisfiable constraints *recursively enumerable*? What is the computational (worst-case) complexity of the satisfiability problem?

Note that the *deduction* (or *entailment*) *problem* 'given a finite set Γ of constraints and an $\mathcal{L}$-formula φ, decide whether φ holds in all spatio-temporal models where Γ holds?' is usually reducible to the satisfiability problem.

The satisfiability problem can be restricted to certain classes of $\mathcal{L}$-formulas and constraints. Here is a typical example. We describe the behaviour of spatial transition systems by imposing some *local* constraints Γ which specify possible initial states and transitions from each given state to the next ones. This is done in some sublanguage $\mathcal{L}_{loc}$ of $\mathcal{L}$. We can also specify (by means of $\mathcal{L}_{loc}$-formulas) states with some desirable property φ or some 'bad' property ψ. And then we are interested in the algorithmic properties of

- the *reachability problem* relative to $\mathcal{L}_{loc}$: 'is it the case that every model where constraints Γ hold contains a state satisfying φ?' or
- the *safety problem* relative to $\mathcal{L}_{loc}$: 'is it the case that no model where constraints Γ hold contains a state satisfying ψ?'

In the extreme case, when $\mathcal{L}_{loc}$ is expressive enough to describe (up to isomorphism) any particular spatial transition system, checking for reachability, safety or some other properties are instances of classical *model checking* problems (see, e.g., Clarke et al., 2000, and references therein).

3. Spatial logics

In this section we introduce the 'mathematical spaces' and spatial logics capable of talking and reasoning about these spaces that will serve as the spatial components of our spatio-temporal formalisms.

We consider spatial logics of two types: (i) those that can represent and reason about topological relations among spatial objects, and (ii) those that can additionally take into account distances between objects. The former are interpreted over *topological spaces* and the latter over *metric* (or more generally, *distance*) *spaces*. The choice of these logics is motivated by the following reasons. First, topological and metric spaces belong to the most important and well-understood structures representing space. Reasoning about topological relations between regions such as 'X is externally connected to Y' or 'X is tangential proper part of Y' has proved to be one of the most successful approaches to *qualitative* spatial knowledge representation and reasoning (KR&R) in artificial intelligence; see, e.g., (Cohn and Hazarika, 2001) and references therein. Extensions of 'topo-logics' with distance operators like $\exists^{\leq a} X$ giving the a-neighbourhood of X or $X \leftleftarrows Y$ giving the set of points that are closer to X than to Y are becoming another interesting research stream (Kutz et al., 2003; Wolter and Zakharyaschev, 2003; Wolter and Zakharyaschev, 2005a) that is especially close to the authors' hearts. Other important aspects of space such as, e.g., orientation have been considered as well; however, no combinations with temporal logics have been constructed so far. We believe that the approach to combining spatial logics of topological and metric spaces with temporal ones to be presented later on in this chapter can be extended to other spatial formalisms as well.

3.1 Metric and topological spaces

Metric spaces. A *metric space* is a pair (Δ, d), where Δ is a nonempty set (of points) and d is a function from $\Delta \times \Delta$ into the set $\mathbb{R}^{\geq 0}$ (of non-negative real numbers) satisfying the following axioms

$$\textit{identity of indiscernibles:} \qquad d(x,y) = 0 \quad \text{iff} \quad x = y, \tag{1.6}$$

$$\textit{symmetry:} \qquad d(x,y) = d(y,x), \tag{1.7}$$

$$\textit{triangle inequality:} \qquad d(x,z) \leq d(x,y) + d(y,z), \tag{1.8}$$

for all $x, y, z \in \Delta$. The value $d(x,y)$ is called the *distance* between points x and y. Given a metric space (Δ, d), a point $x \in \Delta$ and a nonempty $Y \subseteq \Delta$, define the *distance* $d(x, Y)$ between x and Y by taking

$$d(x,Y) = \inf\{d(x,y) \mid y \in Y\}.$$

As usual, $d(y, \emptyset) = \infty$. The distance $d(X,Y)$ between two nonempty sets X and Y is

$$d(X,Y) = \inf\{d(x,y) \mid x \in X,\ y \in Y\}.$$

Distance spaces. Although acceptable in many cases, the defined concept of metric space is not universally applicable to all interesting measures of distance between points, especially those used in everyday life. Consider, for instance, the following two examples:

(i) If $d(x, y)$ is the flight-time from x to y then, as we know it too well, d is not necessarily symmetric, even approximately (just take a plane from London to Tokyo and back).

(ii) Often we do not measure distances by means of real numbers but rather using more fuzzy notions such as 'short,' 'medium' and 'long.' To represent these measures we can, of course, take functions d from $\Delta \times \Delta$ into the subset $\{1, 2, 3\}$ of $\mathbb{R}^{\geq 0}$ and define *short* $:= 1$, *medium* $:= 2$, and *long* $:= 3$. So we can still regard these distances as real numbers. However, for measures of this type the triangle inequality (1.8) does not make sense (short plus short can still be short, but it can also be medium or long).

Spaces (Δ, d) satisfying only the axiom (1.6) are called *distance spaces*.

Topological spaces. A *topological space* is a pair $(U, \mathbb{I})$ in which U is a nonempty set, the *universe* of the space, and $\mathbb{I}$ is the *interior operator* on U satisfying the *Kuratowski axioms*: for all $X, Y \subseteq U$,

$$\mathbb{I}(X \cap Y) = \mathbb{I}X \cap \mathbb{I}Y, \qquad \mathbb{I}X \subseteq \mathbb{I}\mathbb{I}X, \qquad \mathbb{I}X \subseteq X \quad \text{and} \quad \mathbb{I}U = U.$$

The operator dual to $\mathbb{I}$ is called the *closure operator* and denoted by $\mathbb{C}$: for every $X \subseteq U$, we have $\mathbb{C}X = U - \mathbb{I}(U - X)$. Thus, $\mathbb{I}X$ is the *interior* of a set X, while $\mathbb{C}X$ is its *closure*. X is called *open* if $X = \mathbb{I}X$ and *closed* if $X = \mathbb{C}X$. The complement of an open set is closed and vice versa. The *boundary* of a set $X \subseteq U$ is defined as $\mathbb{C}X - \mathbb{I}X$. Note that X and $U - X$ have the same boundary.

Topological spaces are often (equivalently) defined as pairs $(U, \mathcal{O})$, where $\mathcal{O}$ is a family of (open) subsets of U such that $\mathcal{O}$ is closed under arbitrary unions and finite intersections.

Metric spaces and topology. Each metric space (Δ, d) gives rise to the *interior operator* $\mathbb{I}_d$ on Δ: for all $X \subseteq \Delta$,

$$\mathbb{I}_d X = \{x \in X \mid \exists \varepsilon > 0 \ \forall y \ (d(x, y) < \varepsilon \to y \in X)\}.$$

The pair $(\Delta, \mathbb{I}_d)$ is called the *topological space induced by* the metric space (Δ, d). The dual *closure operator* $\mathbb{C}_d$ in this space can be defined by the equality

$$\mathbb{C}_d X = \{x \in \Delta \mid \forall \varepsilon > 0 \ \exists y \in X \ d(x, y) < \varepsilon\}.$$

We briefly remind the reader of a few standard examples of metric and topological spaces that will be used in what follows.

Euclidean spaces. The *one-dimensional Euclidean space* is the set of real numbers $\mathbb{R}$ equipped with the following metric on it

$$d_1(x, y) = |x - y|.$$

Let $X \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is said to be *interior* in X if there is some $\varepsilon > 0$ such that the whole open interval $(x - \varepsilon, x + \varepsilon)$ belongs to X. The interior $\mathbb{I}X$ of X is defined then as the set of all interior points in X. It is not hard to check that $(\mathbb{R}, \mathbb{I})$ is the topological space induced by the Euclidean metric d_1. Open sets in $(\mathbb{R}, \mathbb{I})$ are (possibly infinite) unions of open intervals (a, b), where $a \leq b$. The closure of (a, b), for $a < b$, is the closed interval $[a, b]$, with the end points a and b being its boundary.

In the same manner one can define *n-dimensional Euclidean spaces* based on the universes $\mathbb{R}^n$ with the metric

$$d_n(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

where x and y are n-dimensional vectors $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, respectively (in the definition of interior points x one should take n-dimensional ε-neighbourhoods of x).

Metric spaces on graphs. Another well known example is *metric spaces on graphs*: the distance between two nodes of a graph is defined as the length of the shortest path between them.

Aleksandrov spaces. A topological space is called an *Aleksandrov space* (Alexandroff, 1937) if arbitrary (not only finite) intersections of open sets are open. Aleksandrov spaces are closely related to *quasi-ordered sets*, that is, pairs $\mathfrak{G} = (V, R)$, where V is a nonempty set and R a transitive and reflexive relation on V. Every such quasi-order $\mathfrak{G}$ induces the interior operator $\mathbb{I}_{\mathfrak{G}}$ on V: for $X \subseteq V$,

$$\mathbb{I}_{\mathfrak{G}} X = \{x \in X \mid \forall y \in V \ (xRy \rightarrow y \in X)\}.$$

In other words, the open sets of the topological space $\mathfrak{T}_{\mathfrak{G}} = (V, \mathbb{I}_{\mathfrak{G}})$ are the *upward closed* (or *R-closed*) subsets of V. The *minimal neighbourhood* of a point x in $\mathfrak{T}_{\mathfrak{G}}$ (that is, the minimal open set to contain x) consists of all those points that are R-accessible from x. It is well-known (see, e.g., Bourbaki, 1966) that $\mathfrak{T}_{\mathfrak{G}}$ is an Aleksandrov space and, conversely, every Aleksandrov space is induced by a quasi-order.

For various generalisations of metric and topological spaces (like semi-metrics, closure spaces and digital topology) see Ch. **??**.

3.2 Topo-logics

In this section, we introduce and discuss a number of logical formalisms which can represent and reason about topological relations among spatial objects interpreted over topological spaces. Our choice of logics was guided by two criteria: (i) they should be sufficiently expressive to represent interesting and useful topological knowledge as identified in the qualitative spatial reasoning community; (ii) on the other hand, reasoning with such logics should be decidable and, if possible, of low computational complexity. Another important constraint on logics in the framework described in Sec. 2 is that change in time is modelled by changing the extensions of *unary predicates* representing spatial objects.

The most developed and systemically studied spatial logics satisfying our criteria are fragments of a 'propositional' logic in which 'propositional variables' (= unary predicates) denote spatial objects, and topological relations among them are represented by means of the interior and closure operators, the universal and existential quantifiers over space, as well as the Booleans. This logic, originally introduced as a *modal* logic, is known as $\mathcal{S}4_u$. As we shall see below, it can be regarded as the logic of topological spaces providing a common roof to some other formalisms developed by the spatial community such as the $\mathcal{RCC}$-8 or 9-intersection region connection calculi (where topological relations between regions are regarded as primitive).

Our exposition basically follows (Gabelaia et al., 2005a) where the reader can find more details, references and proofs. For historical references and motivation see Ch. **??** of this Handbook.

Modal logic of topological spaces. $\mathcal{S}4_u$ is the well known *propositional modal logic* $\mathcal{S}4$ extended with the universal modalities. The 'pedigree' of $\mathcal{S}4$ is quite unusual. It was introduced independently by Orlov (1928), Lewis in (Lewis and Langford, 1932), and Gödel (1933), without any intention to reason about space. Orlov and Gödel understood it as a logic of 'provability' (in order to provide a classical interpretation for the intuitionistic logic of Brouwer and Heyting) and Lewis as a logic of necessity and possibility, that is, as a *modal logic*. That it can be regarded as the logic of topological spaces was discovered by Stone (1937), Tarski (1938), Tsao-Chen (1938) and McKinsey (1941).

In the spatial context it is useful to distinguish between spatial terms and spatial formulas of $\mathcal{S}4_u$ as explained below. *Spatial terms* are expressions of the form:

$$\tau \quad ::= \quad p_i \quad | \quad \overline{\tau} \quad | \quad \tau_1 \sqcap \tau_2 \quad | \quad \tau_1 \sqcup \tau_2 \quad | \quad \mathbf{I}\tau \quad | \quad \mathbf{C}\tau, \qquad (1.9)$$

where

- the p_i are *spatial variables*,
- $\overline{}$, $\sqcap$ and $\sqcup$ are the standard Boolean operators (to be interpreted by the set-theoretic complement, intersection and union),
- $\mathbf{I}$ and $\mathbf{C}$ are the *interior* and *closure operators*, respectively (they correspond to the box and diamond of the modal logic $\mathcal{S}4$ but are denoted differently to emphasise their topological nature).

A *topological model* is a structure of the form

$$\mathfrak{M} = (\mathfrak{T}, p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}, \dots), \tag{1.10}$$

where $\mathfrak{T} = (U, \mathbb{I})$ is a topological space and $p_i^{\mathfrak{M}} \subseteq U$ for all i. The *extension* (or *interpretation*) $\tau^{\mathfrak{M}}$ of an arbitrary spatial term τ in $\mathfrak{M}$ is defined inductively by taking:

$$\overline{\tau}^{\mathfrak{M}} = U - \tau^{\mathfrak{M}}, \qquad (\tau_1 \sqcap \tau_2)^{\mathfrak{M}} = \tau_1^{\mathfrak{M}} \cap \tau_2^{\mathfrak{M}}, \qquad (\mathbf{I}\tau)^{\mathfrak{M}} = \mathbb{I}\tau^{\mathfrak{M}},$$
$$(\tau_1 \sqcup \tau_2)^{\mathfrak{M}} = \tau_1^{\mathfrak{M}} \cup \tau_2^{\mathfrak{M}} \qquad \text{and} \qquad (\mathbf{C}\tau)^{\mathfrak{M}} = \mathbb{C}\tau^{\mathfrak{M}}.$$

To be able to express how spatial terms τ_1 and τ_2 are related to each other we require (at least) the atomic formula $\tau_1 \sqsubseteq \tau_2$ with the obvious intended meaning: (the extension of) τ_1 is a subset of (the extension of) τ_2. By taking Boolean combinations of such atoms, we arrive at what will be called *spatial formulas* (or $\mathcal{S}4_u$*-formulas*):

$$\varphi \quad ::= \quad \tau_1 \sqsubseteq \tau_2 \quad | \quad \neg\varphi \quad | \quad \varphi_1 \wedge \varphi_2 \quad | \quad \varphi_1 \vee \varphi_2,$$

where the τ_i are spatial terms. Formally, the language of $\mathcal{S}4_u$ as defined above is weaker than the standard one, say, from (Goranko and Passy, 1992). However, one can easily show that they have precisely the same expressive power: see, e.g., (Hughes and Cresswell, 1996) or (Aiello and van Benthem, 2002).

Spatial formulas can be either true or false in topological models. The *truth-relation* $\mathfrak{M} \models \varphi$—a spatial formula φ is true in a topological model $\mathfrak{M}$—is defined in the following way:

- $\mathfrak{M} \models \tau_1 \sqsubseteq \tau_2$ iff $\tau_1^{\mathfrak{M}} \subseteq \tau_2^{\mathfrak{M}}$,
- $\mathfrak{M} \models \neg\varphi$ iff $\mathfrak{M} \not\models \varphi$,
- $\mathfrak{M} \models \varphi_1 \wedge \varphi_2$ iff $\mathfrak{M} \models \varphi_1$ and $\mathfrak{M} \models \varphi_2$,
- $\mathfrak{M} \models \varphi_1 \vee \varphi_2$ iff $\mathfrak{M} \models \varphi_1$ or $\mathfrak{M} \models \varphi_2$.

Clearly, the traditional *universal modalities* $\forall$ and $\exists$ of $\mathcal{S}4_u$ are expressible in the above language: $\forall\tau$ can be regarded as an abbreviation for

$(\top \sqsubseteq \tau)$ and $\exists\tau$ for $\neg(\tau \sqsubseteq \bot)$, where $\top$ and $\bot$ are constant terms denoting the whole space and the empty set, respectively. In what follows we will also freely use two other 'atomic' formulas $\tau_1 = \tau_2$ and $\tau_1 \neq \tau_2$ standing for $(\tau_1 \sqsubseteq \tau_2) \wedge (\tau_2 \sqsubseteq \tau_1)$ and $\neg(\tau_1 = \tau_2)$, respectively.

Say that a spatial formula φ is *satisfiable* (*in a class $\mathcal{K}$ of topological models*) if there is a topological model $\mathfrak{M}$ (from $\mathcal{K}$) such that $\mathfrak{M} \models \varphi$. A spatial formula φ is *satisfiable in a class of topological spaces* if there is a topological model $\mathfrak{M}$ based on a space from this class such that $\mathfrak{M} \models \varphi$.

This seemingly simple spatial language $\mathcal{S}4_u$ can express rather complex relations between sets in topological spaces. For example, the formula

$$(q \sqsubseteq p) \quad \wedge \quad (p \sqsubseteq \mathbf{C}q) \quad \wedge \quad (p \neq \bot) \quad \wedge \quad (\mathbf{I}q = \bot)$$

says that a set q is dense in a nonempty set p, but has no interior. As an example one can take q to be the rationals $\mathbb{Q}$ and p to be $\mathbb{R}$ in the Euclidean space $(\mathbb{R}, \mathbb{I})$.

In the following theorem we collected the most important facts about $\mathcal{S}4_u$; for proofs and discussions see, e.g., (Nutt, 1999; Areces et al., 2000) and references therein.

THEOREM 1.1 (i) *A spatial formula is satisfiable iff it is satisfiable in an Aleksandrov space.*

(ii) $\mathcal{S}4_u$ *enjoys the exponential finite model property in the sense that every satisfiable spatial formula φ is satisfiable in a topological space whose size is at most exponential in the size of φ.*

(iii) *Satisfiability of spatial formulas in topological models is* PSPACE-*complete.*

The language of the modal logic $\mathcal{S}4$ mentioned above coincides with the language of $\mathcal{S}4_u$-terms. Say that a spatial term (= $\mathcal{S}4$-formula) is *satisfiable* if there is a topological model where the term is interpreted as a nonempty set. Although being of the same computational complexity as $\mathcal{S}4$ (which is also PSPACE-complete), the logic $\mathcal{S}4_u$ is more expressive. For example, spatial formulas can distinguish between arbitrary and connected topological spaces (we remind the reader that a topological space is *connected* if its universe cannot be represented as the union of two disjoint nonempty open sets). Consider the formula

$$(\mathbf{C}p \sqsubseteq p) \quad \wedge \quad (p \sqsubseteq \mathbf{I}p) \quad \wedge \quad (p \neq \bot) \quad \wedge \quad (p \neq \top) \tag{1.11}$$

saying that (the extension of) p is both closed and open, nonempty and does not coincide with the whole space. It can only be satisfied in a model based on a disconnected topological space, while all satisfiable

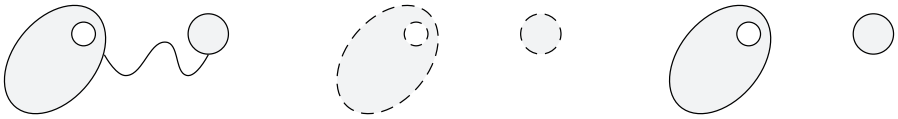


Figure 1.2. Regular closure.

$\mathcal{S}4$-terms are satisfied in connected (e.g., Euclidean) spaces. For we have the following result (McKinsey and Tarski, 1944):

THEOREM 1.2 *An $\mathcal{S}4$-formula is satisfiable iff it is satisfiable in any of (and so in all) $\mathbb{R}^n$, $n > 0$.*

Another example illustrating the expressive power of $\mathcal{S}4_u$ is the formula

$$(p \neq \bot) \quad \wedge \quad (p \sqsubseteq \mathbf{C}\overline{p}) \quad \wedge \quad (\overline{p} \sqsubseteq \mathbf{C}p) \tag{1.12}$$

defining a nonempty set p such that both p and its complement $\overline{p}$ have empty interiors. In fact, the second and the third conjuncts say that both p and $\overline{p}$ consist of boundary points only.

Regions = regular closed sets. In qualitative spatial KR&R, it is quite often assumed that spatial terms can only be interpreted by regular closed (or open) sets of topological spaces (see, e.g., Davis, 1990; Asher and Vieu, 1995; Gotts, 1996). One of the reasons for imposing this restriction is to exclude from consideration such 'pathological' sets as in (1.12). Recall that a set X is *regular closed* if $X = \mathbb{C}\mathbb{I}X$, which clearly does not hold for any set satisfying (1.12). Another reason is to ensure that the space occupied by a physical body is homogeneous in the sense that it does not contain parts of 'different dimensionality.' For example, the one-dimensional curve in Fig. 1.2 disappears from the subset X of the Euclidean plane $(\mathbb{R}^2, \mathbb{I})$ if we form the set $\mathbb{C}\mathbb{I}X$. The latter is regular closed because $\mathbb{C}\mathbb{I}\mathbb{C}\mathbb{I}X = \mathbb{C}\mathbb{I}X$, for every X and every topological space.

In this section, we will consider several fragments of $\mathcal{S}4_u$ dealing with *regular closed sets*. From now on we will call such sets *regions*.

$\mathcal{RCC}$-8. Perhaps the best known language devised for speaking about regions is $\mathcal{RCC}$-8 which was introduced in the area of Geographical Information Systems (Egenhofer and Franzosa, 1991; Smith and Park, 1992) and as a decidable subset of Region Connection Calculus $\mathcal{RCC}$

(Randell et al., 1992). The syntax of $\mathcal{RCC}$-8 contains *region variables* $\mathsf{r}, \mathsf{s}, \ldots$ and eight binary predicates:

- $\mathsf{DC}(\mathsf{r}, \mathsf{s})$ — regions r and s are disconnected,
- $\mathsf{EC}(\mathsf{r}, \mathsf{s})$ — r and s are externally connected,
- $\mathsf{EQ}(\mathsf{r}, \mathsf{s})$ — r and s are equal,
- $\mathsf{PO}(\mathsf{r}, \mathsf{s})$ — r and s partially overlap,
- $\mathsf{TPP}(\mathsf{r}, \mathsf{s})$ — r is a tangential proper part of s,
- $\mathsf{NTPP}(\mathsf{r}, \mathsf{s})$ — r is a nontangential proper part of s,
- the inverses of the last two—$\mathsf{TPPi}(\mathsf{r}, \mathsf{s})$ and $\mathsf{NTPPi}(\mathsf{r}, \mathsf{s})$,

which can be combined using the Boolean connectives.

The arguments of the $\mathcal{RCC}$-8 predicates, that is, region variables, are interpreted by regular closed sets—i.e., regions—of topological spaces. The following was shown in (Renz, 1998; Renz and Nebel, 1999):

THEOREM 1.3 (i) *Every satisfiable $\mathcal{RCC}$-8 formula is satisfiable in any of $\mathbb{R}^n$, for $n \geq 1$ (with region variables interpreted by connected regions only, if $n \geq 3$).*

(ii) *The satisfiability problem for $\mathcal{RCC}$-8 formulas in topological models is* NP-*complete.*

The expressive power of $\mathcal{RCC}$-8 is rather limited. It only operates with 'simple' regions and does not distinguish between connected and disconnected ones, regions with and without holes, etc. (Egenhofer and Herring, 1991). Nor can $\mathcal{RCC}$-8 represent complex relations between more than two regions. Consider, for example, three countries (say, Russia, Lithuania and Poland) such that not only each one of them is adjacent to the others, but there is a point where all the three meet (see Fig. 1.3). It can easily be shown that a ternary predicate like

$$\mathsf{EC3}(\mathit{Russia}, \mathit{Lithuania}, \mathit{Poland}) \tag{1.13}$$

cannot be expressed in $\mathcal{RCC}$-8.

To analyse possible ways of extending $\mathcal{RCC}$-8, it will be convenient to view it as a fragment of $\mathcal{S}4_u$ (that $\mathcal{RCC}$-8 can be embedded into $\mathcal{S}4_u$ was first shown by Bennett (1994); we present here a slightly different embedding and the purpose of changes will become clear in the context of $\mathcal{BRCC}$-8 and $\mathcal{RC}$). Observe first that, for every spatial variable p, the spatial term

$$\mathbf{CI}p \tag{1.14}$$

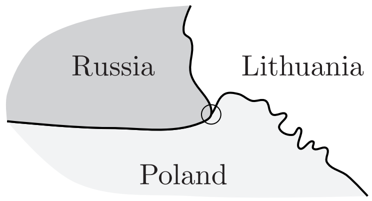


Figure 1.3. Russia, Lithuania and Poland.

is interpreted as a region (i.e., a regular closed set) in every topological model. So with every region variable r of $\mathcal{RCC}$-8 we can associate the spatial term $\varrho_\mathsf{r} = \mathbf{CI}p_\mathsf{r}$, where p_r is a spatial variable representing r, and then translate the $\mathcal{RCC}$-8 predicates into spatial formulas by taking

$$\begin{aligned}
\mathsf{EC}(\mathsf{r},\mathsf{s}) &= \neg(\varrho_\mathsf{r} \sqcap \varrho_\mathsf{s} = \bot) \ \wedge\ (\mathbf{I}\varrho_\mathsf{r} \sqcap \mathbf{I}\varrho_\mathsf{s} = \bot),\\
\mathsf{DC}(\mathsf{r},\mathsf{s}) &= (\varrho_\mathsf{r} \sqcap \varrho_\mathsf{s} = \bot),\\
\mathsf{EQ}(\mathsf{r},\mathsf{s}) &= (\varrho_\mathsf{r} \sqsubseteq \varrho_\mathsf{s}) \ \wedge\ (\varrho_\mathsf{s} \sqsubseteq \varrho_\mathsf{r}),\\
\mathsf{PO}(\mathsf{r},\mathsf{s}) &= \neg(\mathbf{I}\varrho_\mathsf{r} \sqcap \mathbf{I}\varrho_\mathsf{s} = \bot) \ \wedge\ \neg(\varrho_\mathsf{r} \sqsubseteq \varrho_\mathsf{s}) \ \wedge\ \neg(\varrho_\mathsf{s} \sqsubseteq \varrho_\mathsf{r}),\\
\mathsf{TPP}(\mathsf{r},\mathsf{s}) &= (\varrho_\mathsf{r} \sqsubseteq \varrho_\mathsf{s}) \ \wedge\ \neg(\varrho_\mathsf{s} \sqsubseteq \varrho_\mathsf{r}) \ \wedge\ \neg(\varrho_\mathsf{r} \sqsubseteq \mathbf{I}\varrho_\mathsf{s}),\\
\mathsf{NTPP}(\mathsf{r},\mathsf{s}) &= (\varrho_\mathsf{r} \sqsubseteq \mathbf{I}\varrho_\mathsf{s}) \ \wedge\ \neg(\varrho_\mathsf{s} \sqsubseteq \varrho_\mathsf{r})
\end{aligned}$$

(TPPi and NTPPi are the mirror images of TPP and NTPP, respectively). It should be clear that as a result we obtain the following:

Theorem 1.4 *An $\mathcal{RCC}$-8 formula is satisfiable in a topological space iff its translation into $\mathcal{S}4_u$ defined above is satisfiable in the same topological space.*

This translation shows that in $\mathcal{RCC}$-8 any two regions can be related only in terms of truth/falsity of atomic spatial formulas of the form

$$(\varrho_1 \sqcap \varrho_2 = \bot), \qquad (\mathbf{I}\varrho_1 \sqcap \mathbf{I}\varrho_2 = \bot), \qquad (\varrho_1 \sqsubseteq \varrho_2) \quad \text{and} \quad (\varrho_1 \sqsubseteq \mathbf{I}\varrho_2),$$

where ϱ_1 and ϱ_2 are *atomic region terms*, that is, spatial terms of the form (1.14). This observation suggests two ways of increasing the expressive power of $\mathcal{RCC}$-8:

(i) by allowing the formation of *complex region terms* from atomic region terms, and

(ii) by allowing more ways of relating them (i.e., richer languages of atomic spatial formulas).

From now on we will not distinguish between a region variable r and the atomic region term ϱ_r representing it, and use expressions like $\mathsf{DC}(\mathsf{r},\mathsf{s})$ and $(\varrho_\mathsf{r} \sqcap \varrho_\mathsf{s} = \bot)$ as synonymous.

$\mathcal{BRCC}$-8. The language $\mathcal{BRCC}$-8 of (Wolter and Zakharyaschev, 2000; see also Balibiani et al., 2004) extends $\mathcal{RCC}$-8 in direction (i). It uses the same eight binary predicates as $\mathcal{RCC}$-8 and allows not only atomic regions but also their intersections, unions and complements. For instance, in $\mathcal{BRCC}$-8 we can express the fact that a region (say, the Swiss Alps) is the intersection of two other regions (Switzerland and the Alps in this case):

$$\mathsf{EQ}(\mathit{SwissAlps}, \mathit{Switzerland} \sqcap \mathit{Alps}). \tag{1.15}$$

We can embed $\mathcal{BRCC}$-8 into $\mathcal{S}4_u$ by using almost the same translation as in the case of $\mathcal{RCC}$-8. The only difference is that now, since Boolean combinations of regular closed sets are not necessarily regular closed, we should prefix compound spatial terms with **CI**. In this way we can obtain, for example, the spatial term

$$\mathbf{CI}\,(\mathit{Switzerland} \sqcap \mathit{Alps})$$

representing the Swiss Alps. In the same manner we can treat other set-theoretic operations, which leads us to the following definition of *Boolean region terms*:

$$\varrho \quad ::= \quad \mathbf{CI}p \quad | \quad \mathbf{CI}\overline{\varrho} \quad | \quad \mathbf{CI}(\varrho_1 \sqcap \varrho_2) \quad | \quad \mathbf{CI}(\varrho_1 \sqcup \varrho_2).$$

Thus $\mathcal{BRCC}$-8 can be regarded as a syntactically restricted subset of $\mathcal{S}4_u$-formulas. It follows from the above definition that Boolean region terms denote precisely the members of the well-known Boolean algebra of regular closed sets.

It is of interest to note that Boolean region terms do not increase the complexity of reasoning in arbitrary topological models: the satisfiability problem for $\mathcal{BRCC}$-8 formulas is still NP-complete. However, it becomes PSpace-complete if all intended models are based on connected spaces ($\mathcal{BRCC}$-8 can distinguish between connected and disconnected spaces because we can express that regions r_1 and r_2 are nonempty non-tangential proper parts of a region $\mathsf{s} \neq \top$, and the union of r_1 and r_2 is precisely s:

$$\bigwedge_{i=1,2} \Big(\neg\mathsf{DC}(\mathsf{r}_i, \mathsf{r}_i) \land \mathsf{NTTP}(\mathsf{r}_i, \mathsf{s})\Big) \quad \land \quad \mathsf{NTTP}(\mathsf{s}, \mathsf{s}') \;\land\; \mathsf{EQ}(\mathsf{r}_1 \sqcup \mathsf{r}_2, \mathsf{s}).$$

To satisfy this formula, it suffices to take a discrete topological space with three points. But if these constraints are satisfied then both s and its complement are open and nonempty, which means that the space cannot be connected.)

On the other hand, $\mathcal{BRCC}$-8 allows some restricted comparisons of more than two regions as, e.g., in (1.15). Nevertheless, as we shall see below, ternary relations like (1.13) are still unavailable in $\mathcal{BRCC}$-8: they require different ways of comparing regions; see (ii).

$\mathcal{RC}$. Egenhofer and Herring (1991), proposed to relate any *two* regions in terms of the 9-intersections—3×3-matrix specifying emptiness/nonemptiness of all (nine) possible intersections of the interiors, boundaries and exteriors of the regions. Recall that, for a region X, these three disjoint parts of the space $(U, \mathbb{I})$ can be represented as

$$\mathbb{I}X, \qquad X \cap (U - \mathbb{I}X) \qquad \text{and} \qquad U - X,$$

respectively. By generalising this approach to any finite number of regions, we obtain the fragment $\mathcal{RC}$ of $\mathcal{S}4_u$: its terms are defined as follows

$$\begin{array}{lcl} \varrho & ::= & \mathbf{CI}p \quad | \quad \mathbf{CI}\overline{\varrho} \quad | \quad \mathbf{CI}(\varrho_1 \sqcap \varrho_2) \quad | \quad \mathbf{CI}(\varrho_1 \sqcup \varrho_2), \\ \tau & ::= & \varrho \quad | \quad \mathbf{I}\varrho \quad | \quad \overline{\tau} \quad | \quad \tau_1 \sqcap \tau_2 \quad | \quad \tau_1 \sqcup \tau_2, \end{array}$$

and spatial formulas are constructed from atoms of the form $\tau_1 \sqsubseteq \tau_2$ using the Booleans (as in the full $\mathcal{S}4_u$). In other words, in $\mathcal{RC}$ we can define relations between regions in terms of inclusions of sets formed by using arbitrary set-theoretic operations on regions and their interiors. However, nested applications of the topological operators are not allowed (an example where such applications are required can be found below).

Clearly, both $\mathcal{RCC}$-8 and $\mathcal{BRCC}$-8 are fragments of $\mathcal{RC}$. Moreover, unlike $\mathcal{BRCC}$-8, the language of $\mathcal{RC}$ allows us to consider more complex relations between regions. For instance, the ternary relation required in (1.13) can now be defined as follows:

$$\begin{aligned} \mathsf{EC3}(\mathsf{r}_1, \mathsf{r}_2, \mathsf{r}_3) = \; & \neg(\varrho_{\mathsf{r}_1} \sqcap \varrho_{\mathsf{r}_2} \sqcap \varrho_{\mathsf{r}_3} = \bot) \;\; \wedge \;\; (\mathbf{I}\varrho_{\mathsf{r}_1} \sqcap \mathbf{I}\varrho_{\mathsf{r}_2} = \bot) \;\; \wedge \\ & (\mathbf{I}\varrho_{\mathsf{r}_2} \sqcap \mathbf{I}\varrho_{\mathsf{r}_3} = \bot) \;\; \wedge \;\; (\mathbf{I}\varrho_{\mathsf{r}_3} \sqcap \mathbf{I}\varrho_{\mathsf{r}_1} = \bot). \end{aligned}$$

Another, more abstract, example is the formula

$$\varrho_1 \sqcap \dots \sqcap \varrho_i \;\sqcap\; \mathbf{I}\varrho'_1 \sqcap \dots \sqcap \mathbf{I}\varrho'_j \;\sqcap\; \overline{\varrho''_1} \sqcap \dots \sqcap \overline{\varrho''_k} \;\sqcap\; \overline{\mathbf{I}\varrho'''_1} \sqcap \dots \sqcap \overline{\mathbf{I}\varrho'''_n} \;\neq \bot$$

which says that

> *regions $\varrho_1, \dots, \varrho_i$ <u>meet</u> somewhere <u>inside</u> the region occupied jointly by all $\varrho'_1, \dots, \varrho'_j$, but <u>outside</u> the regions $\varrho''_1, \dots, \varrho''_k$ and <u>not inside</u> $\varrho'''_1, \dots, \varrho'''_n$.*

Although $\mathcal{RC}$ is more expressive than both $\mathcal{RCC}$-8 and $\mathcal{BRCC}$-8, reasoning in this language is still of the same computational complexity (Gabelaia et al., 2005a):

Theorem 1.5 *The satisfiability problem for $\mathcal{RC}$-formulas in arbitrary topological models is* NP*-complete.*

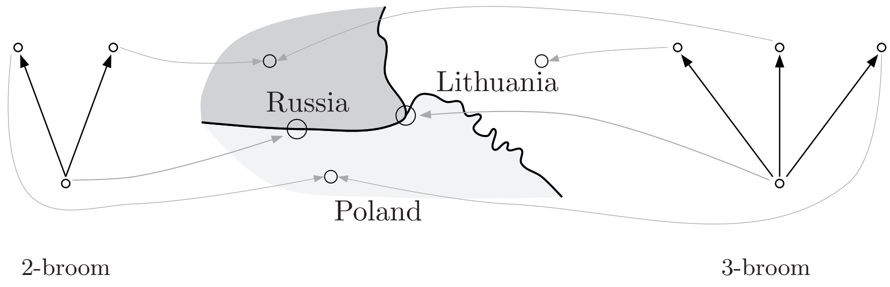


Figure 1.4. Satisfying $\mathsf{EC}(\mathit{Russia}, \mathit{Poland})$ and $\mathsf{EC3}(\mathit{Russia}, \mathit{Lithuania}, \mathit{Poland})$ in 2- and 3-brooms.

The proof follows from the fact that every satisfiable $\mathcal{RC}$-formula can be satisfied in an Aleksandrov space that is induced by a disjoint union of n-*brooms*—i.e., quasi-orders of the form depicted in Fig. 1.4. Topological spaces of this kind have a rather primitive structure satisfying the following property:

(rc) only the roots of n-brooms can be boundary points, and the minimal neighbourhood of every boundary point—i.e., the n-broom containing this point—must contain *at least one* internal point and *at least one* external point.

For example, spatial formula (1.12) cannot be satisfied in a model with this property, and so it is not in $\mathcal{RC}$.

Given a satisfiable $\mathcal{RC}$-formula φ, we can always satisfy it in a model of this kind the size of which is a polynomial (in fact, quadratic) in the length of φ, and so we have a nondeterministic polynomial time algorithm. Actually, the proof is a straightforward generalisation of the complexity proof for $\mathcal{BRCC}$-8 (Wolter and Zakharyaschev, 2000): the only difference is that in the case of $\mathcal{BRCC}$-8 it was sufficient to consider 2-brooms (which were called *forks*). This means, in particular, that ternary relation (1.13)—which is satisfiable only in a model with an n-broom, for $n \geq 3$—is indeed not expressible in $\mathcal{BRCC}$-8 (see Fig. 1.4).

Remark 1.6 In topological terms, n-brooms are examples of so-called *door spaces* where every subset is either open or closed. However, the modal theory of n-brooms defines a wider and more interesting topological class known as *submaximal spaces* in which every dense subset is open. Submaximal spaces have been around since the early 1960s and have generated interesting and challenging problems in topology. For

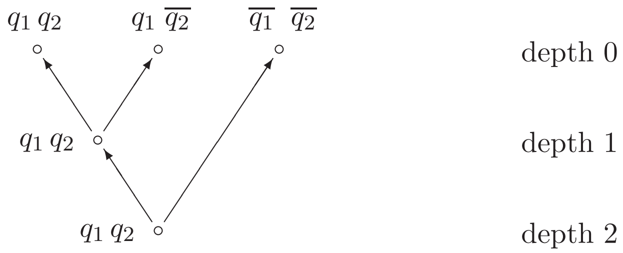


Figure 1.5. Model satisfying formula (1.16).

a survey and a systematic study of these spaces see (Arhangel'skii and Collins, 1995) and references therein.

$\mathcal{RC}^{max}$. One could go even further in direction (ii) and impose no restrictions whatsoever on the ways of relating Boolean atomic region terms. This leads us to the *maximal* fragment $\mathcal{RC}^{max}$ of $\mathcal{S}4_u$ in which spatial terms are interpreted by regular closed sets. The syntax of its spatial terms is defined as follows:

$$\tau \quad ::= \quad \mathbf{CI}p \quad | \quad \overline{\tau} \quad | \quad \tau_1 \sqcap \tau_2 \quad | \quad \tau_1 \sqcup \tau_2 \quad | \quad \mathbf{I}\tau \quad | \quad \mathbf{C}\tau$$

and spatial formulas are constructed as in $\mathcal{S}4_u$. To understand the difference between $\mathcal{RC}^{max}$ and $\mathcal{RC}$, consider the following $\mathcal{RC}^{max}$-formula

$$\big(\mathbf{CI}q_1 \sqcap \overline{\mathbf{ICI}q_1} \neq \bot\big) \quad \wedge \quad \big((\mathbf{CI}q_1 \sqcap \overline{\mathbf{ICI}q_1}) \quad \sqsubseteq \quad \mathbf{C}(\mathbf{ICI}q_1 \sqcap \mathbf{CI}q_2 \sqcap \overline{\mathbf{ICI}q_2})\big). \tag{1.16}$$

It says that the boundary of $\mathbf{CI}q_1$ is not empty and that every neighbourhood of every point in this boundary contains an internal point of $\mathbf{CI}q_1$ that belongs to the boundary of $\mathbf{CI}q_2$ (compare with property (rc) above). The simplest Aleksandrov model satisfying this formula is of depth 2 (whereas n-brooms are of depth 1); it is shown in Fig. 1.5.

The price we have to pay for this expressivity is that the complexity of $\mathcal{RC}^{max}$ is the same as that of full $\mathcal{S}4_u$ (Gabelaia et al., 2005a):

THEOREM 1.7 *The satisfiability problem for $\mathcal{RC}^{max}$-formulas is* PSPACE-*complete.*

This logic can also be regarded as a fragment of $\mathcal{S}4_u$ with all variables interpreted by regular closed sets.

$\mathcal{S}4_u$ with component counting. There are many ways of increasing the expressive power of $\mathcal{S}4_u$ itself. For instance, Pratt-Hartmann (2002) proposes an extension with component counting. We remind the reader that a subset X of a topological space $(U, \mathbb{I})$ is said to be *connected* if there do not exist two sets $Y_1, Y_2 \subseteq U$ such that $X \subseteq Y_1 \cup Y_2$, $X \cap Y_i \neq \emptyset$, for $i = 1, 2$, and $X \cap \mathbb{C}Y_1 \cap \mathbb{C}Y_2 = \emptyset$. Intuitively, connected sets can be thought of as consisting of 'one piece.' Then a *component* of a set X is a maximal connected subset of X. For example, the subset X of the Euclidean plane $(\mathbb{R}^2, \mathbb{I})$ in Fig. 1.2 has only one component and so is connected, whereas its regular closure $\mathbb{C}\mathbb{I}X$ is not connected and has two components.

The language $\mathcal{TCC}$ of (Pratt-Hartmann, 2002) extends the set of atomic spatial formulas of $\mathcal{S}4_u$ with the following construct:

$$c^{\leq k}\tau,$$

where τ is a spatial term (as on p. 14) and $k \in \mathbb{N}$. The formula $c^{\leq k}\tau$ is true iff the interpretation of τ has at most k components. In particular, $c^{\leq 1}\tau$ is true iff τ is connected and $\neg c^{\leq k}\tau$ is true iff τ has at least $k+1$ components (sometimes denoted by $c^{\geq k+1}\tau$). This extension turns out to be quite expressive: for example, the $\mathcal{TCC}$-formula

$$\big(c^{\leq 1}p_1 \quad \wedge \quad c^{\leq 1}p_2 \quad \wedge \quad (p_1 \sqcap p_2 \neq \bot)\big) \quad \rightarrow \quad c^{\leq 1}(p_1 \sqcup p_2)$$

says that the union of two connected intersecting sets is also connected (here, $\varphi_1 \rightarrow \varphi_2$ is an abbreviation for $\neg\varphi_1 \vee \varphi_2$). As usual, the increased expressivity results in higher complexity. The following was proved by Pratt-Hartmann (2002):

THEOREM 1.8 *The satisfiability problem for $\mathcal{TCC}$-formulas in topological models is* NEXPTIME*-complete for the binary coding of the numerical parameters.*

To conclude this section, we summarise the inclusions between the (propositional) spatial languages introduced above:

$$\mathcal{RCC}\text{-}8 \quad \subsetneqq \quad \mathcal{BRCC}\text{-}8 \quad \subsetneqq \quad \mathcal{RC} \quad \subsetneqq \quad \mathcal{RC}^{max} \quad \subsetneqq \quad \mathcal{S}4_u \quad \subsetneqq \quad \mathcal{TCC}.$$

3.3 Logics of distance spaces

Suppose now that we are interested in spatial logics that are capable of reasoning about spatial models based on various *distance spaces*, i.e., models of the form

$$\mathfrak{M} = (\mathfrak{D}, p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}, \dots), \tag{1.17}$$

where $\mathfrak{D} = (\Delta, d)$ is a distance space introduced in Sec. 3.1. If $\mathfrak{D}$ is actually a metric space then we can still use $\mathcal{S}4_u$ or its fragments interpreted on the topological space induced by $\mathfrak{D}$. However, the topological interior and closure operators $\mathbb{I}_d$ and $\mathbb{C}_d$ only deal with points that are 'infinitely close' to the given spatial object (cf. the definitions in Sec 3.1). Being equipped with the distance function over the space, we can extend (or replace) qualitative topological reasoning by means of reasoning about distances between spatial objects. In addition to (or instead of) operators interpreted by the topological interior and closure, we can introduce operators capable of expressing, say, that the distance from a region X to a region Y is not more than 17.

Following the 'operator-based' approach from topological logic, we arrive then to languages with 'bounded quantifiers' like $\exists^{<a}$ 'somewhere at distance $< a$' or $\forall_{>a}^{\leq b}$ 'everywhere within distance d for $a < d < b$,' where a and b are some numbers from $\mathbb{R}^{\geq 0}$ (or rather $\mathbb{Q}^{\geq 0}$ to avoid the problem of representing the reals).

Given a spatial model $\mathfrak{M}$ of the form (1.17), we interpret such operators in the natural way:

$$\begin{aligned}
(\exists^{<a}\tau)^{\mathfrak{M}} &= \{x \in \Delta \mid \exists y\ (d(x,y) < a \wedge y \in \tau^{\mathfrak{M}})\},\\
(\exists^{>a}\tau)^{\mathfrak{M}} &= \{x \in \Delta \mid \exists y\ (d(x,y) > a \wedge y \in \tau^{\mathfrak{M}})\},\\
(\forall_{>a}^{\leq b}\tau)^{\mathfrak{M}} &= \{x \in \Delta \mid \forall y\ (a < d(x,y) < b \rightarrow y \in \tau^{\mathfrak{M}})\},\\
&\text{etc.}
\end{aligned}$$

Before introducing formal languages based on these operators, it is worth having a closer look at some of them. One might be tempted to assume that the 'doughnut'-operator $\exists_{>a}^{\leq b}$ can be expressed via $\exists^{<b}$ and $\exists^{>a}$ by the equivalence $\exists_{>a}^{\leq b}\tau = \exists^{<b}\tau \sqcap \exists^{>a}\tau$. Fig. 1.6 shows that this is not the case. In the figure, we depict the regions $\exists^{<2}X$, $\exists^{>1.9}X$ and $\exists_{>1.9}^{\leq 2}X$ for the region X consisting of the two black boxes. In particular, of all points on the plane only those in the white diamond in Fig. 1.6 (b) do not belong to $\exists^{>1.9}X$. $\exists_{>1.9}^{\leq 2}X$ is $\exists^{<2}X$ without the three white areas in Fig. 1.6 (c). As follows from this example, $\exists_{>1.9}^{\leq 2}X \neq \exists^{<2}X \sqcap \exists^{>1.9}X$.

In our discussion of languages for distance spaces we will formulate most results for metric spaces only. The reader is invited to consult the literature cited below to obtain detailed information about the behaviour of those languages over more general distance spaces and over Euclidean spaces.

Full 'modal' logic of distance spaces. The logic $\mathcal{MS}$ of distance spaces with the operators $\exists^{=a}$, $\exists^{<a}$, $\exists^{>a}$, $\exists_{>a}^{\leq b}$ (and their duals $\forall^{=a}$, $\forall^{<a}$, etc.) interpreted as defined above was introduced and analysed in (Kutz et al., 2003). Formally the *spatial terms* of this logic are defined as

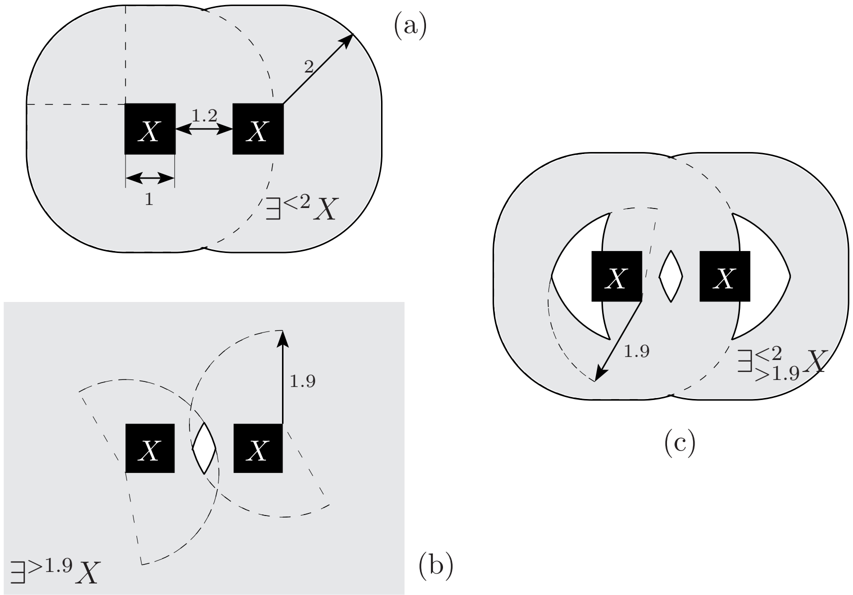


Figure 1.6. Distance operators.

follows:

$$\tau \quad ::= \quad p_i \mid \{\ell_i\} \mid \overline{\tau} \mid \tau_1 \sqcap \tau_2 \mid \exists^{=a}\tau \mid \exists^{<a}\tau \mid \exists^{>a}\tau \mid \exists^{<b}_{>a}\tau,$$

where $a, b \in \mathbb{Q}^{\geq 0}$ with $a < b$, and the ℓ_i are *location constants* (or *nominals*) interpreted by single points, so that the $\{\ell_i\}$ are interpreted by singleton sets. As before, the *formulas* are constructed from atoms of the form $\tau_1 \sqsubseteq \tau_2$ using the Booleans ($\neg$, $\wedge$, etc.); we use $\tau_1 = \tau_2$ as an abbreviation for $\tau_1 \sqsubseteq \tau_2 \wedge \tau_2 \sqsubseteq \tau_1$.

Considering first the expressive power of $\mathcal{MS}$, one can show that over models of the form (1.17) based on metric spaces it is as expressive as the two-variable fragment of first-order logic with equality, individual constants, unary predicate symbols $p_i(x)$ corresponding to spatial variables, and binary relation symbols

$$d(x, y) < a, \qquad d(x, y) = a,$$

for $a \in \mathbb{Q}^{\geq 0}$, which are interpreted in metric spaces in the obvious way (Kutz et al., 2003). Moreover, the translation between the two languages is effective.

This expressive completeness result indicates already that $\mathcal{MS}$ is indeed quite expressive. Analysing its computational properties, Kutz et al.

(2003) proved that the satisfiability problem for $\mathcal{MS}$-formulas over arbitrary metric spaces is undecidable. In fact, the following much stronger theorem holds:

THEOREM 1.9 *No algorithm can decide whether an arbitrarily given $\mathcal{MS}$-formula all of whose distance operators are of the form $\exists^{<a}_{>0}$, for $a \in \mathbb{N}^{>0}$, is satisfiable in a model based on a metric space.*

The proof of this result is based on the observation that one can 'enforce' the $\mathbb{N} \times \mathbb{N}$ grid using the 'punctured' centres of circles provided by $\exists^{<a}_{>0}$.

It is worth noting that in contrast to the undecidability result above, the satisfiability problem for $\mathcal{MS}$-formulas in arbitrary distance spaces and symmetric distances spaces is *decidable*. This observation follows from the standard translation of $\mathcal{MS}$ into the two-variable fragment of first-order logic (which is decidable in NExpTime) and the fact that reflexivity and symmetry of relations can be expressed in first-order logic using two variables only. This argument does not work for satisfiability in metric spaces because the triangle inequality cannot be expressed in first-order logic with two variables.

The logic with $\exists^{\leq a}$ and $\exists^{>a}$. Without the doughnut operators $\mathcal{MS}$ often becomes decidable and has the finite model property with respect to the intended models, that is, a formula satisfiable in a (possibly infinite) metric model is satisfiable in a finite metric model. For example, denote by $\mathcal{MS}^{\leq,>}$ the fragment of $\mathcal{MS}$ with *spatial terms* of the form

$$\tau \quad ::= \quad p_i \quad | \quad \{\ell_i\} \quad | \quad \overline{\tau} \quad | \quad \tau_1 \sqcap \tau_2 \quad | \quad \exists^{\leq a}\tau \quad | \quad \exists^{>a}\tau,$$

where $a \in \mathbb{Q}^{\geq 0}$. Kutz et al. (2003) proved that this logic has the finite model property and that the satisfiability problem for its formulas is decidable in NExpTime under the unary coding of parameters. Actually, this result was improved in (Wolter and Zakharyaschev, 2005b):

THEOREM 1.10 *The satisfiability problem for $\mathcal{MS}^{\leq,>}$-formulas in metric spaces is* ExpTime*-complete under the unary coding of numeric parameters in distance operators.*

The complexity of $\mathcal{MS}^{\leq,>}$-satisfiability under the binary coding of parameters remains an open research problem.

The logic with $\exists^{\leq a}$ and $\exists^{<a}$. Another interesting fragment of $\mathcal{MS}$ is based on the operators $\exists^{<a}$ and $\exists^{\leq a}$ (Wolter and Zakharyaschev, 2003). The *spatial terms* of the resulting logic $\mathcal{MS}^{\leq,<}$ are defined as follows:

$$\tau \quad ::= \quad p_i \quad | \quad \{\ell_i\} \quad | \quad \overline{\tau} \quad | \quad \tau_1 \sqcap \tau_2 \quad | \quad \exists^{\leq a}\tau \quad | \quad \exists^{<a}\tau,$$

where $a \in \mathbb{Q}^{>0}$ (by including 0 in the parameter set we would not increase the expressive power of the language, but some formulations may become awkward). The logic $\mathcal{MS}^{\leq,<}$ has the finite model property, and ExpTime-completeness can now be proved even for the binary coding of parameters:

Theorem 1.11 *The satisfiability problem for $\mathcal{MS}^{\leq,<}$-formulas in metric spaces is* ExpTime*-complete under both unary and binary coding of parameters in distance operators.*

The crucial observation in the proof of this result is that (modulo the interpretation of nominals) the logic turns out to be complete with respect to *tree metric spaces*, a feature not shared by the languages considered above. Completeness with respect to tree metric spaces makes this language also amenable to tableau-based decision procedures (Wolter and Zakharyaschev, 2003) which are not yet available for the language $\mathcal{MS}^{\leq,>}$. An intriguing fact is that the fragments with only strict operators $\exists^{<a}$ and only non-strict ones $\exists^{\leq a}$ behave similarly in the following sense:

Theorem 1.12 *Let φ be a formula whose only distance operators are of the form $\exists^{<a}$. Let φ' be the result of replacing occurrences of $\exists^{<a}$ in φ with $\exists^{\leq a}$. Then φ is satisfiable in a metric space iff φ' is satisfiable in a metric space.*

Of course, in the theorem above one cannot always choose the same metric space. In fact, it is worth noting that the language $\mathcal{MS}^{\leq,<}$ is properly more expressive than its fragments with only the operators $\exists^{<a}$ and $\exists^{\leq a}$, respectively. Namely, using both operators we can say that the distance between two sets p and q is precisely a:

$$\big(p \sqcap \exists^{\leq a}q \neq \bot\big) \quad \wedge \quad \big(p \sqcap \exists^{<a}q = \bot\big).$$

'Modal' logics of metric and topology. The logics of metric spaces we have considered so far can represent certain knowledge about distances between spatial objects, but are not suitable for reasoning about the induced topology. To see this for $\mathcal{MS}^{\leq,>}$ and $\mathcal{MS}^{\leq,<}$, recall that both of them have the finite model property: every satisfiable formula is satisfiable in a finite metric space. Thus, these languages cannot distinguish between finite and infinite metric spaces. On the other hand, every finite metric space induces the trivial topology in which *every set* is both closed and open. It follows that every satisfiable formula is satisfiable in a metric space with a trivial topology and that therefore the languages cannot represent anything interesting about the topology induced by a metric space. A similar argument can be used to show that $\mathcal{MS}$ itself cannot be used for representing topological knowledge.

To be able to reason about both metric and topology we can combine one of the metric logics above with one of the topo-logics considered in Sec. 3.2. Only one such combination has been investigated in detail so far: the extension of $\mathcal{S}4_u$ with the metric operators $\exists^{<a}$ and $\exists^{\leq a}$ of $\mathcal{MS}^{\leq,<}$ (Wolter and Zakharyaschev, 2005a). The terms of the resulting language we call $\mathcal{MT}$ are defined as follows:

$$\tau \quad ::= \quad p_i \quad | \quad \overline{\tau} \quad | \quad \tau_1 \sqcap \tau_2 \quad | \quad \exists^{\leq a}\tau \quad | \quad \exists^{<a}\tau \quad | \quad \mathbf{I}\tau \quad | \quad \mathbf{C}\tau,$$

where $a \in \mathbb{Q}^{>0}$. Notice that $\mathcal{MT}$ does not contain contain nominals $\{\ell_i\}$. Although it would be definitely useful to have nominals in the language, we do not include them into the signature because nothing is known about the algorithmic properties of $\mathcal{MT}$ extended by nominals. Unlike its parts $\mathcal{S}4_u$ and $\mathcal{MS}^{\leq,<}$, the logic $\mathcal{MT}$ does not have the finite model property with respect to metric spaces because the topology induced by a finite metric space is trivial. For example, the term

$$p \sqcap \mathbf{C}\overline{p}$$

is not satisfiable in any finite metric model, yet it is satisfiable in every Euclidean space.

It turns out, however, that the intended metric models for this logic can be represented in the form of relational structures (or Kripke frames), which can be regarded as partial descriptions of metric models. This representation theorem—in fact a generalisation of the McKinsey and Tarski (1944) representation theorem for topological spaces—reduces reasoning with infinite metric models to reasoning with finite relational models and can be used to show the following

Theorem 1.13 *The satisfiability problem for $\mathcal{MT}$-formulas in metric spaces is* ExpTime-*complete under the binary coding of parameters.*

To understand the interaction between the topological and distance operators, it is worth taking a look at the axioms required to describe this interaction. It turns out that to axiomatise the $\mathcal{MT}$-formulas that are valid in all metric models, we need the axioms governing the behaviour of the distance operators, those for the topological operators, and only two axioms where both are involved:

$$\mathbf{C}\tau \sqsubseteq \exists^{<a}\tau,$$
$$\exists^{<a}\mathbf{C}\tau \sqsubseteq \exists^{<a}\tau.$$

The logic $\mathcal{MT}$ is also decidable over the real line, where it has been considered in the framework of reasoning about real-time systems (Hirshfeld and Rabinovich, 1999). It becomes undecidable, however, when we take $\mathbb{R}^2$ as the intended metric space.

Closer operator. The representation of knowledge about distances in (fragments of) $\mathcal{MS}$ is restricted to absolute distances. In particular, in $\mathcal{MS}$ it is not possible to compare distances between spatial objects without estimating the absolute values for the distances. A purely comparative approach to representing and reasoning about distance spaces would need predicates like 'X is closer to Y than it is to Z' which are quite common in our everyday life ('the body was in the middle of the room, rather closer to the door than to the window'). In the framework of spatial logics we have considered so far this predicate can be represented using the binary *closer operator* $\leftleftarrows$ with the following interpretation in distance models $\mathfrak{M} = (\mathfrak{D}, p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}, \dots)$:

$$(\tau_1 \leftleftarrows \tau_2)^{\mathfrak{M}} \quad = \quad \{x \in \Delta \mid d(x, \tau_1^{\mathfrak{M}}) < d(x, \tau_2^{\mathfrak{M}})\}. \tag{1.18}$$

In other words, $\tau_1 \leftleftarrows \tau_2$ is (interpreted by) the set containing those objects of Δ that are 'closer' (or 'more similar') to τ_1 than to τ_2. Formally, the terms of the language $\mathcal{CSL}$ of comparative distances (or similarity) are defined as follows:

$$\tau \quad ::= \quad p_i \quad \mid \quad \overline{\tau} \quad \mid \quad \tau_1 \sqcap \tau_2 \quad \mid \quad \tau_1 \leftleftarrows \tau_2.$$

The language $\mathcal{CSL}$ turns out to be quite powerful. Using it we can express the interior (and so the closure) operator by taking

$$\mathbf{I}\tau \;=\; \top \leftleftarrows \overline{\tau}.$$

Indeed, by the definition above, we have

$$(\mathbf{I}\tau)^{\mathfrak{M}} \;=\; \{x \in \Delta \mid d(x, \Delta - \tau^{\mathfrak{M}}) > 0\}.$$

We can also express the existential (and so the universal) modality:

$$\exists\tau \;=\; \tau \leftleftarrows \bot$$

because $d(x, \emptyset) = \infty$. Thus, $\mathcal{CSL}$ contains $\mathcal{S}4_u$ and can be regarded as a qualitative spatial formalism for reasoning about comparative distances and topology. One more interesting operator is

$$\tau_1 \leftrightarrows \tau_2 \;=\; \overline{(\tau_1 \leftleftarrows \tau_2)} \sqcap \overline{(\tau_2 \leftleftarrows \tau_1)}$$

which defines the set of points located at the same distance from τ_1 and τ_2.

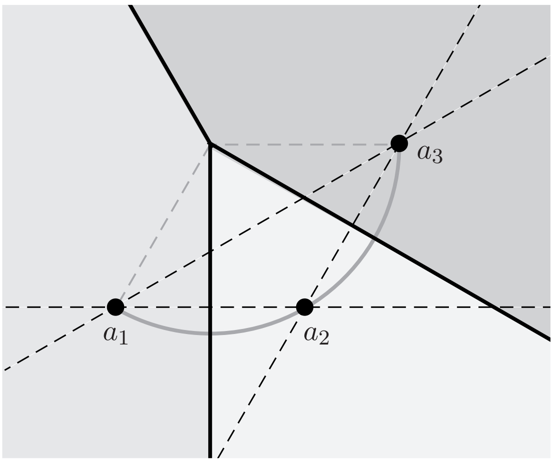
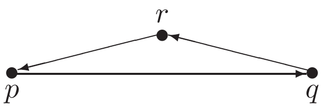


Figure 1.7. Closer operator and Voronoi tessellation.

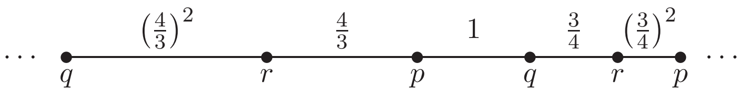
As a small illustrating example consider the formula

$$p \sqsubseteq (q \leftleftarrows r) \quad \wedge \quad q \sqsubseteq (r \leftleftarrows p) \quad \wedge \quad r \sqsubseteq (p \leftleftarrows q) \quad \wedge \quad p \neq \bot. \tag{1.19}$$

One can readily check that it is satisfiable in a three-point non-symmetrical model, say, in the one depicted below where the distance from x to y is the length of the shortest directed path from x to y.



On the other hand, it can be satisfied in the following subspace of $\mathbb{R}$



The following result has been obtained in (Sheremet et al., 2006):

THEOREM 1.14 *The satisfiability problem for $\mathcal{CSL}$-formulas in metric spaces is* EXPTIME-*complete.*

Investigating the algorithmic properties of the combination of $\mathcal{CSL}$ with fragments of $\mathcal{MS}$ in order to facilitate reasoning about topology, comparative distances, and absolute distances in one formalism is a challenging research problem. Define the terms of the language $\mathcal{CMS}$ by taking

$$\tau \quad ::= \quad p_i \;\mid\; \{\ell_i\} \;\mid\; \overline{\tau} \;\mid\; \tau_1 \sqcap \tau_2 \;\mid\; \exists^{\leq a}\tau \;\mid\; \exists^{<a}\tau \;\mid\; \tau_1 \leftleftarrows \tau_2,$$

where $a \in \mathbb{Q}^{>0}$. $\mathcal{CSL}$ enriched with nominals can represent *Voronoi tessellations* of various spaces. For example, let location constants ℓ_1, ℓ_2, ℓ_3 be interpreted by the points a_1, a_2, a_3 of $\mathbb{R}^2$ in Fig. 1.7. Then the $\mathcal{CMS}$-terms

$$\{\ell_i\} \leftleftarrows \{\ell_j\} \sqcup \{\ell_k\}, \quad \text{for } \{i,j,k\} = \{1,2,3\},$$

define the Voronoi tessellation of $\mathbb{R}^2$ corresponding to the set $\{a_1, a_2, a_3\}$.

Nothing is known about the algorithmic properties of $\mathcal{CMS}$ interpreted over arbitrary metric spaces. However, if one considers metric spaces satisfying the *min condition*

$$d(X,Y) \;=\; \min\{d(x,y) \mid x \in X, y \in Y\},$$

for all sets X and Y, then the topology induced by the metric space is trivial again and $\mathcal{CMS}$ can represent knowledge about comparative and absolute distances only (note that, by the definition, $d(X,Y) = \inf\{d(x,y) \mid x \in X, y \in Y\}$). Then we have the following result of (Sheremet et al., 2005a, 2005b):

THEOREM 1.15 *The satisfiability problem for $\mathcal{CMS}$-formulas in metric spaces with the min-condition is* EXPTIME-*complete under the binary coding of parameters.*

Rather unexpectedly, over the real line $\mathbb{R}$ the logic $\mathcal{CSL}$ turns out to be undecidable, which can be proved by reduction of the (undecidable) 10th Hilbert problem on the existence of an algorithm solving arbitrary Diophantine equations; see, e.g., (Barwise, 1977) and references therein. A proof can be found in (Sheremet et al., 2005b).

4. Temporal logics

Now we briefly remind the reader of the two basic propositional temporal logics that will be used for speaking about the temporal dimension of spatio-temporal models introduced in Sec. 6: the linear temporal logic $\mathcal{LTL}$ and its branching time extension $\mathcal{BTL}$ (a variant of the well-known computation tree logic $\mathcal{CTL}^*$).

4.1 Linear temporal logic $\mathcal{LTL}$

Temporal logic, as opposed to first-order logic, is an approach to reasoning about time (and computation) using temporal connectives and without explicit quantification over time. Its most popular variant, the *propositional linear temporal logic $\mathcal{LTL}$*, is successfully applied in model checking as well as program verification and specification; see e.g., (Clarke et al., 2000; Manna and Pnueli, 1992; Manna and Pnueli, 1995).

The intended *flow of time* for $\mathcal{LTL}$ is any strict linear order $(W, <)$ with *time points* $w \in W$ and the *precedence relation* $<$. In what follows we will be mainly interested in $(\mathbb{N}, <)$ and arbitrary finite flows of time. *$\mathcal{LTL}$-formulas* are constructed from *propositional variables* $p_0, p_1, \dots$ using the Booleans and the binary *temporal operator* $\mathcal{U}$ ('until'), the intended meaning of which is as follows:

- $\varphi \,\mathcal{U}\, \psi$ stands for 'φ holds true until ψ holds.'

Other temporal connectives like $\Diamond_F$ ('sometime in the future'), $\Box_F$ ('always in the future'), and $\bigcirc$ ('at the next moment') can be defined via $\mathcal{U}$:

$$\Diamond_F \varphi = \top \,\mathcal{U}\, \varphi, \qquad \Box_F \varphi = \neg \Diamond_F \neg \varphi \qquad \text{and} \qquad \bigcirc \varphi = \bot \,\mathcal{U}\, \varphi.$$

It should be noted that we adopt the 'strict' interpretation of temporal operators, i.e., $\Box_F$, $\Diamond_F$ and $\mathcal{U}$ do not not include the present. We will use abbreviations $\Box_F^+ \varphi$ and $\Diamond_F^+ \varphi$ for $\Box_F \varphi \wedge \varphi$ and $\Diamond_F \varphi \vee \varphi$, respectively. (Note also that 'past' operators like 'since' and 'sometime in the past' can be added to the language of $\mathcal{LTL}$ as well. Here we only deal with the 'future fragment' of $\mathcal{LTL}$, as this restriction does not influence any of the results throughout.)

To evaluate $\mathcal{LTL}$-formulas in a flow of time $\mathfrak{F} = (W, <)$, we have to specify first at which time points the propositional variables hold. An *$\mathcal{LTL}$-model* is a structure of the form

$$\mathfrak{M} = (\mathfrak{F}, p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}, \dots),$$

where $p_i^{\mathfrak{M}} \subseteq W$ for all i. The *truth-relation* $(\mathfrak{M}, w) \models \varphi$, or simply $w \models \varphi$ if understood (which says that an $\mathcal{LTL}$-formula φ holds at moment w in $\mathfrak{M}$) is defined as follows (we omit the clauses for the Booleans):

- $w \models p_i$ iff $w \in p_i^{\mathfrak{M}}$,
- $w \models \varphi \,\mathcal{U}\, \psi$ iff there is $v > w$ such that $v \models \psi$ and $u \models \varphi$ for all $u \in (w, v)$,

where $(w, v) = \{u \in W \mid w < u < v\}$. Other temporal operators (expressible via $\mathcal{U}$) are interpreted according to their meaning. For example,

- $w \models \bigcirc \varphi$ iff $w+1 \models \bigcirc \varphi$ (where $w+1$ is the immediate successor of w),
- $w \models \Diamond_F \varphi$ iff there is $v > w$ such that $v \models \varphi$.

A formula φ is *satisfiable* if there is a model $\mathfrak{M}$ over $(\mathbb{N}, <)$ and a time point $n \in \mathbb{N}$ such that $(\mathfrak{M}, n) \models \varphi$. We say that φ is *finitely satisfiable* if there is a finite strict linear order $\mathfrak{F}$ and a model $\mathfrak{M}$ over it such that $(\mathfrak{M}, n) \models \varphi$ for some n in $\mathfrak{F}$.

The following results are due to Sistla and Clarke (1985):

THEOREM 1.16 *The satisfiability problem for $\mathcal{LTL}$-formulas is* PSPACE-*complete. The problem whether an $\mathcal{LTL}$-formula is finitely satisfiable is* PSPACE-*complete as well.*

This complexity result might suggest that the expressive power of $\mathcal{LTL}$ is rather limited. Surprisingly enough, this is not the case. According to the famous Kamp theorem (Kamp, 1968), the propositional temporal language with both 'until' and 'since' is as expressive as the monadic first-order language over $(\mathbb{N}, <)$ (which of course is considerably more *succinct* than $\mathcal{LTL}$).

We will also consider the fragment $\mathcal{LTL}_\Box$ of $\mathcal{LTL}$ containing only $\Box_F$ and $\Diamond_F$ as its temporal operators. The following results are due to Ono and Nakamura (1980) and Sistla and Clarke (1985):

THEOREM 1.17 *The satisfiability problem for $\mathcal{LTL}_\Box$-formulas is* NP-*complete. The problem whether an $\mathcal{LTL}_\Box$-formula is finitely satisfiable is* NP-*complete as well.*

4.2 Branching time temporal logic $\mathcal{BTL}$

The temporal logic considered above is not able to express the following statements (due to Aristotle):

- it is *necessary* that there will be a sea-battle tomorrow,
- it is *possible* that there will be a sea-battle tomorrow.

$\mathcal{LTL}$ can only say

- $\bigcirc$*sea-battle*, i.e., there will be a sea-battle tomorrow.

In other words, it does not distinguish between possible, actual, or necessary future developments. A natural way to formalise assertions of this sort is to add two more operators A and E to the temporal language and understand them as quantifiers over 'possible histories.' For example, by interpreting E as 'it is possible that' and A as 'it is necessary that,' we can express the two Aristotle's statements by the formulas $\mathsf{A}\bigcirc$sea-battle and $\mathsf{E}\bigcirc$*sea-battle*, respectively.

Numerous extensions of $\mathcal{LTL}$ by means of such kind of operators have been introduced in various disciplines, in particular, computer science and artificial intelligence (Lamport, 1980; Clarke and Emerson, 1981; Emerson and Halpern, 1986) or philosophy (Prior, 1968); for more references and discussions see (Thomason, 1984; Gabbay et al., 2000). Here we only outline the essential ideas using the simple extension of $\mathcal{LTL}$ with A and E; it will be called $\mathcal{BTL}$, *branching temporal logic*.

Having fixed the language, we need to choose time structures that could allow for non-trivial interpretations. Clearly, if the flow of time is linear then at every moment the future is fixed, and so both $\mathsf{A}\varphi$ and $\mathsf{E}\varphi$ are equivalent to φ. The flows of time we need should be able to represent different evolutions of history. Since, on the other hand, it is natural to assume that, in contrast to the future, the past is fixed, *trees* as defined below appear to be perfect structures for modelling different histories (in particular, they correspond to the discrete tree of evolutions (1.5) of spatial transition systems).

A *tree* is a flow of time $\mathfrak{F} = (W, <)$ containing a point r, called the *root* of $\mathfrak{F}$, for which $W = \{v \mid r < v\} \cup \{r\}$, and such that for every $w \in W$, the set $\{w \mid v < w\}$ is well-founded and (strictly) linearly ordered by $<$. A *history* in $\mathfrak{F}$ is a maximal linearly $<$-ordered subset of W. Finally, an ω-tree is such a tree where every history is order isomorphic to $(\mathbb{N}, <)$.

By a *branching time model* we understand a structure

$$\mathfrak{B} = (\mathfrak{F}, \mathcal{H}, p_0^{\mathfrak{B}}, p_1^{\mathfrak{B}}, \dots),$$

where $\mathfrak{F} = (W, <)$ is an ω-tree, $\mathcal{H}$ a set of histories in $\mathfrak{F}$—the set of possible flows of time in the model—and $p_i^{\mathfrak{B}} \subseteq W$ for all i. Formulas are evaluated relative to pairs (h, w) consisting of an *actual history* $h \in \mathcal{H}$ and a time point $w \in h$. In such a pair (h, w), the temporal operators are interpreted along the actual history h as in the linear time framework, while the operators E and A quantify over the set of all histories

$$\mathcal{H}(w) = \{h' \in \mathcal{H} \mid w \in h'\}$$

coming through w. More precisely, the *truth-relation* $\models$ between models $\mathfrak{B}$ with pairs (h, w) and $\mathcal{BTL}$-formulas φ is defined inductively in the following way (we omit the clauses for the Booleans):

- $(h, w) \models p_i$ iff $w \in p_i^{\mathfrak{B}}$,
- $(h, w) \models \varphi\,\mathcal{U}\,\psi$ iff there is $v \in h$ such that $v > w$, $(h, v) \models \psi$ and $(h, u) \models \varphi$ for all $u \in (w, v)$,
- $(h, w) \models \mathsf{E}\varphi$ iff there is $h' \in \mathcal{H}(w)$ such that $(h', w) \models \varphi$,
- $(h, w) \models \mathsf{A}\varphi$ iff $(h', w) \models \varphi$ for all $h' \in \mathcal{H}(w)$.

Note that propositional variables are assumed to have no temporal aspect in the sense that their truth-values at (h, w) do not depend on the actual history h. We say that a $\mathcal{BTL}$-formula is *satisfiable* if there exists a branching time model $\mathfrak{B}$ such that $(\mathfrak{B}, h, w) \models \varphi$ for some history $h \in \mathcal{H}$ and some time point $w \in h$.

The branching time model defined above reflects the 'Ockhamist view' of time. We refer the reader to (Burgess, 1979; Zanardo, 1996; Gabbay et al., 2000; Reynolds, 2002) for more information about this and related approaches. Here we only note that our branching time logic is closely related to the computational tree logics $\mathcal{CTL}$ and $\mathcal{CTL}^*$ that are widely used in model checking and program verification and specification (Clarke and Emerson, 1981; Emerson and Halpern, 1986; Clarke et al., 2000).

It might seem more natural to quantify with E and A over the set of *all* histories in the tree rather than its subset $\mathcal{H}$. But then we would be forced to accept possibly unintended histories in $\mathfrak{F}$ as possible flows of time. Here is an example of a formula satisfiable in a branching time model as defined above, but not in a branching time model in which $\mathcal{H}$ is the set of all histories. The formula is a conjunction of the following three $\mathcal{BTL}$-formulas:

$$\mathsf{P}(\textit{Scotland}, \textit{UK}), \tag{1.20}$$

$$\mathsf{A}\Diamond_F \Box_F \mathsf{EC}(\textit{Scotland}, \textit{UK}), \tag{1.21}$$

$$\mathsf{A}\Box_F^+\big(\mathsf{P}(\textit{Scotland}, \textit{UK}) \to \mathsf{E}\bigcirc\mathsf{P}(\textit{Scotland}, \textit{UK})\big). \tag{1.22}$$

The first formula means that at present Scotland is part of the U.K. The second says that in all possible histories, there will be a time starting from which Scotland will be externally connected to the U.K. And the last formula claims that in all possible histories, it is always the case that if Scotland is part of the U.K. then it is still possible that it will remain in U.K. for at least one more day. (Since we do not have a combined spatio-temporal language yet, the $\mathcal{RCC}$-8 predicates $\mathsf{P}(\textit{Scotland}, \textit{UK})$ and $\mathsf{EC}(\textit{Scotland}, \textit{UK})$ should be regarded as a propositional variable and its negation, respectively.)

The following result can be obtained using a reduction to satisfiability in $\mathcal{CTL}^*$ (see Hodkinson et al., 2001):

THEOREM 1.18 *The satisfiability problem for $\mathcal{BTL}$-formulas is decidable in* 2EXPTIME.

It seems that the lower bound for the computational complexity of this problem is still unknown.

REMARK 1.19 Similarly to $\mathcal{RCC}$-8, instead of time points one can take extended time entities, i.e., intervals, as primitives. This approach to temporal representation and reasoning reflects the fact that certain assertions can be evaluated only at periods of time (e.g., 'John often drinks beer'). It was developed by Allen (1983; 1984), who observed, in particular, that relative positions of any two intervals i and j of a strict linear order can be described by precisely one of the thirteen basic interval relations: $\mathsf{before}(i,j)$, $\mathsf{meets}(i,j)$, $\mathsf{overlaps}(i,j)$, $\mathsf{during}(i,j)$, $\mathsf{starts}(i,j)$, $\mathsf{finishes}(i,j)$, their inverses ($\mathsf{before}(j,i)$, $\mathsf{meets}(j,i)$, etc.), and $\mathsf{equal}(i,j)$.

We will not consider interval temporal logics in this chapter and refer the interested reader to (Vilain et al., 1989; Blackburn, 1992; Gabbay et al., 2000; Goranko et al., 2004).

5. Combination principles

We have defined how the intended models of spatio-temporal logics (yet to be constructed) should look like. We have also identified a stock of available spatial and temporal logics to be integrated into spatio-temporal formalisms. However, we have not discussed yet how the component logics are supposed to interact with each other.

The expressive power (and consequently the computational complexity) of combined spatio-temporal formalisms obviously depends on three parameters:

- the expressiveness of the spatial component,
- the expressiveness of the temporal component, and
- the *interaction* between the two components allowed in the combined logic.

Regardless of the chosen component languages, the minimum requirement for a spatio-temporal combination to be useful is the following:

The language should be able to express changes in time of the truth-values of purely spatial propositions. (**PC**)

Languages satisfying (PC) can capture, for instance, some aspects of the *continuity of change principle* (see, e.g., Cohn, 1997) such as example (A) from Sec 2: 'if two clouds are disconnected now, then at the next moment they either remain disconnected or become externally connected.' A natural way to express this principle is to encode it into the following 'spatio-temporal formula'

$$\mathsf{DC}(cloud_1, cloud_2) \to \bigcirc\mathsf{DC}(cloud_1, cloud_2) \vee \bigcirc\mathsf{EC}(cloud_1, cloud_2). \quad \text{(A)}$$

We may also need to impose some constrains on possible movements of spatial objects by comparing their positions at different moments of

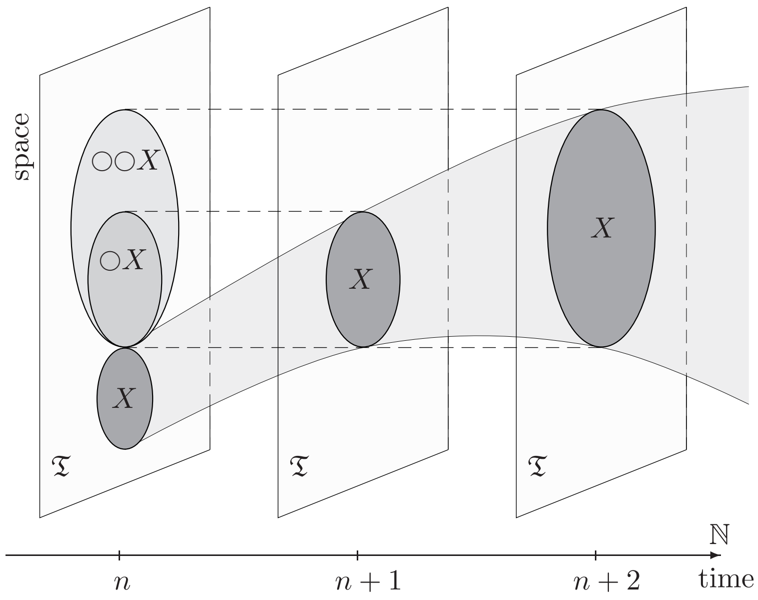


Figure 1.8. Temporal operators on regions.

time. For example, the continuity principle above can be further refined by saying that the current cloud's position overlaps with its positions at the next two moments, which requires a spatio-temporal formula of the form

$$\mathsf{O}(\mathit{cloud}, \bigcirc \mathit{cloud}) \wedge \mathsf{O}(\mathit{cloud}, \bigcirc\bigcirc \mathit{cloud}), \tag{1.23}$$

where the predicate $\mathsf{O}(\mathsf{r}, \mathsf{s})$ means that regions r and s have at least one common interior point; it can be expressed as a disjunction of all $\mathcal{RCC}$-8 relations but DC and EC (see Fig. 1.8 where $\bigcirc X$ at moment n denotes the state of X at moment $n+1$).

The difference between (A) and (1.23) is that in the former case we apply temporal operators to spatial formulas, while in the latter to regions.

Consider now example (G) from Sec. 2: 'it will be raining over every part of England ever and ever again.' This gives rise to the formula

$$\mathsf{P}(\mathit{England}, \Box_F \Diamond_F \mathit{Rain}) \tag{G}$$

where $\mathsf{P}(\mathsf{r}, \mathsf{s}) = \mathsf{TPP}(\mathsf{r}, \mathsf{s}) \vee \mathsf{NTPP}(\mathsf{r}, \mathsf{s}) \vee \mathsf{EQ}(\mathsf{r}, \mathsf{s})$. Formula (G) can be understood as follows: all bits (points) of England will infinitely often occur in region *Rain*, but not necessarily all at the same time. Note that the formula

$$\Box_F \Diamond_F \mathsf{P}(\mathit{England}, \mathit{Rain})$$

means that it will be raining over the whole England ever and ever again.

There is an essential difference between examples (1.23) and (G). In the former, we want to control the movements of objects over a fixed finite number of steps, while in the latter example we impose restrictions on their 'asymptotic' behaviour. This leads us to two fundamental principles which will be called *local spatial object change principle* (LOC) and *asymptotic spatial object change principle* (AOC).

The language should be able to express changes or evolutions of spatial objects over some fixed finite periods of time. (**LOC**)

The language should be able to express changes or evolutions of spatial objects over the whole duration of time. (**AOC**)

In logical terms, (PC) refers to the change of truth-values of propositions, while (LOC) and (AOC) to the change of extensions of predicates.

As we shall see later on in this chapter, different combination principles result in spatio-temporal logics of different expressive power and computational complexity.

6. Combining topo-logics with temporal logics

In this section we introduce and discuss various ways of combining topo-logics and temporal logics. First we consider combinations with (fragments of) linear temporal logic $\mathcal{LTL}$ and then with branching time temporal logic $\mathcal{BTL}$.

6.1 Combinations with linear temporal logic $\mathcal{LTL}$

First we construct 'maximal' combinations with (fragments of) $\mathcal{LTL}$ meeting all three combination principles (PC), (LOC) and (AOC), and see that such a straightforward approach results in undecidable logics. Then we systematically weaken the component languages and their interaction. The result is a hierarchy of spatio-temporal logics whose complexity ranges from NP via PSpace, ExpSpace and 2ExpSpace to undecidable. All omitted proofs and further details can be found in (Gabelaia et al., 2005a).

As outlined in the introduction, we represent the motion of spatial objects in time using the following kind of 'snapshot' models. A *topological-temporal model* (a *tt-model*, for short) is a pair of the form $\mathfrak{M} = (\mathfrak{T}, \mathfrak{V})$, where $\mathfrak{T} = (U, \mathbb{I})$ is a topological space, and $\mathfrak{V}$, a *valuation*, is a map associating with every spatial variable p and every time point $n \in \mathbb{N}$ a set $\mathfrak{V}(p,n) \subseteq U$—the 'space' occupied by p at moment n. Such a pair $\mathfrak{M} = (\mathfrak{T}, \mathfrak{V})$ is simply a shorthand for the representation (1.3) of spatio-temporal models as a sequence of spatial models:

$$\mathfrak{M}_0 = (\mathfrak{T}, \mathfrak{V}(p_0, 0), \mathfrak{V}(p_1, 0), \dots),\ \mathfrak{M}_1 = (\mathfrak{T}, \mathfrak{V}(p_0, 1), \mathfrak{V}(p_1, 1), \dots),\ \dots$$

Combinations with (PC), (LOC) and (AOC). A 'maximalist' approach to constructing spatio-temporal logics is to allow unrestricted applications of the Booleans, the topological and the temporal operators to form spatio-temporal terms.

Denote by $\mathcal{LTL} \times \mathcal{S}4_u$ the spatio-temporal language given by the following definition:

$$\begin{aligned} \tau &::= p \mid \overline{\tau} \mid \tau_1 \sqcap \tau_2 \mid \tau_1 \sqcup \tau_2 \mid \mathbf{I}\tau \mid \mathbf{C}\tau \mid \tau_1 \mathcal{U} \tau_2, \\ \varphi &::= \tau_1 \sqsubseteq \tau_2 \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \mathcal{U} \varphi_2. \end{aligned} \tag{1.24}$$

Expressions of the form τ and φ will be called $\mathcal{LTL} \times \mathcal{S}4_u$ *terms* and *formulas*, respectively. Most of the languages we consider in this subsection are fragments of $\mathcal{LTL} \times \mathcal{S}4_u$.

As before, we can introduce the temporal operators $\Box_F$, $\Diamond_F$, and $\bigcirc$ applicable to $\mathcal{LTL} \times \mathcal{S}4_u$ formulas. Moreover, these operators can now be used to form $\mathcal{LTL} \times \mathcal{S}4_u$ terms: for example,

$$\Diamond_F \tau = \top \mathcal{U} \tau, \qquad \Box_F \tau = \overline{\Diamond_F \overline{\tau}} \qquad \text{and} \qquad \bigcirc\tau = \bot \mathcal{U} \tau,$$

where the intended meaning of $\bot$ and $\top$ is the empty set and the whole space, respectively.

$\mathcal{LTL} \times \mathcal{S}4_u$ formulas are supposed to represent propositions speaking about moving spatial objects represented by $\mathcal{LTL} \times \mathcal{S}4_u$ terms. The intended truth-values of propositions in tt-models can vary in time, but do not depend on points of spaces. But how are we to understand 'temporalised' terms?

The meaning of $\bigcirc\tau$ should be clear: at moment n, it denotes the space occupied by τ at the next moment $n+1$ (see (1.23) and Fig. 1.8). The formula

$$\mathsf{EQ}(\bigcirc\bigcirc EU, EU \sqcup \mathit{Romania} \sqcup \mathit{Bulgaria}) \tag{F}$$

formalises sentence (F) from Sec. 2. It says that in two years the EU (as it is today) will be extended with Romania and Bulgaria. Note that $\bigcirc\bigcirc\mathsf{EQ}(EU, EU \sqcup \mathit{Romania} \sqcup \mathit{Bulgaria})$ has a different meaning because the EU may expand or shrink in a year. It is also not hard to formalise sentences (D), (E) and (H):

$$\mathsf{EQ}(\bigcirc \mathsf{X}, \mathsf{Y}) \to \neg \mathsf{EQ}(\mathsf{Y}, \bigcirc \mathsf{Y}), \tag{D}$$

$$\Box_F^+ \mathsf{EQ}(\bigcirc Europe, Europe), \tag{E}$$

$$\Box_F^+ \big(\mathsf{EQ}(Earth, W \sqcup L) \land \mathsf{EC}(W, L)\big) \land \mathsf{P}(W, \bigcirc W) \to \mathsf{P}(\bigcirc L, L). \tag{H}$$

The intended interpretation of terms of the form $\Diamond_F \tau$ and $\Box_F \tau$ is a bit more sophisticated. It reflects the standard temporal meanings of propositions '$\Diamond_F\, x \in \tau$' and '$\Box_F\, x \in \tau$,' for all points x in the topological space:

- at moment n, term $\Diamond_F \tau$ is interpreted as the union of all spatial extensions of τ at moments $m > n$;
- at moment n, term $\Box_F \tau$ is interpreted as the intersection of all spatial extensions of τ at moments $m > n$.

For example, consider Fig. 1.8 with moving cloud X depicted on it at three consecutive moments of time, and suppose X does not change after $n+2$. Then $\Diamond_F X$ at n is the union of $\bigcirc X$ and $\bigcirc\bigcirc X$ at n and $\Box_F X$ at n is the intersection of $\bigcirc X$ and $\bigcirc\bigcirc X$ at n (i.e., $\bigcirc X$).

As another example, let us interpret the term $\Box_F \Diamond_F Rain$ occurring in formula (G) on page 38:

- $\Diamond_F Rain$ at moment n occupies the space where it will be raining at *some* time points $m > n$ (which may be different for different places). $\Box_F Rain$ at n occupies the space where it will *always* be raining after n.
- $\Box_F \Diamond_F Rain$ at n is the space where it will be raining ever and ever again after n, while $\Diamond_F \Box_F Rain$ comprises all places where it will always be raining starting from some future moments of time.

Now, what can be the meaning of $Rain\, \mathcal{U}\, Snow$? Similarly to the readings of $\Box_F \tau$ and $\Diamond_F \tau$ above, we adopt the following definition:

- at moment n, the spatial extension of $\tau_1\, \mathcal{U}\, \tau_2$ consists of those points x of the topological space for which there is $m > n$ such that x belongs to τ_2 at moment m and x is in τ_1 at all k whenever $n < k < m$.

The past counterpart of $\mathcal{U}$—i.e., the operator 'since' $\mathcal{S}$—can be used to say that the part of Russia that has been remaining Russian since 1917 is not connected to the part of Germany (Königsberg) that became Russian after the Second World War (Kaliningrad):

$$\mathsf{DC}(Russia\, \mathcal{S}\ Russian\_Empire, Russia\, \mathcal{S}\ Germany).$$

Summing up, the valuation $\mathfrak{V}$ in tt-models can be inductively extended to arbitrary $\mathcal{LTL} \times \mathcal{S}4_u$ terms:

- $\mathfrak{V}(\overline{\tau}, n) = U - \mathfrak{V}(\tau, n), \qquad \mathfrak{V}(\tau_1 \sqcap \tau_2, n) = \mathfrak{V}(\tau_1, n) \cap \mathfrak{V}(\tau_2, n),$
- $\mathfrak{V}(\mathbf{I}\tau, n) = \mathbb{I}\,\mathfrak{V}(\tau, n), \qquad \mathfrak{V}(\mathbf{C}\tau, n) = \mathbb{C}\,\mathfrak{V}(\tau, n),$
- $\mathfrak{V}(\tau_1 \,\mathcal{U}\, \tau_2, n) = \bigcup\limits_{m>n} \Big(\mathfrak{V}(\tau_2, m) \cap \bigcap\limits_{k\in(n,m)} \mathfrak{V}(\tau_1, k)\Big).$

Then we also have:

- $\mathfrak{V}(\Diamond_F \tau, n) = \bigcup\limits_{m>n} \mathfrak{V}(\tau, m), \qquad \mathfrak{V}(\Box_F \tau, n) = \bigcap\limits_{m>n} \mathfrak{V}(\tau, m),$
- $\mathfrak{V}(\bigcirc \tau, n) = \mathfrak{V}(\tau, n+1).$

The truth-values of $\mathcal{LTL} \times \mathcal{S}4_u$ formulas in tt-models are defined as follows:

- $(\mathfrak{M}, n) \models \tau_1 \sqsubseteq \tau_2$ iff $\mathfrak{V}(\tau_1, n) \subseteq \mathfrak{V}(\tau_2, n)$,
- $(\mathfrak{M}, n) \models \neg\varphi$ iff $(\mathfrak{M}, n) \not\models \varphi$,
- $(\mathfrak{M}, n) \models \varphi_1 \wedge \varphi_2$ iff $(\mathfrak{M}, n) \models \varphi_1$ and $(\mathfrak{M}, n) \models \varphi_2$,
- $(\mathfrak{M}, n) \models \varphi_1 \,\mathcal{U}\, \varphi_2$ iff there is $m > n$ such that $(\mathfrak{M}, m) \models \varphi_2$ and $(\mathfrak{M}, k) \models \varphi_1$ for all $k \in (n, m)$.

An $\mathcal{LTL} \times \mathcal{S}4_u$ formula φ is called *satisfiable* if there exists a tt-model $\mathfrak{M}$ such that $(\mathfrak{M}, n) \models \varphi$ for some time point $n \in \mathbb{N}$.

Observe that $\mathcal{LTL} \times \mathcal{S}4_u$ contains both $\mathcal{LTL}$ and $\mathcal{S}4_u$. At first sight it may appear that the computational properties of this combination should not be too bad—after all, its spatial and temporal components are PSpace-complete. It turns out, however, that this is very far from being the case:

Theorem 1.20 *The satisfiability problem for $\mathcal{LTL} \times \mathcal{S}4_u$ formulas in tt-models is Σ_1^1-complete.*

It follows from Theorem 1.20 that if we strengthen the topological component to $\mathcal{TCC}$ (by allowing terms of the form $c^{\leq k}\tau$, see Sec. 3.2), then the satisfiability problem for the resulting language $\mathcal{LTL} \times \mathcal{TCC}$ is also Σ_1^1-hard. However, Theorem 1.20 is proved by a reduction of the Σ_1^1-complete *recurrent tiling problem* (see Gabelaia et al., 2005b), and the terms used in its proof can denote arbitrary (i.e., not necessarily *connected*) sets. It would be interesting to know the complexity of the satisfiability problem for $\mathcal{LTL} \times \mathcal{TCC}$ formulas in tt-models where spatial variables can be interpreted at each time point by connected sets or sets containing at most k connected components for some fixed k.

One might conjecture that it is the use of the *infinitary* operators $\mathcal{U}$, $\Box_F$ and $\Diamond_F$ in the construction of $\mathcal{LTL} \times \mathcal{S}4_u$ terms that makes logics like $\mathcal{LTL} \times \mathcal{S}4_u$ 'over-expressive.' Moreover, the whole idea of tt-models based on an *infinite* flow of time may look counterintuitive in the context of spatio-temporal representation and reasoning (unlike, say, models used to represent the behaviour of reactive computer systems).

There are different approaches to avoid infinity in tt-models:

- The most radical one is to allow only *finite flows* of time. A *finite tt-model* is a triple of the form $\mathfrak{M} = (\mathfrak{T}, \mathfrak{V}, N)$, where $\mathfrak{T}$ is a topological space, $N \in \mathbb{N}$, and $\mathfrak{V}$ is a map associating with every spatial variable p and every time point $n \leq N$ a subset $\mathfrak{V}(p, n)$ of the topological space $\mathfrak{T}$.
- A more cautious approach is to impose the following *finite state assumption* on models:

FSA *Every spatial variable may have only finitely many possible states (although it may change its states infinitely often).*

Say that a (possibly infinite) tt-model $(\mathfrak{T}, \mathfrak{V})$ *satisfies* **FSA** if, for every spatial variable p, there are finitely many sets $A_1, \ldots, A_m$ in the space $\mathfrak{T}$ such that $\{\mathfrak{V}(p, n) \mid n \in \mathbb{N}\} = \{A_1, \ldots, A_m\}$. (Such models can be used, for instance, to capture periodic fluctuations due to season or climate changes, say, a daily tide.)

One can actually show (see Gabelaia et al., 2005a) that an $\mathcal{LTL} \times \mathcal{S}4_u$ formula is satisfiable in a model with **FSA** iff it is satisfiable in a model based on a *finite* (Aleksandrov) topological space.

Unfortunately, none of these 'finitising' approaches improves the computational behaviour of the combinations too much. We can even try to weaken the temporal component to $\mathcal{LTL}_\Box$ (by allowing only the temporal operators $\Box_F$ and $\Diamond_F$ in terms and formulas), and still we have:

THEOREM 1.21 (i) *The satisfiability problem for $\mathcal{LTL}_\Box \times \mathcal{S}4_u$ formulas in (arbitrary) finite tt-models is undecidable.*

(ii) *The satisfiability problem for $\mathcal{LTL}_\Box \times \mathcal{S}4_u$ formulas in tt-models satisfying* **FSA** *is undecidable.*

However, if we weaken the spatial component further, the combinations can become decidable, with high but gradually decreasing complexity. Recall the hierarchy of topo-logics from Sec. 3.2. It suggests that next we should consider $\mathcal{RC}^{max}$ as the spatial component. In this case $\mathcal{LTL} \times \mathcal{RC}^{max}$ *terms* τ are defined by

$$\tau \quad ::= \quad \mathbf{CI}p \;\mid\; \overline{\tau} \;\mid\; \tau_1 \sqcap \tau_2 \;\mid\; \tau_1 \sqcup \tau_2 \;\mid\; \mathbf{I}\tau \;\mid\; \mathbf{C}\tau \;\mid\; \tau_1 \mathcal{U} \tau_2,$$

and $\mathcal{LTL} \times \mathcal{RC}^{max}$-formulas as in (1.24). Unfortunately, it is an open problem whether the satisfiability problems for $\mathcal{LTL} \times \mathcal{RC}^{max}$-formulas in arbitrary or in finite tt-models, or in tt-models satisfying **FSA** are decidable.

Let us move one more step down and denote by $\mathcal{LTL} \times \mathcal{RC}$ the language given by the following definition:

$$\begin{array}{lll} \varrho & ::= & \mathbf{CI}p \mid \mathbf{CI}\overline{\varrho} \mid \mathbf{CI}(\varrho_1 \sqcap \varrho_2) \mid \mathbf{CI}(\varrho_1 \sqcup \varrho_2) \mid \mathbf{CI}(\varrho_1 \,\mathcal{U}\, \varrho_2), \\ \tau & ::= & \varrho \mid \mathbf{I}\varrho \mid \overline{\tau} \mid \tau_1 \sqcap \tau_2 \mid \tau_1 \sqcup \tau_2 \end{array}$$

and formulas as in (1.24). Expressions of the form ϱ will be called *$\mathcal{LTL} \times \mathcal{RC}$ region terms.* Now the complexity of reasoning decreases indeed:

THEOREM 1.22 *The satisfiability problem for $\mathcal{LTL} \times \mathcal{RC}$ formulas in tt-models satisfying* **FSA**, *and in those based on* (*arbitrary*) *finite flows of time is* 2EXPSPACE-*complete.*

The existence of a 2EXPSPACE decision algorithm follows from the fact that, similarly to the case of topo-logic $\mathcal{RC}$ (without a temporal component), it is enough to deal with *Aleksandrov* topological spaces. More precisely, it can be shown that an $\mathcal{LTL} \times \mathcal{RC}$ formula is satisfiable in a tt-model with **FSA** iff it is satisfiable in a tt-model $(\mathfrak{T}, \mathfrak{V})$ where $\mathfrak{T}$ is an Aleksandrov space induced by a finite disjoint union of finite brooms (cf. Theorem 1.5). The lower bound is established by showing that 'yardsticks' of double-exponential length (similar to those used by Stockmeyer, 1974; Halpern and Vardi, 1989) can be encoded by $\mathcal{LTL} \times \mathcal{RC}$ formulas of polynomial length. These yardsticks can then be used to encode any Turing machine computation over double-exponential space.

By restricting the language further we obtain $\mathcal{LTL} \times \mathcal{BRCC}$-8:

$$\varphi \ ::= \ Q(\varrho_1, \varrho_2) \quad | \quad \neg\varphi \quad | \quad \varphi_1 \wedge \varphi_2 \quad | \quad \varphi_1 \vee \varphi_2 \quad | \quad \varphi_1 \,\mathcal{U}\, \varphi_2,$$

where the ϱ_i are $\mathcal{LTL} \times \mathcal{RC}$ region terms and Q ranges over (the translations of) the eight $\mathcal{RCC}$-8 predicates.

THEOREM 1.23 *The satisfiability problem for $\mathcal{LTL} \times \mathcal{BRCC}$-8 formulas in tt-models with* **FSA**, *and those that are based on* (*arbitrary*) *finite flows of time is* EXPSPACE-*complete.*

The exponential decrease in the complexity is due to the fact that now we can have a bound (linear in the size of the given formula φ) on the size of the brooms inducing the underlying Aleksandrov space of a tt-model in which an $\mathcal{LTL} \times \mathcal{BRCC}$-8 formula φ is satisfied. The lower bound can be proved by reduction of a 2^n-corridor tiling problem.

Finally, by replacing the available region terms of $\mathcal{LTL} \times \mathcal{BRCC}$-8 with

$$\varrho \quad ::= \quad \mathbf{CI}p \quad | \quad \mathbf{CI}(\varrho_1 \,\mathcal{U}\, \varrho_2)$$

we obtain the product $\mathcal{LTL} \times \mathcal{RCC}$-8. The exact complexity of the satisfiability problem for $\mathcal{LTL} \times \mathcal{RCC}$-8 formulas in tt-models satisfying **FSA**, and in (arbitrary) finite tt-models is not known. These problems are PSPACE-hard by Theorem 1.16 and in EXPSPACE by Theorem 1.23.

It is also an open problem whether satisfiability of $\mathcal{LTL} \times \mathcal{L}$ formulas in (arbitrary) tt-models is decidable, whenever $\mathcal{L} \in \{\mathcal{RCC}\text{-8}, \mathcal{BRCC}\text{-8}, \mathcal{RC}, \mathcal{RC}^{max}\}$.

Combinations with (PC) and (LOC). We can try to obtain decidable spatio-temporal combinations over *infinite* time lines by omitting the (AOC) principle and allowing only 'local control' of evolutions of spatial objects. To begin with, let us consider the fragment $\mathcal{LTL} \circ \mathcal{S}4_u$ of $\mathcal{LTL} \times \mathcal{S}4_u$ with terms of the form:

$$\tau \quad ::= \quad p \quad | \quad \overline{\tau} \quad | \quad \tau_1 \sqcap \tau_2 \quad | \quad \tau_1 \sqcup \tau_2 \quad | \quad \mathbf{I}\tau \quad | \quad \mathbf{C}\tau \quad | \quad \bigcirc\tau.$$

In other words, $\mathcal{LTL} \circ \mathcal{S}4_u$ does not allow applications of temporal operators different from $\bigcirc$ to form *terms* (but they are still available as *formula* constructors). This means that the language still satisfies (LOC), but (AOC) is no longer available.

This fragment is definitely less expressive than full $\mathcal{LTL} \times \mathcal{S}4_u$. For instance, on the one hand one can show that $\mathcal{LTL} \circ \mathcal{S}4_u$ formulas do not distinguish between arbitrary tt-models and those based on Aleksandrov topological spaces. On the other hand, the set of $\mathcal{LTL} \times \mathcal{S}4_u$ formulas satisfiable in tt-models based on Aleksandrov spaces is a proper subset of those satisfiable in arbitrary tt-models. Consider, for example, the $\mathcal{LTL} \times \mathcal{S}4_u$ formula

$$\Box_F \mathbf{I}p \;\sqsubseteq\; \mathbf{I}\Box_F p.$$

One can readily see that it is true in every tt-model based on an Aleksandrov space, but its negation can be satisfied in a tt-model. For it suffices to take the topology $\mathfrak{T} = (\mathbb{R}, \mathbb{I})$ with the standard interior operator $\mathbb{I}$ on the real line, select a sequence X_n of open sets such that $\bigcap_{n\in\mathbb{N}} X_n$ is not open, e.g., $X_n = (-1/n, 1/n)$, and put $\mathfrak{U}(p, n) = X_n$.

However, even this seemingly weak interaction between topological and temporal operators turns out to be dangerous:

THEOREM 1.24 *The satisfiability problem for $\mathcal{LTL} \circ \mathcal{S}4_u$ formulas in tt-models is undecidable. It is undecidable as well for tt-models satisfying* **FSA**, *and for (arbitrary) finite tt-models.*

We can try to weaken again the topological component. The language $\mathcal{LTL} \circ \mathcal{RC}^{max}$ can be obtained from $\mathcal{LTL} \times \mathcal{RC}^{max}$ by replacing the constructor $\tau_1 \, \mathcal{U} \, \tau_2$ with $\bigcirc\tau$ in the definition of terms. It is not known whether this helps, that is, whether the satisfiability problem for $\mathcal{LTL} \circ \mathcal{RC}^{max}$-formulas in tt-models or in (arbitrary) finite tt-models is decidable.

If we weaken the spatial component even further, then this kind of combination turns out to be decidable. Consider the languages $\mathcal{LTL} \circ \mathcal{L}$, for $\mathcal{L} \in \{\mathcal{RC}, \mathcal{BRCC}\text{-}8, \mathcal{RCC}\text{-}8\}$, which differ from $\mathcal{LTL} \times \mathcal{L}$ only in the following aspect: in the corresponding definition of *region terms* ϱ the constructor $\mathbf{CI}(\varrho_1 \, \mathcal{U} \, \varrho_2)$ is replaced with $\mathbf{CI}\bigcirc\varrho$. Then again we have a hierarchy of gradually decreasing complexity:

THEOREM 1.25 *The satisfiability problem for $\mathcal{LTL} \circ \mathcal{RC}$ formulas in tt-models is* 2EXPSPACE-*complete. It is* 2EXPSPACE-*complete as well for tt-models satisfying* **FSA** *and for (arbitrary) finite tt-models.*

THEOREM 1.26 *The satisfiability problem for $\mathcal{LTL} \circ \mathcal{BRCC}$-8 formulas in tt-models is* EXPSPACE-*complete. It is* EXPSPACE-*complete as well for tt-models satisfying* **FSA** *and for (arbitrary) finite tt-models.*

The proofs of Theorems 1.25 and 1.26 are essentially the same as those of Theorems 1.22 and 1.23. The difference is that now the correspondence between arbitrary satisfiability and satisfiability in tt-models based on Aleksandrov spaces holds not only for tt-models satisfying **FSA**, but for arbitrary tt-models as well.

THEOREM 1.27 *The satisfiability problem for $\mathcal{LTL} \circ \mathcal{RCC}$-8 formulas in tt-models is* PSPACE-*complete.*

The idea of the proof is to separate the topological and temporal parts of a given formula, and then use available satisfiability checking algorithms for the component logics (see also Theorem 1.28 below). In order to take into account the interaction between the topological and temporal parts, one has to use the so-called 'completion property' of $\mathcal{RCC}$-8 (cf. Balbiani and Condotta, 2002) with respect to a certain class $\mathfrak{C}$ of models: given a satisfiable set Φ of $\mathcal{RCC}$-8 formulas and a model in $\mathfrak{C}$ satisfying a subset of Φ, one can extend this 'partial' model to a model in $\mathfrak{C}$ satisfying the whole Φ.

The exact complexity of the satisfiability problem for $\mathcal{LTL} \circ \mathcal{RCC}$-8 formulas in tt-models satisfying **FSA**, and in (arbitrary) finite tt-models is not known. These problems are PSPACE-hard by Theorem 1.16 and in EXPSPACE by Theorem 1.26.

Combinations with (PC) only. If we want to keep the complexity low but to use an expressive topological component, then the interaction between space and time has to be weakened. One way of doing this is to consider combined languages in which the temporal operators can be applied to spatial formulas but not to spatial terms. The resulting combinations will satisfy (PC), but neither (LOC) nor (AOC) is expressible. (This way of 'temporalising' a logic was first introduced by Finger and Gabbay, 1992).

Denote by $\mathcal{LTL}[\mathcal{S}4_u]$ the spatio-temporal language given by the following definition:

$$\begin{aligned} \tau \; &::= \; p \quad | \quad \overline{\tau} \quad | \quad \tau_1 \sqcap \tau_2 \quad | \quad \tau_1 \sqcup \tau_2 \quad | \quad \mathbf{I}\tau \quad | \quad \mathbf{C}\tau, \\ \varphi \; &::= \; \tau_1 \sqsubseteq \tau_2 \quad | \quad \neg\varphi \quad | \quad \varphi_1 \wedge \varphi_2 \quad | \quad \varphi_1 \vee \varphi_2 \quad | \quad \varphi_1 \,\mathcal{U}\, \varphi_2. \end{aligned}$$

Note that the definition of $\mathcal{LTL}[\mathcal{S}4_u]$ terms coincides with the definition of spatial terms in $\mathcal{S}4_u$ which reflects the fact that $\mathcal{LTL}[\mathcal{S}4_u]$ cannot capture (LOC) or (AOC). We have imposed no restrictions upon the temporal operators in formulas—so the combined language still contains $\mathcal{LTL}$. (Clearly, $\mathcal{S}4_u$ is a fragment of $\mathcal{LTL}[\mathcal{S}4_u]$.)

THEOREM 1.28 *The satisfiability problem for $\mathcal{LTL}[\mathcal{S}4_u]$ formulas in tt-models and in (arbitrary) finite tt-models is* PSPACE*-complete.*

The proof of this theorem is based on the fact that the interaction between spatial and temporal components of $\mathcal{LTL}[\mathcal{S}4_u]$ is rather limited. In fact, for every $\mathcal{LTL}[\mathcal{S}4_u]$ formula φ one can construct an $\mathcal{LTL}$ formula φ^* by replacing every occurrence of a (spatial) subformula $\tau_1 \sqsubseteq \tau_2$ in φ with a fresh propositional variable p_{τ_1,τ_2}. Then, given an $\mathcal{LTL}$-model $\mathfrak{M}$ for φ^* (based on $(\mathbb{N}, <)$ or a finite flow of time) and a moment n, we take the set

$$\Phi_n = \{\tau_1 \sqsubseteq \tau_2 \mid (\mathfrak{M}, n) \models p_{\tau_1,\tau_2}\} \;\cup\; \{\neg\,(\tau_1 \sqsubseteq \tau_2) \mid (\mathfrak{M}, n) \models \neg p_{\tau_1,\tau_2}\}$$

of spatial formulas. It is not hard to see that if Φ_n is satisfiable for every n, then there is a tt-model satisfying φ (simply because extensions of a spatial variable at different time moments are independent). Now, to check whether φ is satisfiable, it suffices to use a suitable nondeterministic algorithm (see, e.g., Sistla and Clarke, 1985) which guesses an $\mathcal{LTL}$-model for φ^* and then, for each time point n, to check satisfiability of Φ_n. This can be done using polynomial space in the length of φ.

Theorem 1.28 (together with Theorem 1.16) shows that the satisfiability problem for each of the spatio-temporal logics of the form $\mathcal{LTL}[\mathcal{L}]$, where $\mathcal{L} \in \{\mathcal{RCC}\text{-8}, \mathcal{BRCC}\text{-8}, \mathcal{RC}, \mathcal{RC}^{max}\}$, is also PSPACE-complete.

However, if—instead of $\mathcal{LTL}$—we consider its NP-complete fragment $\mathcal{LTL}_\Box$, the complexity of 'temporalisations' can even be lower. On the one hand, by Theorems 1.1, 1.7 and 1.28, $\mathcal{LTL}_\Box[\mathcal{S}4_u]$ and $\mathcal{LTL}_\Box[\mathcal{RC}^{max}]$ are still PSPACE-complete. On the other, by considering NP-complete topological components, the same argument as in the proof of Theorem 1.28 gives us:

THEOREM 1.29 *The satisfiability problem for $\mathcal{LTL}_\Box[\mathcal{RC}]$ formulas in tt-models is* NP*-complete.*

It follows from Theorem 1.29 that the satisfiability problems for the weaker $\mathcal{LTL}_\Box[\mathcal{RCC}\text{-}8]$ and $\mathcal{LTL}_\Box[\mathcal{BRCC}\text{-}8]$ are NP-complete as well.

6.2 Combinations with branching time temporal logic $\mathcal{BTL}$

In the framework of linear time spatio-temporal logics, we can say, for instance, that the U.K. will join the euro-zone: $\Diamond_F \mathsf{P}(\mathit{UK}, \mathit{Eurozone})$. We can also say that this will never happen. But we are not able to convey the reality, viz., that both variants are possible, that is, something like

$$\mathsf{E}\Diamond_F \mathsf{P}(\mathit{UK}, \mathit{Eurozone}) \quad \wedge \quad \mathsf{E}\neg\Box_F \mathsf{P}(\mathit{UK}, \mathit{Eurozone}). \tag{1.25}$$

In this section we summarise the results of (Wolter and Zakharyaschev, 2002) on the combinations of the branching time temporal logic $\mathcal{BTL}$ with the topo-logic $\mathcal{BRCC}$-8.

The combined languages are interpreted in the following modification of tt-models. A *branching time topological model* (a *btt-model*, for short) is a quadruple $\mathfrak{M} = (\mathfrak{F}, \mathcal{H}, \mathfrak{T}, \mathfrak{V})$, where $\mathfrak{F} = (W, <)$ is an ω-tree, $\mathcal{H}$ a set of histories in $\mathfrak{F}$, $\mathfrak{T} = (U, \mathbb{I})$ a topological space, and $\mathfrak{V}$, a *valuation*, is a map associating with every spatial variable p and every time point $w \in W$ a set $\mathfrak{V}(p, w) \subseteq U$. (Observe that according to this definition, $\mathfrak{V}(p, w)$—the 'space' occupied by p at moment w—does not depend on the actual history of events.)

As concerns the languages, for each choice of topological/linear-time combination $\mathcal{L} \in \{\mathcal{LTL}[\mathcal{BRCC}\text{-}8], \mathcal{LTL} \circ \mathcal{BRCC}\text{-}8, \mathcal{LTL} \times \mathcal{BRCC}\text{-}8\}$, we have two options: to allow applications of A and E to $\mathcal{L}$-formulas only, or to both $\mathcal{L}$-formulas and $\mathcal{L}$-region terms. The resulting languages will be denoted by $\mathcal{L}^b$ (the former option) and $\mathcal{L}^{b^x}$ (the latter one).

For example, (1.21), (1.22) and (1.25) are $\mathcal{LTL}[\mathcal{BRCC}\text{-}8]^b$-formulas. The following $(\mathcal{LTL} \circ \mathcal{BRCC}\text{-}8)^{b^x}$-formula

$$\mathsf{A}\Box_F^+\big(\mathsf{EQ}(\mathit{Europe}, \bigcirc\mathit{Europe}) \wedge \mathsf{P}(\mathit{EU}, \mathit{Europe})\big) \wedge \\ \mathsf{P}(\mathit{Europe}, \mathsf{E}\bigcirc\mathit{EU}) \wedge \mathsf{P}(\mathsf{A}\bigcirc\mathit{EU}, \mathit{EU})$$

says that, whatever happens, the region occupied by Europe will always remain the same and the EU will be part of Europe; moreover, every part of Europe has a possibility to join the EU next year, while, on the hand, what will certainly belong to the EU next year, is only part of the EU as it is today.

Now the valuation $\mathfrak{V}$ in btt-models can be inductively extended to arbitrary region terms in a way similar to the linear case: we only have to add a history as parameter. Given a region term ϱ, a history $h \in \mathcal{H}$, and a time point $w \in h$, define the *value* $\mathfrak{V}(\varrho, h, w)$ *of* ϱ *at* w *relative to* h inductively by taking

- $\mathfrak{V}(\mathbf{CI}p, h, w) = \mathbb{CI}\mathfrak{V}(p, w)$, p a spatial variable,
- $\mathfrak{V}(\mathbf{CI}\overline{\varrho}, h, w) = \mathbb{CI}(U - \mathfrak{V}(\varrho, h, w))$,
- $\mathfrak{V}(\mathbf{CI}(\varrho_1 \sqcap \varrho_2), h, w) = \mathbb{CI}(\mathfrak{V}(\varrho_1, h, w) \cap \mathfrak{V}(\varrho_2, h, w))$,
- $\mathfrak{V}(\mathbf{CI}(\varrho_1 \sqcup \varrho_2), h, w) = \mathbb{CI}(\mathfrak{V}(\varrho_1, h, w) \cup \mathfrak{V}(\varrho_2, h, w))$,
- $\mathfrak{V}(\mathbf{CI}(\varrho_1 \mathcal{U} \varrho_2), h, w) = \mathbb{CI} \bigcup\limits_{v>w,\, v\in h} \Big(\mathfrak{V}(\varrho_2, h, v) \cap \bigcap\limits_{u\in(w,v)} \mathfrak{V}(\varrho_1, h, u)\Big)$,
- $\mathfrak{V}(\mathbf{CI}\mathsf{E}\varrho, h, w) = \mathbb{CI} \bigcup\limits_{h'\in\mathcal{H}(w)} \mathfrak{V}(\varrho, h', w)$,
- $\mathfrak{V}(\mathbf{CI}\mathsf{A}\varrho, h, w) = \mathbb{CI} \bigcap\limits_{h'\in\mathcal{H}(w)} \mathfrak{V}(\varrho, h', w)$.

Now, for a formula φ and a pair (h, w), the *truth-value* of φ at (h, w) in $\mathfrak{M}$ is defined inductively as follows:

- $(\mathfrak{M}, h, w) \models Q(\varrho_1, \varrho_2)$ iff $Q(\mathfrak{V}(\varrho_1, h, w), \mathfrak{V}(\varrho_2, h, w))$ holds in $\mathfrak{T}$, for $\mathcal{RCC}$-8 predicates Q,
- $(\mathfrak{M}, h, w) \models \neg\varphi$ iff $(\mathfrak{M}, h, w) \not\models \varphi$,
- $(\mathfrak{M}, h, w) \models \varphi_1 \wedge \varphi_2$ iff $(\mathfrak{M}, h, w) \models \varphi_1$ and $(\mathfrak{M}, h, w) \models \varphi_2$,
- $(\mathfrak{M}, h, w) \models \varphi_1 \,\mathcal{U}\, \varphi_2$ iff there is $v > w$, $v \in h$, such that $(\mathfrak{M}, h, v) \models \varphi_2$ and $(\mathfrak{M}, h, u) \models \varphi_1$ for all $u \in (w, v)$,
- $(\mathfrak{M}, h, w) \models \mathsf{E}\varphi$ iff there is $h' \in \mathcal{H}(w)$ such that $(\mathfrak{M}, h', w) \models \varphi$,
- $(\mathfrak{M}, h, w) \models \mathsf{A}\varphi$ iff for all $h' \in \mathcal{H}(w)$, we have $(\mathfrak{M}, h', w) \models \varphi$.

A formula φ is called *satisfiable* if there exists a btt-model $\mathfrak{M}$ such that $(\mathfrak{M}, h, w) \models \varphi$ for some history $h \in \mathcal{H}$ and time point $w \in h$.

THEOREM 1.30 *The satisfiability problem for* $(\mathcal{LTL} \circ \mathcal{BRCC}\text{-}8)^b$ *formulas in btt-models is decidable.*

No significant result on the computational complexity of this satisfiability problem has been obtained yet.

As to satisfiability of $\mathcal{L}^{b^x}$-formulas, we again face the problem of infinitary temporal operations on region terms. Now, besides the linear temporal operators, the region terms can also be affected by the 'branch' operators A and E. In fact, at least for *discrete* topological spaces (i.e., spaces $\mathfrak{T} = (U, \mathbb{I})$ in which $\mathbb{I}$ is the identity function) we have the following negative result:

THEOREM 1.31 *The satisfiability problem for* $(\mathcal{LTL} \times \mathcal{BRCC}\text{-}8)^{b^x}$ *formulas in btt-models based on discrete topological spaces is undecidable.*

We conjecture that the satisfiability problem for $(\mathcal{LTL} \times \mathcal{BRCC}\text{-}8)^{b^x}$ formulas in btt-models based on arbitrary topological and Euclidean spaces is undecidable as well.

A natural way to search for decidable variants of the undecidable logics discussed above is to restrict the class of btt-models to those having finite sets of histories and where each history satisfies the finite state assumption. We conjecture that the satisfiability problem for $(\mathcal{LTL} \times \mathcal{BRCC}\text{-}8)^{b^x}$ formulas in this kind of btt-models is decidable.

REMARK 1.32 Temporalisations of $\mathcal{RCC}$-8 and $\mathcal{BRCC}$-8 with the help of Allen's interval calculus (see Remark 1.19) were considered in (Bennett et al., 2002; Gerevini and Nebel, 2002; Gabbay et al., 2003).

7. Combining distance logics with temporal logics

Unfortunately, not so much is known about temporal extensions of logics of distance spaces. Of course, some of the 'negative' results from Sec. 6 hold for similar combinations with those distance logics that contain $\mathcal{S}4_u$ as a sub-logic (for example, $\mathcal{MT}$ or $\mathcal{CSL}$). The technique of the proof of Theorem 1.28 can be used to show that the temporalisations $\mathcal{LTL}[L]$ of the logics L of distance spaces from Sec. 3.3 (where the temporal operators can only be applied to formulas but not to spatial terms) inherit the complexity of L (see the end of this section). And finally, some of the methods developed to deal with products of modal (in particular, temporal) logics (see Gabbay et al., 2003 and references therein) can be applied to analyse the computational behaviour of combinations of $\mathcal{LTL}$ with distance logics like $\mathcal{MS}^{\leq}$ which only contains distance operators of the form $\exists^{\leq a}$ (and their duals).

Denote by $\mathcal{LTL} \times \mathcal{MS}^{\leq}$ the spatio-temporal language satisfying the (PC), (LOC) and (AOC) principles and given by the following definition:

$$\begin{array}{lcl} \tau & ::= & p_i \mid \overline{\tau} \mid \tau_1 \sqcap \tau_2 \mid \tau_1 \sqcup \tau_2 \mid \exists^{\leq a}\tau \mid \tau_1 \,\mathcal{U}\, \tau_2, \\ \varphi & ::= & \tau_1 \sqsubseteq \tau_2 \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \,\mathcal{U}\, \varphi_2. \end{array}$$

As before, expressions of the form τ and φ are called $\mathcal{LTL} \times \mathcal{MS}^{\leq}$ *terms* and *formulas*, respectively.

A *metric temporal model* (*mt-model*, for short) is a pair of the form $\mathfrak{M} = (\mathfrak{D}, \mathfrak{V})$, where $\mathfrak{D} = (\Delta, d)$ is a metric space and $\mathfrak{V}$, a valuation, is a map associating with each spatial variable p and each time instant n a set $\mathfrak{V}(p, n) \subseteq \Delta$. The valuation can be inductively extended to arbitrary $\mathcal{LTL} \times \mathcal{MS}^{\leq}$ terms in a straightforward way:

- $\mathfrak{V}(\overline{\tau}, n) = \Delta - \mathfrak{V}(\tau, n), \qquad \mathfrak{V}(\tau_1 \sqcap \tau_2, n) = \mathfrak{V}(\tau_1, n) \cap \mathfrak{V}(\tau_2, n),$
- $\mathfrak{V}(\exists^{\leq a}\tau, n) = \big\{x \in \Delta \mid \exists y(d(x, y) < a \wedge y \in \mathfrak{V}(\tau, n))\big\},$
- $\mathfrak{V}(\tau_1 \,\mathcal{U}\, \tau_2, n) = \bigcup_{m>n} \Big(\mathfrak{V}(\tau_2, m) \cap \bigcap_{k\in(n,m)} \mathfrak{V}(\tau_1, k)\Big).$

The truth-values of $\mathcal{LTL} \times \mathcal{MS}^{\leq}$ formulas in mt-models are defined in precisely the same way as for spatio-temporal logics from Sec. 6.1. As before, we freely use the temporal operators $\bigcirc$, $\Diamond_F$ and $\Box_F$ (as well as their non-strict versions $\Diamond_F^+$ and $\Box_F^+$).

As an example of an $\mathcal{LTL} \times \mathcal{MS}^{\leq}$ formula, consider the following formalisation of (I) from Sec. 2:

$$\bigwedge_{i=1,2} \big((\mathit{desert}_i \neq \bot) \;\wedge\; \Box_F^+(\exists^{\leq a}\mathit{desert}_i \sqsubseteq \bigcirc \mathit{desert}_i)\big) \;\rightarrow\; \Diamond_F\Box_F\big(\mathit{desert}_1 \sqcap \mathit{desert}_2 \neq \bot\big).$$

It says that two nonempty deserts (say, the Kalahari and the Sahara) increasing their size in all directions by at least some $a \in \mathbb{Q}^{>0}$ each year will eventually intersect. Notice that this formula is valid in mt-models based on Euclidean spaces, but not in models based on disconnected or discrete metric spaces.

Unfortunately, the complexity of this combination of a PSpace-complete and an ExpTime-complete logics turns out to be too high. Using an almost straightforward encoding of the recurring tiling problem (see, e.g., the proof of Theorem 11.1 in Gabbay et al., 2003) one can prove the following:

Theorem 1.33 *The satisfiability problem for $\mathcal{LTL} \times \mathcal{MS}^{\leq}$ formulas in mt-models is Σ_1^1-complete.*

This result might suggest that combinations of $\mathcal{LTL}$ with $\mathcal{MS}^{\leq}$ have the same computational properties as combinations with $\mathcal{S}4_u$ in Sec. 6.1. However, this is not the case. To see the difference, let us consider the problem of *term satisfiability* for both languages: a term τ is *satisfiable* if there is a model $\mathfrak{M}$ for the language where $\mathfrak{V}(\tau, n)$ is not empty for some time moment n. It can be shown (similarly to the proof of Theorem 1.20) that the satisfiability problem for $\mathcal{LTL} \times \mathcal{S}4_u$ terms is Σ_1^1-complete. In the case of $\mathcal{MS}^{\leq}$ the picture is slightly better—the problem is decidable but not in time bounded by any 'tower' of exponents:

THEOREM 1.34 *The satisfiability problem for $\mathcal{LTL} \times \mathcal{MS}^{\leq}$ terms in mt-models is decidable, but not in elementary time.*

The proof requires three ingredients. First, one can show (similarly to the proof of Theorem 1.11) that any satisfiable $\mathcal{LTL} \times \mathcal{MS}^{\leq}$ term is satisfiable in an mt-model based on a *tree metric space*. This observation makes it possible to apply the methods developed to analyse the product modal logic $\mathbf{PTL}_{\Box\bigcirc} \times \mathbf{K}$. In particular, the decidability result is proved analogously to Theorem 13.6 from (Gabbay et al., 2003): first, mt-models are represented in the form of *quasimodels*, and then the existence of a quasimodel for a given term is encoded in monadic second-order logic. The non-elementary lower bound can be established by a polynomial reduction of the satisfiability problem for $\mathbf{PTL}_{\Box\bigcirc} \times \mathbf{K}$ (which is non-elementary by Theorem 6.37 and Claim 6.25 of Gabbay et al., 2003) to satisfiability of $\mathcal{LTL} \times \mathcal{MS}^{\leq}$ terms.

It is worth noting that the language of $\mathcal{LTL} \times \mathcal{MS}^{\leq}$ terms is 'local' in the sense that every term refers to a bounded area of the metric space, and the size of this area can be effectively computed. (In particular, statement (I) refers to the whole space, and so cannot be expressed in the language of $\mathcal{LTL} \times \mathcal{MS}^{\leq}$ terms.) In fact, this is the crucial observation required for the decidability result. $\mathcal{LTL} \times \mathcal{MS}^{\leq}$ formulas, on the contrary, can speak about the whole space which makes it possible to simulate tilings.

In the same way one can explain the computational behaviour of the combination $\mathcal{LTL} \circ \mathcal{MS}^{\leq}$ satisfying both (PC) and (LOC), but not (AOC). It is defined analogously to the spatio-temporal case by replacing $\tau_1 \mathcal{U} \tau_2$ with $\bigcirc\tau$ in the definition of terms:

$$\tau \quad ::= \quad p_i \quad | \quad \overline{\tau} \quad | \quad \tau_1 \sqcap \tau_2 \quad | \quad \tau_1 \sqcup \tau_2 \quad | \quad \exists^{\leq a}\tau \quad | \quad \bigcirc\tau.$$

The formulas of $\mathcal{LTL} \circ \mathcal{MS}^{\leq}$ are defined in the same way as above.

As Theorem 1.24 might suggest, the same negative result holds even for this restricted combination:

THEOREM 1.35 *The satisfiability problem for $\mathcal{LTL} \circ \mathcal{MS}^{\leq}$ formulas in mt-models is undecidable.*

On the other hand, $\mathcal{LTL} \circ \mathcal{MS}^{\leq}$ terms are basically 'harmless' because they can only speak about limited time, at most $\ell(\tau)$ time moments, to be more precise (where $\ell(\tau)$ is the length of τ). So we have the following:

THEOREM 1.36 *The satisfiability problem for $\mathcal{LTL} \circ \mathcal{MS}^{\leq}$ terms in mt-models is* EXPTIME*-complete under both unary and binary coding of parameters in distance operators.*

Finally, the temporalisations of distance logics satisfying only the (PC) principle, and so containing no temporal operators in spatial terms, inherit the higher complexity of the spatial component, which is proved similarly to the proof of Theorem 1.28:

THEOREM 1.37 *The satisfiability problem for $\mathcal{LTL}[\mathcal{MS}^{\leq,<}]$, $\mathcal{LTL}[\mathcal{MT}]$, and $\mathcal{LTL}[\mathcal{CMS}]$ formulas in mt-models is* EXPTIME*-complete for both unary and binary coding of parameters.*

8. Logics for dynamical systems

The snapshot models and the corresponding spatio-temporal logics discussed above are a convenient tool for representing and reasoning about evolutions of spatial configurations of regions such as the political (geographical, weather, etc.) map of the changing world, where we are interested in keeping track of the relations between regions.

On the other hand, if we want to model how an object moves over an otherwise stable space and keep track of its asymptotic trajectory then different models of space and time may be preferable, namely models corresponding to dynamical systems (see, e.g., Brown, 1976; Katok and Hasselblatt, 1995).

A *dynamical model* is a pair of the form

$$\mathfrak{A} = (\mathfrak{M}, g), \tag{1.26}$$

where $\mathfrak{M} = (\mathfrak{S}, p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}, \dots)$ is a spatial model and g is a total function on the space $\mathfrak{S}$. Often g is required to satisfy certain constraints depending on the structure of $\mathfrak{S}$. For example, if $\mathfrak{S}$ is a topological space, then g is often required to be continuous or even a bijective continuous and open mapping (that is, a homeomorphism).

In the framework of such models, we are interested in the *orbits*

$$Orb_g(w) = \{g(w),\ g^2(w),\ \dots\}$$

of certain points w from $\mathfrak{S}$ (representing moving objects). The model $\mathfrak{M}$ describes a spatial environment in which w moves according to the rule (law) g. A typical question in the framework of dynamical models is whether a point from a region $p_0^{\mathfrak{M}}$ will eventually reach $p_1^{\mathfrak{M}}$ without visiting $p_2^{\mathfrak{M}}$, or whether the rule g is such that w will be returning to $p_1^{\mathfrak{M}}$ infinitely often.

The aim of this section is to discuss the existing logics capable of talking about some aspects of dynamical models. In particular, we consider the relation between the spatio-temporal logics above and logics for dynamical models. Before reading this section the reader is recommended to have a look at Ch. **??**.

Let us begin with two illuminative examples.

A physical system. Consider a physical system with a single degree of freedom, say, a body having mass m and moving along some axis. The movement of the body in a force field $f(x,t)$ can be described by the following system of differential equations:

$$\begin{aligned} \dot{x}(t) &= v(t), \\ \dot{v}(t) &= f(x,t)/m, \end{aligned}$$

where $x(t)$ and $v(t)$ are, respectively, the position and the velocity of the body at time t. For every initial point $(x_0, v_0) \in \mathbb{R}^2$, the differential equations determine the trajectory $\pi_{(x_0,v_0)}(t)$ of the body (more precisely, its position and velocity) that starts with the velocity v_0 at the position x_0 and moves according to the above equations. The collection of all those trajectories for different initial conditions form the *phase portrait* of the differential equation (depicted in the left-hand side of Fig. 1.9).

Now consider the function $\phi((x,v),t)$, called the *flow* of the equations, defined by taking $\phi((x,v),t) = \pi_{(x,v)}(t)$. Note that

- $\phi((x,v),0) = (x,v)$ and
- $\phi((x,v),t+s) = \phi(\phi((x,v),t),s)$.

The graph of this function represents trajectories in $\mathbb{R}^2 \times \mathbb{R}$ and the phase portrait can be considered as the projection of $\phi((x,v),t)$ onto $\mathbb{R}^2$; see Fig. 1.9. Note also that $\phi((x,v),t)$ is continuous in all coordinates.

Given such a physical system, we usually want to know answers to the following standard questions. Suppose that the initial conditions (position and velocity) of the body are restricted by some set $I \subseteq \mathbb{R}^2$. Is it the case that starting from any point of I the body will eventually reach some point in another set F? Will it be visiting F infinitely often? Is it the case that the body will never hit some 'danger zone' $D \subseteq \mathbb{R}^2$?

A dynamical model $(\mathfrak{M}, g)$ for the differential equations above can be defined as follows. The underlying space $\mathfrak{S}$ of $\mathfrak{M}$ is the Euclidean plane

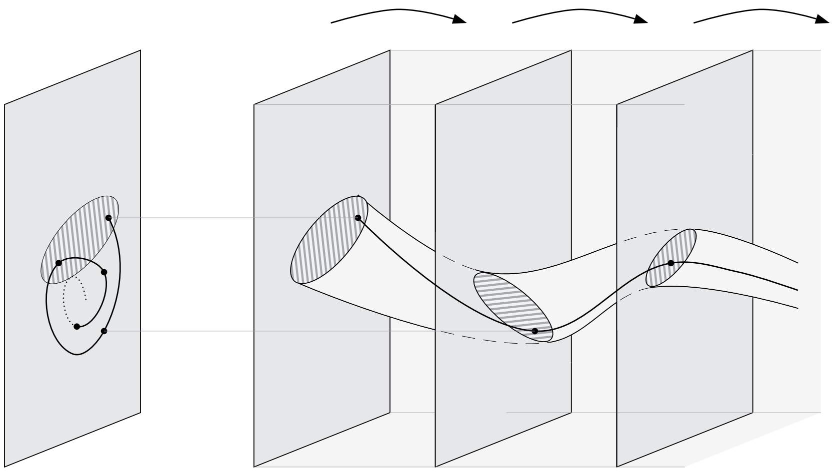


Figure 1.9. Dynamical system.

$\mathbb{R}^2$. Let $g(x, v) = \phi((x, v), \delta)$, for some fixed small time unit $\delta > 0$. The predicates $p_i^{\mathfrak{M}}$ can model the initial and final conditions I and F, the danger zone D, etc. As g is easily seen to be continuous, $(\mathfrak{M}, g)$ is a dynamical model with a continuous function on $\mathbb{R}^2$. The three questions above can then be formalised as whether we have

- $I \subseteq \bigcup_{i>0} g^{-i}(F)$,
- $I \subseteq \bigcap_{j>0} g^{-j}\big(\bigcup_{i>0} g^{-i}(F)\big)$,
- $I \cap \bigcup_{i>0} g^{-i}(D) = \emptyset$.

It is to be noted that, on the other hand, the flow $\phi((x, v), t)$ can be regarded as a snapshot spatio-temporal model

$$\mathfrak{M}_0, \quad \mathfrak{M}_1, \quad \dots,$$

where $\mathfrak{M}_i = \big(\mathfrak{S}, g^{-i}(p_0^{\mathfrak{M}}), g^{-i}(p_1^{\mathfrak{M}}), \dots\big)$, for $i \geq 0$. The intuition behind this definition is as follows: a point (x, v) belongs to a set $Y \subseteq \mathbb{R}^2$ at time point i iff (x, v) is moved to Y by i consecutive applications of g, that is, $(x, v) \in g^{-i}(Y)$.

Game of Life. Our second example is the *Game of Life* invented by J.H. Conway in the 1970s (see, e.g., Allouche et al., 2001). The game is defined as follows. We have a finite $\{1, \dots, n\} \times \{1, \dots, n\}$ or an infinite $\mathbb{Z} \times \mathbb{Z}$ board. Each point on the board is either occupied or vacant (living or dead). At each regular time step the points of the board simultaneously change according to the following rules:

(birth) a vacant point with exactly three occupied neighbours becomes an occupied cell,

(survival) an occupied point with two or three occupied neighbours stays occupied,

(death) in all other cases, the point becomes or remains vacant.

Thus, at each step $i \geq 0$ the state of the game can be represented by the spatial model

$$\mathfrak{M}_i = (\mathfrak{S}, o^{\mathfrak{M}_i}, v^{\mathfrak{M}_i}), \tag{1.27}$$

where $\mathfrak{S}$ is the board, $o^{\mathfrak{M}_i}$ is the set of occupied points at step i and $v^{\mathfrak{M}_i}$ the set of vacant ones.

The Game of Life can be represented by the spatial transition system which consists of all possible models $\mathfrak{M}$ of the form (1.27), and $\mathfrak{M} \rightarrowtail \mathfrak{M}'$ holds iff $\mathfrak{M}'$ is obtained from $\mathfrak{M}$ by one step of the game according to the rules above. As the Game of Life is deterministic (for every $\mathfrak{M}$ there is exactly one $\mathfrak{M}'$ such that $\mathfrak{M} \rightarrowtail \mathfrak{M}'$), there is exactly one evolution for any spatial transition system representing it. In other words, for every initial state of the Game we obtain exactly one snapshot model.

The Game of Life (on, say, $\mathbb{Z}\times\mathbb{Z}$) can also be formalised as a dynamical model

$$\mathfrak{N} = \big((\mathfrak{T}, p_0^{\mathfrak{N}}, p_1^{\mathfrak{N}}, \dots), g\big).$$

The underlying space $\mathfrak{T}$ is comprised of all functions from $\mathbb{Z} \times \mathbb{Z}$ into $\{o, v\}$ representing distributions of occupied and vacant points, that is, $\mathfrak{T} = \{o, v\}^{\mathbb{Z}\times\mathbb{Z}}$. The function g maps every $\eta \in \{o, v\}^{\mathbb{Z}\times\mathbb{Z}}$ to the function $g(\eta) \in \{o, v\}^{\mathbb{Z}\times\mathbb{Z}}$ representing the next distribution of occupied and vacant points. In other words, the underlying space can be regarded as the set of all models $(\mathfrak{S}, o^{\mathfrak{M}}, v^{\mathfrak{M}})$ with the function g given by the transition relation (rule) $\rightarrowtail$. Finally, define a metric d on $\{o, v\}^{\mathbb{Z}\times\mathbb{Z}}$ so that g becomes a continuous function for the induced topology $\mathbb{I}_d$ as follows. Set, for $\eta_1, \eta_2 \in \{o, v\}^{\mathbb{Z}\times\mathbb{Z}}$,

$$d(\eta_1, \eta_2) = \frac{1}{k}$$

iff η_1 and η_2 agree on all points within the $k \times k$ square

$$I_k = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} \mid \max\{n, m\} < k\}$$

but disagree on at least one point in I_{k+1}. One can show that the metric d defines a compact topological space on $\{o, v\}^{\mathbb{Z}\times\mathbb{Z}}$ with respect to which g is continuous.

Notice that in this dynamical model predicates are not subsets of the board $\mathbb{Z} \times \mathbb{Z}$ but of $\{o, v\}^{\mathbb{Z}\times\mathbb{Z}}$. We can take, for instance, some interesting set $p_0^{\mathfrak{N}}$ of initial states (i.e., models $(\mathfrak{G}, o^{\mathfrak{M}}, v^{\mathfrak{M}})$), say, those with precisely N living points, and check whether all of them (or at least one of them) will eventually 'die out,' that is, reach the singleton set $p_1^{\mathfrak{N}} = \{(\mathfrak{G}, o^{\mathfrak{M}}, v^{\mathfrak{M}})\}$ where $o^{\mathfrak{M}}$ is empty.

The resulting dynamical model $\mathfrak{N} = \big((\mathfrak{T}, p_0^{\mathfrak{N}}, p_1^{\mathfrak{N}}, \dots), g\big)$ can be 'unravelled' into the transition system $s_0 \rightarrowtail s_1 \rightarrowtail \dots$ where

$$\mu(s_n) = \big(\mathfrak{T}, g^{-n}(p_0^{\mathfrak{N}}), g^{-n}(p_1^{\mathfrak{N}}), \dots\big).$$

8.1 Dynamic topological logics

We start our discussion of languages for reasoning about dynamical systems by considering dynamical models based on various topological spaces.

A *dynamic topological model* (DTM, for short) is a pair

$$\mathfrak{A} = (\mathfrak{M}, g),$$

where $\mathfrak{M} = (\mathfrak{T}, p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}, \dots)$ is a topological model based on a topological space $\mathfrak{T} = (U, \mathbb{I})$ and g is a function on $\mathfrak{T}$. The minimum requirement imposed on g in dynamical systems is its continuity. We remind the reader that a function g on $\mathfrak{T}$ is called *continuous* if $g^{-1}(X)$ is open whenever $X \subseteq U$ is open. If $g(X)$ is open whenever X is open, then g is called *open*. Another important type of functions is *homeomorphisms*, that is, bijective continuous and open functions on $\mathfrak{T}$. (It is also usually assumed that the underlying topological spaces are compact. We will not make this assumption in general, but point out when our results hold for compact topological spaces.)

The language we consider for representing and reasoning about dynamic topological systems is slightly different from most of the languages for snapshot models because, as we have already seen, in dynamical systems we are more interested in following the orbit of an object in space and time rather than in comparing the relative positions of regions in space. That is why the language $\mathcal{DTL}$ for reasoning about topological systems is 'local' in the sense that we see the space from the windows of our moving 'car' as opposed to the 'global' language of spatio-temporal logics from Sec. 6.1 where we could observe all moving 'cars' and their relative positions. Formally, this means that we represent knowledge about the evolution of objects by means of terms and do not consider formulas constructed from them.

The set of $\mathcal{DTL}$-*terms* τ is defined as follows:

$$\tau \quad ::= \quad p \quad | \quad \overline{\tau} \quad | \quad \tau_1 \sqcap \tau_2 \quad | \quad \mathbf{I}\tau \quad | \quad \bigcirc\tau \quad | \quad \Box_F\tau \quad | \quad \Diamond_F\tau.$$

It is worth noting that the addition of the operator $\mathcal{U}$ for 'until' to the set of constructors for terms would not affect any of the results presented below. We have omitted 'until' to keep the language as simple as possible.

In a dynamic topological model $\mathfrak{A} = (\mathfrak{M}, g)$, terms τ are interpreted as sets $\tau^{\mathfrak{A}} \subseteq U$, where $\mathfrak{M}$ is based on the topological space $(U, \mathbb{I})$. Clearly, $p_i^{\mathfrak{A}} = p_i^{\mathfrak{M}}$ for every spatial variable p_i. The Boolean operators and the operator $\mathbf{I}$ are interpreted as before. The interpretation of the temporal operators on terms should become clear from the following consideration: for a point $w \in U$ and a term τ, we have

$$w \in (\bigcirc\tau)^{\mathfrak{A}} \quad \text{iff} \quad g(w) \in \tau^{\mathfrak{A}} \quad \text{iff} \quad w \in g^{-1}(\tau^{\mathfrak{A}}). \tag{1.28}$$

Roughly, a time point n in a snapshot model corresponds to n applications of the function g. If we understand $w \in (\Diamond_F\tau)^{\mathfrak{A}}$ as 'eventually w will be moved by g to $\tau^{\mathfrak{A}}$' and $w \in (\Box_F\tau)^{\mathfrak{A}}$ as 'g will always keep w in $\tau^{\mathfrak{A}}$,' then

$$w \in (\Diamond_F\tau)^{\mathfrak{A}} \quad \text{iff} \quad Orb_g(w) \cap \tau^{\mathfrak{A}} \neq \emptyset \quad \text{iff} \quad w \in \bigcup_{i>0} g^{-i}(\tau^{\mathfrak{A}}), \tag{1.29}$$

$$w \in (\Box_F\tau)^{\mathfrak{A}} \quad \text{iff} \quad Orb_g(w) \subseteq \tau^{\mathfrak{A}} \quad \text{iff} \quad w \in \bigcap_{i>0} g^{-i}(\tau^{\mathfrak{A}}). \tag{1.30}$$

For example, $w \in \big(p_1 \sqcap \Diamond_F p_2\big)^{\mathfrak{A}}$ means that w is in $p_1^{\mathfrak{A}}$ and reaches $p_2^{\mathfrak{A}}$ by a finite number of iterations of g.

In this section, we are interested in the satisfiability and validity problem for $\mathcal{DTL}$-terms in some important classes of dynamic topological models:

- A $\mathcal{DTL}$-term τ is *satisfiable* in a class $\mathcal{M}$ of DTMs iff there exists $\mathfrak{A} \in \mathcal{M}$ such that $\tau^{\mathfrak{A}} \neq \emptyset$.
- A $\mathcal{DTL}$-term τ is *valid* in a class $\mathcal{M}$ of DTMs iff $\overline{\tau}$ is not satisfiable in $\mathcal{M}$—i.e., iff $\tau^{\mathfrak{A}}$ coincides with the whole space for every $\mathfrak{A} \in \mathcal{M}$.

DTMs with homeomorphisms. We first connect satisfiability in certain dynamic topological models with satisfiability in snapshot topological temporal models. The discussion of the two examples above indicates already how one can go back and forth between snapshot topological models and dynamic topological models. More precisely, one can show the following:

THEOREM 1.38 *Let $\mathcal{M}$ be any of the following classes of dynamic topological models:*

- *DTMs based on Aleksandrov spaces with homeomorphisms;*
- *DTMs based on topological spaces with homeomorphisms;*
- *DTMs based on $\mathbb{R}^n$ with homeomorphisms, for $n > 1$;*
- *DTMs based on the n-dimensional unit ball with a measure preserving homeomorphism, for $n > 1$.*

Then a $\mathcal{DTL}$-term τ is satisfiable in $\mathcal{M}$ iff the formula $\neg(\tau = \bot)$ is satisfiable in a snapshot tt-model based on a topological space underlying some model from the class $\mathcal{M}$.

It should not come as a surprise now that reasoning with $\mathcal{DTL}$-terms about these classes of DTMs can be extremely complex. The following result was proved in (Konev et al., 2006b) by reduction of Post's correspondence problem:

THEOREM 1.39 *Let $\mathcal{M}$ be any of the classes of DTMs mentioned in Theorem 1.38. Then the set of $\mathcal{DTL}$-terms that are valid in models from $\mathcal{M}$ is not recursively enumerable.*

It is worth noting that the four sets of terms that are valid in the classes of models mentioned in Theorem 1.38 are all different. As was shown by Slavnov (2003), the term

$$\mathbf{I}\Diamond_F\big(p \sqcap \mathbf{CI}\overline{p}\big)$$

is not satisfiable in any DTM based on $(\mathbb{R}^n, g)$, while it is clearly satisfiable in some DTM. According to Kremer and Mints (2005), the term

$$\mathbf{I}p \ \rightarrow\ \mathbf{C}\Diamond_F\mathbf{I}p,$$

where $\tau_1 \rightarrow \tau_2 = \overline{\tau_1} \sqcup \tau_2$, is valid in all unit balls, but refuted in a DTM based on an Aleksandrov space and a DTM based on $\mathbb{R}^n$ with the homeomorphism $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, x_n + 1)$. Finally, the $\mathcal{DTL}$-term

$$\Box_F\mathbf{I}p \ \rightarrow\ \mathbf{I}\Box_F p$$

is valid in DTMs based on Aleksandrov spaces, but refuted in the classes of DTMs based on Euclidean spaces and unit balls.

DTMs with continuous functions. Theorem 1.38 shows that DTMs with *homeomorphisms* behave similarly to topological snapshot models. This situation changes drastically for DTMs with *continuous functions* (which are not necessarily open). In this case, no corresponding snapshot tt-models have been developed. To clarify—at least to some extent—the relation between the two kinds of models, let us consider DTMs based on Aleksandrov spaces.

Suppose that an Aleksandrov topological space $\mathfrak{T}_{\mathfrak{G}} = (W, \mathbb{I}_{\mathfrak{G}})$ is induced by the quasi-order $\mathfrak{G} = (W, R)$ (see Sec. 3.1). Then it is easy to check that a function $g\colon W \to W$ is a *continuous function* on $\mathfrak{T}_{\mathfrak{G}}$ iff for all $u, v \in W$,

$$uRv \qquad \text{implies} \qquad g(u)Rg(v).$$

(A bijection f is a *homeomorphism* on $\mathfrak{T}_{\mathfrak{G}}$ iff both the above implication and its converse hold.)

This observation suggests that DTMs based on Aleksandrov spaces with continuous functions correspond to what may be called *Aleksandrov snapshot models with expanding domains*. Indeed, suppose that $\mathfrak{A} = (\mathfrak{M}, g)$ is a DTM where $\mathfrak{M} = (\mathfrak{T}_{\mathfrak{G}}, p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}, \dots)$, $\mathfrak{G} = (W, R)$ is as above and g is a continuous and surjective map on $\mathfrak{T}_{\mathfrak{G}}$. Consider the sequence of models

$$\mathfrak{M}_0 = \mathfrak{M}, \quad \mathfrak{M}_1 = (\mathfrak{T}_{\mathfrak{G}_1}, p_0^{\mathfrak{M}_1}, \dots), \quad \mathfrak{M}_2 = (\mathfrak{T}_{\mathfrak{G}_2}, p_0^{\mathfrak{M}_2}, \dots), \dots \qquad (1.31)$$

where

- $\mathfrak{G}_n = (W, R_n)$,
- uR_nv iff $g^n(u)Rg^n(v)$ for any $u, v \in W$,
- $u \in p_i^{\mathfrak{M}_n}$ iff $g^n(u) \in p_i^{\mathfrak{M}}$.

The temporal and topological operators on this sequence of models can be interpreted in exactly the same way as in Sec. 6.1. In particular,

- $u \in (\mathbf{C}\tau)^{\mathfrak{M}_n}$ iff there is $v \in W$ such that uR_nv and $v \in \tau^{\mathfrak{M}_n}$,
- $u \in (\Diamond_F \tau)^{\mathfrak{M}_n}$ iff there is $m > n$ such that $u \in \tau^{\mathfrak{M}_m}$.

We then obtain that, for every $\mathcal{DTL}$-term τ, every $w \in W$ and every $n \geq 0$,

$$g^n(w) \in \tau^{\mathfrak{A}} \quad \text{iff} \quad w \in \tau^{\mathfrak{M}_n},$$

and so τ is satisfiable in $\mathfrak{A}$ iff τ is satisfiable in (1.31).

The difference between (1.31) and the snapshot models we have considered before is that the spaces $\mathfrak{T}_{\mathfrak{G}_n}$ or, which is the same, the quasi-orders $\mathfrak{G}_n = (W, R_n)$ do not necessarily coincide. More precisely, using

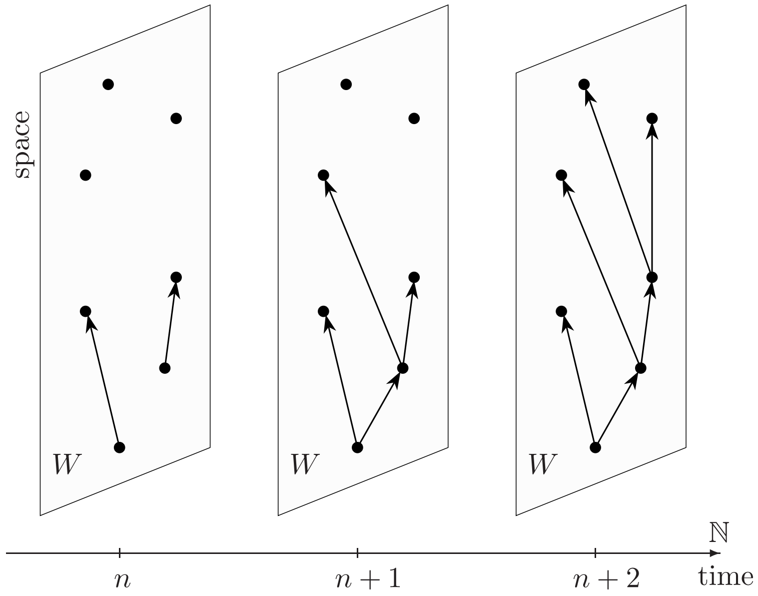


Figure 1.10. Model with expanding domains.

the fact that g is continuous it is easy to see that $R_n \subseteq R_{n+1}$ for every $n \geq 0$; see Fig. 1.10. That is why we call these models *snapshot models with expanding domains*. Fig. 1.10 also shows that the term

$$\bigcirc \mathbf{C}\tau \rightarrow \mathbf{C}\bigcirc \tau$$

is not valid in all DTMs with continuous functions, while it is clearly valid in all DTMs with homeomorphisms. For more details on the connection between such models and DTMs based on Aleksandrov spaces with continuous functions see (Gabelaia et al., 2006).

It is known (see, e.g., Gabbay et al., 2003) that satisfiability in models with expanding domains can be reduced to satisfiability in models with constant domains, but not the other way round as we shall see a bit later. So in principle one could expect that the dynamic topological logics interpreted in DTMs based on arbitrary, Aleksandrov or Euclidean topological spaces with continuous functions behave 'better' than their counterparts with homeomorphisms. Indeed, a fine-grained complexity analysis reveals interesting differences between the logic of homeomorphisms and the logic of continuous functions. We begin with the following 'negative' theorem proved in (Konev et al., 2005; Konev et al., 2006a):

THEOREM 1.40 *Let $\mathcal{M}$ be any of the following classes of dynamic topological models:*

- *DTMs based on Aleksandrov spaces with continuous functions;*
- *DTMs based on topological spaces with continuous functions;*
- *DTMs based on $\mathbb{R}^n$ with continuous functions, for $n \geq 1$.*

Then the satisfiability problem for $\mathcal{DTL}$-terms in $\mathcal{M}$ is undecidable.

Note that, in contrast to DTMs with homeomorphisms, it is still not clear whether any of these logics is recursively enumerable or even finitely axiomatisable. However, the first exciting difference between the algorithmic behaviour of the two models can be observed by considering the fragment of $\mathcal{DTL}$ in which the topological operators are not applied to formulas containing the 'infinitary' temporal operators $\Box_F$ and $\Diamond_F$. This language is still very expressive and the undecidability/non-axiomatisability results of Theorems 1.39 and 1.40 still hold for it. However, the set of formulas from this fragment that are valid in DTMs based on Aleksandrov spaces or arbitrary topological spaces with continuous functions is recursively enumerable. This is proved in (Konev et al., 2006a) by an application of Kruskal's tree theorem.

The proof of Theorem 1.40 proceeds by a rather involved reduction of the ω-reachability problem for lossy channel systems (see Schnoebelen, 2002). It essentially uses the fact that the number of function iterations is infinite. This observation opens a second possibility for a fine-grained complexity analysis: what happens if we consider DTMs where only *finitely* (but unboundedly) many function iterations are allowed. In this case the interpretation of $\mathcal{DTL}$-terms containing temporal operations depends of course on the iteration step of g.

More precisely, let $\mathfrak{A} = (\mathfrak{M}, g)$ be a DTM based on a topological space $(U, \mathbb{I})$, $N > 0$ is the allowed number of iterations of g, and $n \leq N$. Given a $\mathcal{DTL}$-term τ, we define $\tau^{\mathfrak{A},n,N}$, the *extension of τ after n steps in the DTM $\mathfrak{A}$ with N iterations*, inductively as follows:

- $p_i^{\mathfrak{A},n,N} = p_i^{\mathfrak{M}}$,
- $(\tau_1 \sqcap \tau_2)^{\mathfrak{A},n,N} = \tau_1^{\mathfrak{A},n,N} \cap \tau_2^{\mathfrak{A},n,N}$,
- $(\overline{\tau})^{\mathfrak{A},n,N} = U - \tau^{\mathfrak{A},n,N}$,
- $(\mathbf{I}\tau)^{\mathfrak{A},n,N} = \mathbb{I}\tau^{\mathfrak{A},n,N}$,

- $(\bigcirc\tau)^{\mathfrak{A},n,N} = \emptyset$ for $n = N$, and $(\bigcirc\tau)^{\mathfrak{A},n,N} = g^{-1}(\tau^{\mathfrak{A},n+1,N})$ otherwise,
- $(\Diamond_F\tau)^{\mathfrak{A},n,N} = \bigcup_{m=n+1}^{N} g^{n-m}(\tau^{\mathfrak{A},m,N}).$

Say that τ is *satisfiable in DTMs from a class $\mathcal{M}$ with finite iterations*, or *fi-satisfiable in $\mathcal{M}$*, for short, if there exist a DTM $\mathfrak{A}$ in $\mathcal{M}$ and $N > 0$ such that $\tau^{\mathfrak{A},0,N} \neq \emptyset$.

It is not hard to see that the reduction of Post's correspondence problem from the proof of Theorem 1.39 can be also used to prove the following:

THEOREM 1.41 *Let $\mathcal{M}$ be any of the classes of DTMs mentioned in Theorem 1.38. Then fi-satisfiability of $\mathcal{DTL}$-terms in $\mathcal{M}$ is undecidable.*

On the contrary, if we consider the class of DTMs based on arbitrary topological spaces with continuous functions then one can first reduce fi-satisfiability in this class to fi-satisfiability in DTMs based on finite Aleksandrov spaces with continuous functions, and then use Kruskal's tree theorem to prove the following:

THEOREM 1.42 *Let $\mathcal{M}$ be one of the following classes:*

- *DTMs based on Aleksandrov spaces with continuous functions,*
- *DTMs based on topological spaces with continuous functions.*

Then fi-satisfiability of $\mathcal{DTL}$-terms in $\mathcal{M}$ is decidable, but not in primitive recursive time.

The non-primitive recursive lower bound is proved by reduction of the reachability problem for lossy channel systems. All details can be found in (Gabelaia et al., 2006).

8.2 Dynamic metric logics

In this section we fill the missing gap and consider the fourth formal model—dynamic metric systems. Similarly to dynamic topological models from Sec. 8.1, a *dynamic metric model* (DMM, for short) is a pair of the form

$$\mathfrak{A} = (\mathfrak{M}, g),$$

where $\mathfrak{M} = (\mathfrak{D}, p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}, \dots)$ is a metric model based on a metric space $\mathfrak{D} = (\Delta, d)$ and g is a function on $\mathfrak{D}$. We will only consider *isometric functions*, i.e., bijections on Δ such that $d(x, y) = d(g(x), g(y))$, for all $x, y \in \Delta$. For instance, the translation $x \mapsto x + 1$ and reflection $x \mapsto -x$ maps on $\mathbb{R}$, the rotations g_α of the two-dimensional unit ball $B^2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$ by the angle α around $(0, 0)$ are isometric automorphisms on the respective spaces.

We only consider the simplest language $\mathcal{DML}^{\leq}$ of *dynamic metric logic*, which is defined in the same way as $\mathcal{DTL}$ with the exception that the topological operators are now replaced by the *metric operators* $\exists^{\leq a}$ and $\forall^{\leq a}$, for $a \in \mathbb{Q}^{\geq 0}$. Formally, $\mathcal{DML}^{\leq}$-terms are

$$\tau \quad ::= \quad p \quad | \quad \overline{\tau} \quad | \quad \tau_1 \sqcap \tau_2 \quad | \quad \exists^{\leq a}\tau \quad | \quad \bigcirc\tau \quad | \quad \Box_F\tau \quad | \quad \Diamond_F\tau.$$

Again, we omit the 'until' operator to keep our language simple, although all the results can be extended to the language including 'until.'

In a dynamic metric model $\mathfrak{A} = (\mathfrak{M}, g)$, terms τ are interpreted as sets $\tau^{\mathfrak{A}} \subseteq \Delta$, where $\mathfrak{M}$ is based on the metric space $\mathfrak{D} = (\Delta, d)$. For spatial variables we have $p_i^{\mathfrak{A}} = p_i^{\mathfrak{M}}$; the Boolean operators are interpreted as usual, the metric operators $\exists^{\leq a}\tau$ and $\forall^{\leq a}\tau$ as in Sec. 3.3, and the temporal operators as in Sec. 8.1.

The notions of satisfiability and validity of $\mathcal{DML}^{\leq}$-terms are defined in the standard way. The next theorem connects satisfiability in DMMs with satisfiability in snapshot models from Sec. 7:

THEOREM 1.43 *A $\mathcal{DML}^{\leq}$-term τ is satisfiable in a DMM with an isometric function iff the formula $\neg(\tau = \bot)$ is satisfiable in a metric snapshot model based on the same metric space.*

As it happened with metric temporal logics in Sec. 7, dynamic metric logics are slightly simpler then their topological counterparts:

THEOREM 1.44 *The set of $\mathcal{DML}^{\leq}$-terms that are valid in DMMs with isometric functions is decidable. However, the decision problem is not elementary.*

This theorem should not come as a surprise: its claim and the proof are essentially the same as those of Theorem 1.34 (all details can be found in Konev et al., 2006b).

9. Related 'temporalised' formalisms

The logics we have considered in this chapter can be regarded as *temporalisations* of static spatial logics. As many other 'static' logics have also been extended by a temporal dimension, for example, first-order temporal logic (Gabbay et al., 1994), temporal epistemic logic (Fagin et al., 1995), temporal description logic (Gabbay et al., 2003), it makes sense to briefly discuss similarities and differences between these temporalisations.

The most generic approach to the temporalisation of a static logic is of course *first-order temporal logic*. In this logic, temporal operators may occur anywhere in first-order formulas (in particular, in the scope of quantifiers), and the intended models are flows of time where each time point is represented by a relational structure interpreting the first-order part of the language. It is known since the 1960s that the resulting logics are extremely complex, mostly Σ^1_1-complete (see, e.g., Gabbay et al., 1994; Gabbay et al., 2003 and references therein). For example, the two-variable fragment, the monadic fragment, and the guarded fragment of first-order temporal logic over the natural numbers and with constant or expanding domains is Σ^1_1-complete.

Only recently the so-called *monodic* fragments of first-order temporal logics (in which temporal operators are only applied to formulas with at most one free variable) have been identified as expressive yet often decidable (or at least recursively enumerable) fragments (Hodkinson et al., 2000; Hodkinson et al., 2001; Gabbay et al., 2003). The positive results about the monodic fragments rely, however, on the fact that they are not able to express that a *binary* relation does not change over time. In other words, in the monodic fragments one can reason about the change (or non-change) of unary predicates but not about the change (or non-change) of binary relations. This feature of monodic fragments is in sharp contrast with the logics we encounter in the context of spatio-temporal representation and reasoning: as we have seen, in this case we usually expect the underlying space (e.g., a metric or topological space) not to change in time. What changes is the extension of unary predicates. That is to say, we almost always have at least one constant binary relation (or higher-order operator): in metric spaces the relation $R(x, y)$ defined by $d(x, y) < a$, in Aleksandrov spaces the relation R inducing the topological space, in arbitrary topological spaces even the higher-order interior operator, etc. For this reason, *the results on the decidability of monodic fragments do not apply to spatio-temporal logics*. In fact, we have seen that the straightforward combination of spatial and temporal formalisms almost always leads to highly undecidable logics. In the more abstract setting of *products of modal logics* this phenomenon has been recently investigated by Gabbay et al. (2003), Gabelaia et al. (2005b, 2006).

The main message to be deduced from the results on combinations of spatial and temporal formalisms is that a fine-tuned analysis of both the spatial logic and the interaction between spatial and temporal operators is required in order to obtain expressive and still decidable formalisms. There appears to be no general way of translating positive results from other temporalisations to spatio-temporal logics. Actually, this is also the case for temporal epistemic logic and temporal description logic. Again, most of the 'positive' results in those areas depend on the assumption that one cannot reason about the change (and non-change) of binary relations. With one exception, the results in those areas are therefore much closer to the results on monodic fragments of first-order temporal logics than to the results on spatio-temporal logics.

The only exception from this rule we know is the decidability (in non-elementary time) of the satisfiability problem for terms of the metric temporal and dynamic logics. Although this result cannot be obtained as an instance of a known result from other temporalised formalisms, its proof nevertheless closely resembles the proofs of the following results:

- The decidability (in non-elementary time) of the temporal epistemic logic with multi-modal $\mathcal{S}5$ interpreted in synchronous systems with perfect recall and no learning (Halpern and Vardi, 1989).

- In temporal description logic, the decidability (in non-elementary time) of the satisfiability problem for temporalised $\mathcal{ALC}$ where roles (binary predicates) do not change over time (Wolter and Zakharyaschev, 1999; Gabbay et al., 2003).

In all these cases, we deal with models where certain relations do not change over time (in the epistemic case these are the equivalence relations interpreting the epistemic operators, in the description logic case these are the roles interpreting the value restrictions). The crucial property underlying the decidability proofs is that those constant relations can be assumed to form tree-like structures and that the satisfaction relation is 'local' in the sense that the interpretation of terms (propositional variables/concepts) in a certain distance from the root of the tree-like structure does not influence the satisfaction relation in the root.

Notice, however, that in each case one has to consider carefully the constraints on the relations. As we know from Theorem 1.21, a decidability proof does not go through for transitive relations (from Aleksandrov spaces) which do not change over time.

Acknowledgements

The work on this chapter was partially supported by the U.K. EPSRC grants GR/S61966, GR/S63182, GR/S63175, GR/S61973.

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Index