From Points to Regions (and back again)

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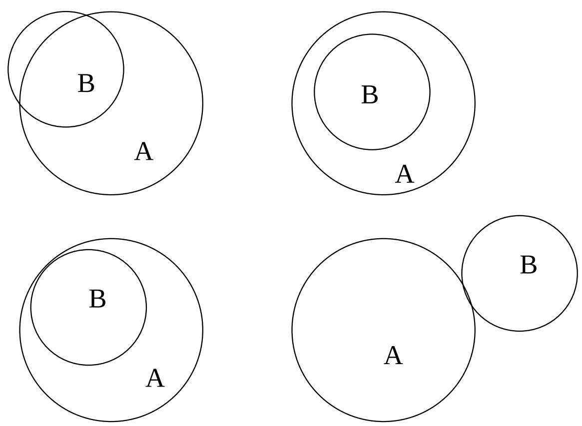
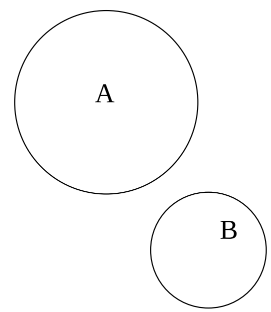
- Most theories of space take points as the primitive entities.
- This postulation has always troubled some philosophers.
- What happens if we take *regions* rather than *points* as the primitive spatial entities, and *qualitative relations involving regions* as the primitive spatial relations?

- Whitehead (1920, 1929) proposed a system of postulates governing a primitive relation of ‘contact’.
- This notion, being primitive, cannot be defined, but it can be illustrated by the following diagrams:



a)



b)

- A generally similar theory was proposed by De Laguna (1922).

- An alternative (and independent) approach to developing a region-based theory of space is illustrated by Tarski (1929):
  - begin with the familiar model of space as $\mathbb{R}^3$;
  - define a formal language with variables ranging over the set of spheres in this space, and the part-whole relation as the only primitive relation;
  - axiomatize the theory (in higher-order logic);
  - prove a categoricity result.
- Then the subject went very quiet, until, ...

- …the revival of region-based theories of space in Artificial Intelligence in the 1990s.
- The primary objective of this work is to devise 'spatial representation languages', enabling quantification over regions while remaining within first-order logic.
- Issues of particular concern
  - computational complexity,
  - efficient practical automation,
  - expressive power.
- Note the shift in underlying motivation: computational inefficiency rather than epistemological hygiene.

- Whatever the motivation, this talk addresses the questions
  - What might region-based theories of space look like?
  - How should we go about developing them?
  - Which mathematical methods can we use to analyse them?
  - Are these theories technically interesting?
- For historical continuity (and simplicity!) we shall focus on topology.
- Remark: *proximity spaces* (Naimpally and Warrack).

- Two methodologies present themselves:
  1. the ‘postulation-based’ methodology of Whitehead and de Laguna (and much of the early AI work);
  2. the ‘model-theoretic’ strategy of Tarski.
- We opt for the latter. In more detail:
  – start with a familiar, point-based model of space;
  – select a collection of ‘regions’, understood as sets of points;
  – select a collection of non-logical constants representing primitive geometrical relations between regions, interpreted over the space in the standard way;
  – investigate the logical theory of these geometrical primitives.

- To fix ideas, let us
  - take our (point-based) model of space to be $\mathbb{R}^2$;
  - take Whitehead's contact relation C to be our only geometrical primitive, where this predicate is given the interpretation:

$$\langle r, s \rangle \text{ satisfies } C \text{ iff } r^- \cap s^- \neq \emptyset;$$

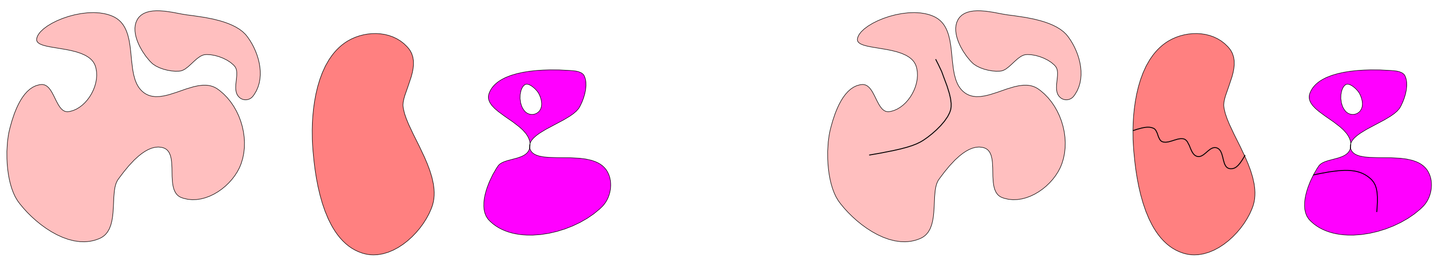
  - take regions to be certain subsets of $\mathbb{R}^2$.
- We need to be clear about *which* subsets of $\mathbb{R}^2$ qualify as regions.

- Allowing *arbitrary* sets of points as regions is unlikely to yield an attractive spatial ontology: we want our regions to be ‘region-like’.
- A subset x of a topological space X is regular open if it is equal to the interior of its closure:

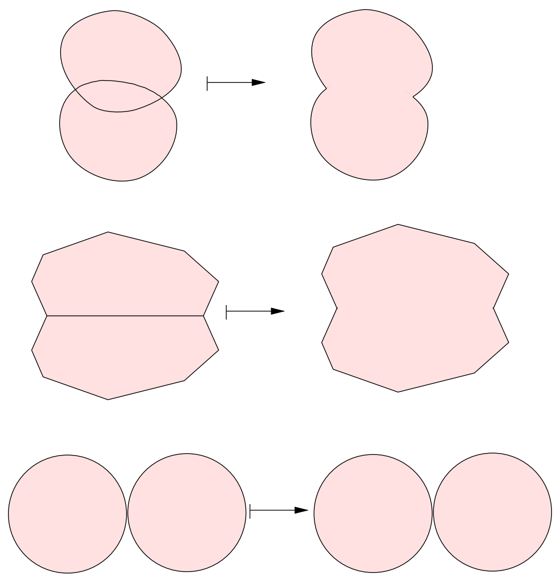
$$x = x^{-0}$$

- We propose (provisionally) that regions be simply the regular open subsets of $\mathbb{R}^2$.

- If X is any topological space, we denote the collection of regular open subsets of X by $\mathrm{RO}(X)$.
- The elements of $\mathrm{RO}(\mathbb{R}^2)$ are nice: they are the open sets with no internal cracks or point-holes



- **Theorem:** Let X be a topological space. Then $\mathrm{RO}(X)$ forms a Boolean algebra with top and bottom defined by $1 = X$ and $0 = \emptyset$, and Boolean operations defined by $x.y = x \cap y$, $x + y = (x \cup y)^{-0}$ and $-x = (X \setminus x)^0$.



- Here, we have selected
  - our domain of regions to be $\mathrm{RO}(\mathbb{R}^2)$,
  - our set of geometrical primitives to be C.
- Notation:
  - denote the first-order language with the single non-logical primitive C by $\mathcal{L}_C$;
  - denote the first-order theory of the structure $(\mathrm{RO}(\mathbb{R}^2), C)$ by $\mathrm{Th}_C(\mathrm{RO}(\mathbb{R}^2))$.
- Question:

What can we say about $\mathrm{Th}_C(\mathrm{RO}(\mathbb{R}^2))$?

- Of course, other sets of geometrical primitives are possible:

$c(x)$	x is connected
$x \leq y$	"x is a part of y"
$\mathrm{conv}(x)$	"x is convex"
$b(x)$	x is bounded (has a compact closure)
$\ldots$	$\ldots$

- Given any collection Σ of such primitives, we can ask:

  What can we say about $\mathrm{Th}_\Sigma(\mathrm{RO}(\mathbb{R}^2))$?

- In fact, given any collection Σ of such primitives and any domain D of regions over some space (for which the primitives Σ are defined), we can ask:

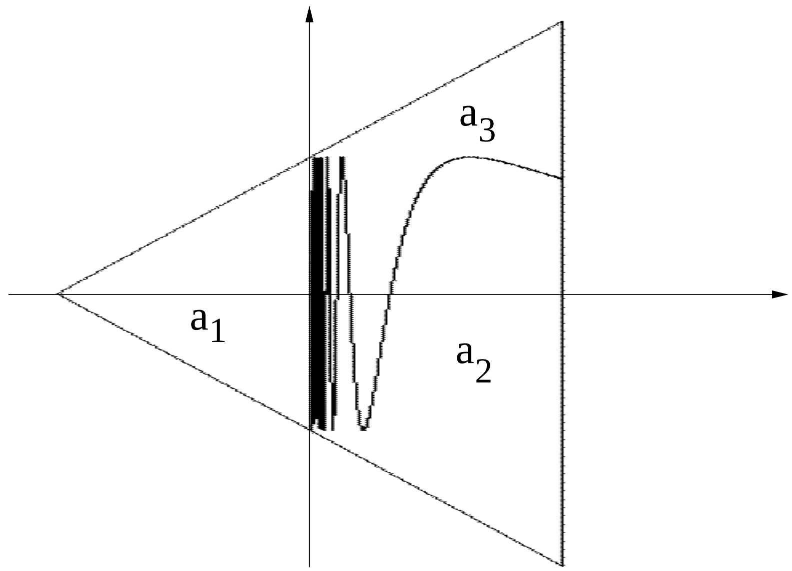
  What can we say about $\mathrm{Th}_\Sigma(D)$?

- It is worth considering a further example. Suppose we select the primitives to be
$$\Sigma = \{c, \leq\},$$
and our domain of regions to be the regular opens of the one-point compactification of the Euclidean plane, i.e.
$$D = \mathrm{RO}(\dot{\mathbb{R}}^2).$$
What can be say about $\mathrm{Th}_{c,\leq}(\mathrm{RO}(\dot{\mathbb{R}}^2))$?

- Let $\psi_{\text{no-wiggle}}$ be the $\mathcal{L}_{c,\leq}$-sentence:
$$\forall x_1 \forall x_2 \forall x_3 \big((\bigwedge_{1 \leq i \leq 3} c(x_i) \wedge c(x_1 + x_2 + x_3)) \rightarrow (c(x_1 + x_2) \vee c(x_1 + x_3)) \big).$$

- In fact, $\mathrm{RO}(\dot{\mathbb{R}}^2) \not\models \psi_{\text{no-wiggle}}$, because ...

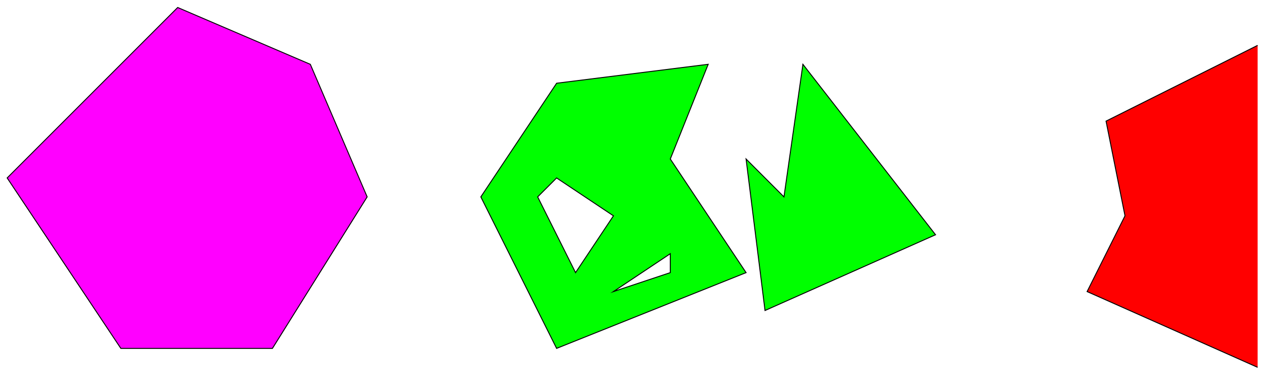
- . . . RO($\dot{\mathbb{R}}^2$) contains very wiggly regions:



$$
\begin{aligned}
a_1 &= \{(x,y) \mid -1 < x < 0 \; ; \; -1 - x < y < 1 + x\} \\
a_2 &= \{(x,y) \mid 0 < x < 1 \; ; \; -1 - x < y < \sin(1/x)\} \\
a_3 &= \{(x,y) \mid 0 < x < 1 \; ; \; \sin(1/x) < y < 1 + x\}
\end{aligned}
$$

- Do we want our spatial ontology to include these regions?

- Define a *half-plane* (in $\dot{\mathbb{R}}^2$) to be one of the residual domains of a straight line; all half-planes are regular open.
- Define a polygon (in $\dot{\mathbb{R}}^2$) to be a finite Boolean combination of half-planes in $\mathrm{RO}(\dot{\mathbb{R}}^2)$.



- Denote by $\mathrm{ROP}(\dot{\mathbb{R}}^2)$ the set of polygons in the closed plane.
- Obviously, $\mathrm{ROP}(\dot{\mathbb{R}}^2)$ is a Boolean subalgebra of $\mathrm{RO}(\dot{\mathbb{R}}^2)$.

- Remark: we might alternatively call the regular open polygons the *regular open semi-linear sets in* $\mathbb{R}^2$.
- It can be shown that $\mathrm{ROP}(\dot{\mathbb{R}}^2) \models \psi_{\text{no-wiggle}}$. That is:

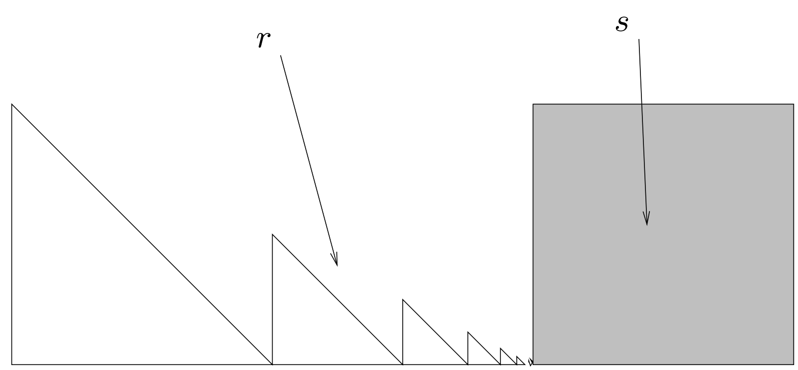
$$\mathrm{RO}(\dot{\mathbb{R}}^2) \not\equiv_{c,\leq} \mathrm{ROP}(\dot{\mathbb{R}}^2).$$

- Moral: it matters what counts as a 'region'.

- Here is a further difference between $\mathrm{RO}(\dot{\mathbb{R}}^2)$ and $\mathrm{ROP}(\dot{\mathbb{R}}^2)$.
- Let $\Sigma = (C, c, \leq)$, and let ψ_{inf} be the L_Σ-sentence

$$\forall x \forall y (C(x, y) \rightarrow \exists z (c(z) \wedge z \leq y \wedge C(x, z))).$$

- Again, we have $\mathrm{RO}(\dot{\mathbb{R}}^2) \not\models \phi$, because of the configuration:



- By contrast, $\mathrm{ROP}(\dot{\mathbb{R}}^2) \models \phi$.

- The regions in $\mathrm{ROP}(\dot{\mathbb{R}}^2)$ are all *tame*, in the following sense:
  - They exhibit curve-selection: if r is a region, and $q \in \mathcal{F}(r)$, then there exists a Jordan arc have end q as one of its endpoints, lying in $r \cup \{q\}$.
  - They are all finitely decomposable: each region is the sum of finitely many connected regions.
- These properties make $\mathrm{ROP}(\dot{\mathbb{R}}^2)$ much easier to work with than $\mathrm{RO}(\dot{\mathbb{R}}^2)$—so let us do that.

- Having (provisionally) chosen our collection of regions, let us revisit the choice of topological primitives.
- The language $\mathcal{L}_C$ has at least as much expressive power over $\text{ROP}(\dot{\mathbb{R}}^2)$ as does $\mathcal{L}_{c,\leq}$:
  - The $\mathcal{L}_C$-formula $\phi_{\leq}(x, y) := \forall z(C(x, z) \rightarrow C(y, z))$ is satisfied in $\text{ROP}(\dot{\mathbb{R}}^2)$ by a pair (a, b) iff $a \leq b$.
  - There exists a (more complicated) $\mathcal{L}_C$-formula $\phi_c(x)$, such that $\phi_c(x)$ is satisfied in $\text{ROP}(\dot{\mathbb{R}}^2)$ by a iff a is connected.
- These results are *robust*—they work for almost any sensible collection of regions (over almost any topological space).
- In particular, over the polygons in the *open* plane (defined analogously), $\mathcal{L}_C$ has at least as much expressive power as $\mathcal{L}_{c,\leq}$.

- Conversely, the language $\mathcal{L}_{c,\leq}$ has at least as much expressive power over $\text{ROP}(\dot{\mathbb{R}}^2)$ as does $\mathcal{L}_C$:
  - There exists a (complicated) $\mathcal{L}_{c,\leq}$-formula $\psi_C(x, y)$, such that $\psi_C(x, y)$ is satisfied in $\text{ROP}(\dot{\mathbb{R}}^2)$ by a pair (a, b) iff $a^- \cap b^- \neq \emptyset$.
- This result is *fragile*: it relies on global topological features of the space $\dot{\mathbb{R}}^2$.
  - In particular, over the polygons in the *open* plane, $\mathcal{L}_{c,\leq}$ is strictly less expressive than $\mathcal{L}_C$. For example, $\mathcal{L}_C$ can define the property of being bounded, but $\mathcal{L}_{c,\leq}$ cannot.
- Nevertheless, we have enough motivatation for studying $\text{Th}_{c,\leq}(\text{ROP}(\dot{\mathbb{R}}^2))$.

- Now we have a question which we can answer:

$$\boxed{\text{What can we say about } \mathrm{Th}_{c,\leq}(\mathrm{ROP}(\dot{\mathbb{R}}^2))?}$$

- As an aside:
  - Considered as $\{c, \leq\}$-strcutures, $\mathrm{ROP}(\dot{\mathbb{R}}^2)$ and $\mathrm{ROP}(\mathbb{R}^2)$ are isomorphic:

$$\mathrm{ROP}(\dot{\mathbb{R}}^2) \simeq_{c,\leq} \mathrm{ROP}(\mathbb{R}^2);$$

  - however, as $\{C\}$-structures, they are not even elementarily equivalent:

$$\mathrm{ROP}(\dot{\mathbb{R}}^2) \not\equiv_C \mathrm{ROP}(\mathbb{R}^2).$$

- We can characterize $\text{Th}_{c,\leq}(\text{ROP}(\dot{\mathbb{R}}^2))$ axiomatically as follows:

1. the usual axioms of non-trivial Boolean algebra;

2. two axioms concerning the interaction between c and $\leq$, e.g. the axiom

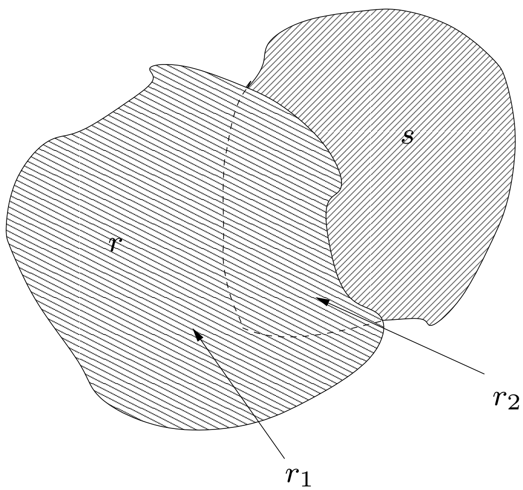
$$\forall x \forall y (c(x) \wedge c(y) \wedge x \cdot y \neq 0 \rightarrow c(x+y));$$

3. two planarity axioms, e.g.

$$\neg \exists x_1 \ldots \exists x_5 \Big(\bigwedge_{1 \leq i \leq 5} (c(x_i) \wedge x_i \neq 0) \wedge \bigwedge_{1 \leq i < j \leq 5} (c(x_i + x_j) \wedge x_i \cdot x_j = 0) \Big);$$

4. two axioms to do with partitioning up regions, e.g.

$$\forall x \forall y \big(x \cdot y = 0 \wedge$$
$$x,\ y \text{ and } -(x+y) \text{ are non-empty and connected} \rightarrow$$
$$\exists u \exists v \big(u_1 \text{ and } u_2 \text{ partition } x$$
$$\wedge c(u_1 + y) \wedge \neg c(u_1 + -(x+y))$$
$$\wedge c(u_2 + -(x+y)) \wedge \neg c(u_2 + y) \big) \big);$$



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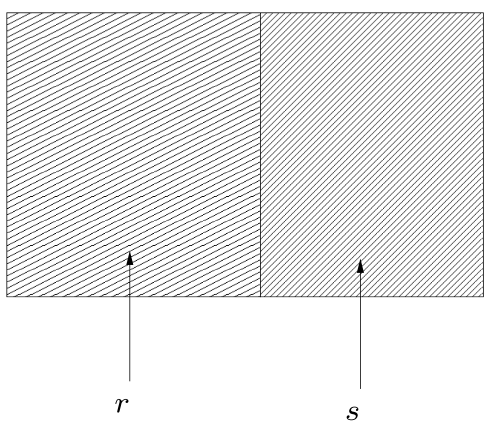


b)

5. the infinitary rule of inference

$$\frac{\{\forall x(\psi_{\mathrm{c}}^{n}(x) \rightarrow \phi(x)) \mid n \geq 1\}}{\forall x \phi(x)},$$

where $\psi_{\mathrm{c}}^{n}(x)$ stands for the formula

$$\exists z_1 \ldots \exists z_n \Big(\bigwedge_{1 \leq i \leq n} c(z_i) \wedge (x = z_1 + \cdots + z_n) \Big)$$

("If a property holds of all n-component regions, for all n, then it holds of all regions").

- Let $T_{c,\leq}$ denote the set of sentences which are consequences of the above axioms and the infinitary proof rule.

  Theorem: $T_{c,\leq} = \mathrm{Th}_{c,\leq}(\mathrm{RO}(\dot{\mathbb{R}}^2))$.

  Proof: Take a model of $T_{c,\leq}$ in which every element is the sum of finitely many elements (using the infinitary rule of inference and the omitting types theorem); embed it in $\mathrm{ROP}(\mathbb{R}^2)$ (using the planarity axioms), and show that the embedding is elementary (using the splitting axioms).

- This gives us a very quick method of checking whether alternative choices of the domain of regions yield the same first-order theory. In particular, we consider
  - The regular open *rational* polygons, $\mathrm{ROQ}(\dot{\mathbb{R}}^2)$.
  - The regular open *semialgebraic* sets, $\mathrm{ROS}(\dot{\mathbb{R}}^2)$.
- Note that $\mathrm{ROQ}(\dot{\mathbb{R}}^2) \subset \mathrm{ROP}(\dot{\mathbb{R}}^2) \subset \mathrm{ROS}(\dot{\mathbb{R}}^2)$.
- These collections of regions also satisfy curve selection and finite decomposability.
- It is also easy to check that they make the above axioms true and validate the infinitary rule of inference. Hence

$$\mathrm{ROQ}(\dot{\mathbb{R}}^2) \equiv_{c,\leq} \mathrm{ROP}(\dot{\mathbb{R}}^2) \equiv_{c,\leq} \mathrm{ROS}(\dot{\mathbb{R}}^2).$$

- We can generalize this observation:

  Definition: Let X be a topological space. A *mereotopology* over X is a Boolean sub-algebra M of $\mathrm{RO}(X)$ such that M forms a basis for the topology on X.

- Thus, $\mathrm{ROQ}(\dot{\mathbb{R}}^2)$, $\mathrm{ROP}(\dot{\mathbb{R}}^2)$ and $\mathrm{ROS}(\dot{\mathbb{R}}^2)$ are all mereotopologies over $\dot{\mathbb{R}}^2$.

- Where M is clear from context, we refer its elements as *regions*.

- A word on etymology:
  - Mereology (Leśniewski): the logic of the part-whole relationship ($\leq$).
  - Mereotopology is simply the study of topological spaces with regions functioning as the primary objects.
- I am not sure where the term 'mereotopology' first appeared in print.

- Now for the generalization:

  Theorem: All finitely decomposable mereotopologies over $\dot{\mathbb{R}}^2$ having curve-selection, and satisfying the above 'splitting axiom' have the same $L_{c,\leq}$-theory (and hence the same L_C-theory).

- Actually, the following can be shown:

  Theorem: All splittable, finitely decomposable mereotopologies over $\dot{\mathbb{R}}^2$ with curve-selection have the same L_Σ-theory for any topological signature Σ.

- Let us return to the issue of expressive power of $\mathcal{L}_{c,\leq}$ over $\mathrm{ROP}(\dot{\mathbb{R}}^2)$:
- Note that any tuple $\bar{r}$ from $\mathrm{ROP}(\dot{\mathbb{R}}^2)$ can be 'triangulated' with finitely many 'triangles'.
- These triangles can be combinatorially described in $\mathcal{L}_{c,\leq}$.
- This (almost) immediately yields the following result.

  Theorem: For every tuple $\bar{r}$ of $\mathrm{ROP}(\dot{\mathbb{R}}^2)$, there exists an $\mathcal{L}_{c,\leq}$-formula $\phi_{\bar{r}}(\bar{x})$ such that, for every tuple $\bar{s}$ of $\mathrm{ROP}(\dot{\mathbb{R}}^2)$, $\bar{s}$ satisfies $\phi_{\bar{r}}(\bar{x})$ iff $\bar{r}$ and $\bar{s}$ are similarly situated.

- Actually, this observation, combined with some elementary model theory, yields a much more interesting result.
- Recall that a structure $\mathfrak{A}$ is said to be *prime* if it is elementarily embeddable in every other model of its theory.
- Prime models (where they exist) are unique up to isomorphism, and are considered 'simplest' models of their respective theories.
- We have:

  Theorem: The $\{c, \leq\}$-structure $\mathrm{ROQ}(\dot{\mathbb{R}}^2)$ is a prime model of $T_{c,\leq}$.

- Yet more follows from the same line of argument.
- We can generalize the notion of *finite decomposability* to general structures interpreting $\mathcal{L}_{c,\leq}$ in the obvious way. Doing so, we obtain:

  Theorem: All countable, finitely decomposable models of $T_{c,\leq}$ are isomorphic.
- Thus, we can get very close to characterizing the rational polygons in the closed Euclidean plane axiomatically, using either the language $\mathcal{L}_{c,\leq}$ or the language $\mathcal{L}_C$.

- The last theorem states that any countable, finitely decomposable model $\mathfrak{A}$ of $T_{c,\leq}$ is, up to isomorphism, the same as the mereotopology $\mathrm{ROQ}(\dot{\mathbb{R}}^2)$.
- In fact, switching to the language L_C (which we may do), it turns out that $\mathrm{ROQ}(\dot{\mathbb{R}}^2)$ is almost the only mereotopology $\mathfrak{A}$ is the same as:

  Theorem: Let M be a countable, finitely decomposable mereotopology over a locally connected, compact, Hausdorff space X, such that $\mathrm{Th}_C(M) = \mathrm{Th}_C(\mathrm{ROQ}(\dot{\mathbb{R}}^2))$. Then there is a homeomorphism $h : X \leftrightarrow \dot{\mathbb{R}}^2$ taking M to $\mathrm{ROQ}(\dot{\mathbb{R}}^2)$.

- We can link this last result back to a completely separate development in topology originating in the study of "proximity spaces".

Definition: A contact algebra is a structure interpreting the signature $(C, \leq, +, \cdot, -, 0, 1)$ satisfying the usual axioms of Boolean algebra together with

(C0) $\forall x \neg C(x, 0)$
(C1) $\forall x (x > 0 \rightarrow C(x, x))$
(C2) $\forall x \forall y (C(x, y) \rightarrow C(y, x))$
(C3) $\forall x \forall y (C(x, y) \wedge y \leq z \rightarrow C(x, z))$
(C4) $\forall x \forall y (C(x, y + z) \rightarrow C(x, y) \vee C(x, z))$

- We consider also the following additional axioms:

(Ext) $\forall x \forall y (\forall z (C(x, z) \rightarrow C(y, z)) \rightarrow x \leq y)$
(Int) $\forall x \forall y (\neg C(x, y) \rightarrow \exists z (\neg C(x, -z) \wedge \neg C(y, z)))$
(Con) $\forall x \forall y (x + y = 1 \wedge x > 0 \wedge y > 0 \rightarrow C(x, y))$.

- A topological space is semi-regular if it has a basis of regular open sets; a topological space is weakly regular if it is semi-regular and, for any non-empty open set u, there exists a non-empty open set v with $v^- \subseteq u$.
- X is regular $\Rightarrow$ X is weakly regular $\Rightarrow$ X is semi-regular.

  Theorem: Let X be a topological space, and let M be a mereotopology over X, regarded as a structure interpreting the signature $(C, \leq, +, \cdot, -, 0, 1)$. Then $M \models$ (C0)–(C4). In addition:

  1. If X is weakly regular, then $M \models$ (Ext).
  2. If X is compact and Hausdorff, then $M \models$ (Int).
  3. If X is connected, then $M \models$ (Con).

  Proof: Routine.

Theorem: (Dimov and Vakarelov, 2006) Let $\mathfrak{A}$ be a structure interpreting $(C, \leq, +, \cdot, -, 0, 1)$, whose reduct to $(\leq, +, \cdot, -, 0, 1)$ is a Boolean algebra. If $\mathfrak{A} \models$ (C0)–(C4), then $\mathfrak{A}$ is isomorphic to a mereotopology over some topological space X. Moreover:

1. if $\mathfrak{A} \models$ (Ext), then X can be chosen to be weakly regular (Düntsch and Winter, 2004);
2. if $\mathfrak{A} \models$ (Int) and (Ext), then X can be chosen to be compact and Hausdorff (Roeper, 1997); and
3. if $M \models$ (Con), then X can be chosen to be connected.

Proof sketch: Define the points of X to be ultrafilter-like subsets of A; define a mapping $g : A \rightarrow \mathbb{P}(X)$ by

$$g(a) = \{x \in X \mid a \in X\};$$

use these sets as the basis of a topology.

- Thus, the classes of mereotopologies over
  - all topological spaces
  - weakly regular topological spaces
  - compact, Hausdorff topological spaces

  can be axiomatically characterized
- Cf. our earlier result that the rational polygons can be *almost* axiomatically characterized.

- Returning to region based theories of familiar spaces, of obvious interest are tame mereotopologies over $\mathbb{R}^3$ (or $\dot{\mathbb{R}}^3$).
- Here, the language $\mathcal{L}_{c,\leq}$ is too inexpressive to be of much interest. However, $\mathcal{L}_C$ is again maximally expressive, in the same sense as it is for $\text{ROP}(\mathbb{R}^2)$:

  Theorem: For every tuple $\bar{r}$ of $\text{ROP}(\mathbb{R}^3)$, there exists an $\mathcal{L}_C$-formula $\phi(\bar{x})$ such that, for every tuple $\bar{b}$ of $\text{ROP}(\mathbb{R}^3)$, $\bar{b}$ satisfies $\phi(\bar{x})$ iff $\bar{a}$ and $\bar{b}$ are similarly situated.

  Proof: Show that 'triangulations' can be combinatorially described by $\mathcal{L}_C$-formulas.
- This means that $\text{Th}(\text{ROP}(\dot{\mathbb{R}}^3))$ must display similar model-theoretic characteristics to $\text{Th}(\text{ROP}(\dot{\mathbb{R}}^2))$. In particular

  Theorem: The $\{C\}$-structure $\text{ROQ}(\dot{\mathbb{R}}^3)$ is a prime model of its theory.
- As for axiomatization, ...

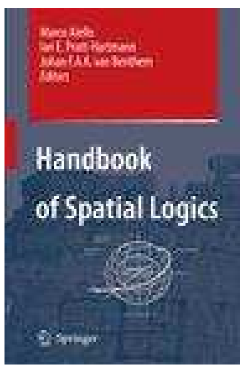
- We have concentrated on topology, but that is largely for historical reasons.
- Consider the langauge $\mathcal{L}(\text{conv}, \leq)$ (Pratt 1999, Davis, Gotts and Cohn, 1999).
- It is simple to show that:
  - $\text{ROQ}(\mathbb{R}^2)$, $\text{ROP}(\mathbb{R}^2)$, $\text{ROS}(\mathbb{R}^2)$ and $\text{RO}(\mathbb{R}^2)$ all have different $\mathcal{L}(\text{conv}, \leq)$-theories;
  - every tuple $\bar{r}$ of regions from $\text{ROQ}(\mathbb{R}^2)$, satisfies a formula which fixes $\bar{r}$ up to an affine transformation;
  - every tuple $\bar{r}$ of regions from $\text{ROP}(\mathbb{R}^2)$, satisfies a set of formulas which fix $\bar{r}$ up to an affine transformation.
- This (and more expressive) region-based theories remain, however, largely unexplored.

- Time to summarize:
  - We recalled early interest in region-based theories of space, which were motivated by concerns about the distance between theories of space and empirical data.
  - We translated this interest into questions of the form

    $$\boxed{\text{What can we say about } \mathrm{Th}_{\Sigma}(D)?}$$

    for a signature of geometrical primitives Σ and domain of regions D.
  - We gave a reasonably full answer to this question for topological signatures of primitives.

- Many of the results reported here can be found in



Ian Pratt-Hartmann: "First-Order Mereotopology", in Aiello, Pratt-Hartmann and van Benthem (eds.), *Handbook of Spatial Logics*, Springer, 2007.