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Towards Uniform Reasoning via Structured Subset Spaces

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ABSTRACT. This paper deals with subset spaces where points are structured as pairs of states. Our aim is to approach a formal model of uniform topological reasoning in this way. That is, in case a uniformity underlies the topological space under consideration, the refined reasoning model is to be sensitive to that. We show completeness and decidability of the basic modal logic arising from this setting, and discuss both possible extensions of the system and limitations.

1 Introduction

Approximately ten years ago, a bimodal logic for *topological reasoning* was proposed by Moss and Parikh [17]. The system developed in that paper connects the notion of knowledge, *cf* Fagin *et al.* [6], with fundamental concepts from general topology. In this way, a basic formal apparatus for reasoning tasks in the respective fields should be supplied.

Pursuing these ideas several classes of structures relevant to topology have been investigated since then; see for example Dabrowski *et al.* [5] (the journal version of Moss and Parikh [17], concerning general subset spaces and topological spaces respectively), Georgatos [9] (about treelike spaces), Heinemann [10] (spaces satisfying certain chain conditions), and Weiss and Parikh [19] (dealing with so-called *directed spaces,* which are closely related to bases of neighbourhood filters of topological spaces).

Why is topological reasoning significant for computer science and AI? First, because of its connection with the logic of knowledge. Summarizing the discussion in Dabrowski *et al.* [5], one can say suggestively that topological reasoning offers a natural way of reasoning about knowledge acquisition. Second, topology provides the solid mathematical framework for notions of location. Spatial objects and their mutual relationships like closeness, separation, or connectedness, are modelled adequately by topological means. Over and above this, there is yet another quite different idea inherent in topolgy: approximation. Approximating objects is topologically realized as a shrinking procedure applied to certain sets, viz *neighbourhoods* of the points of the considered topological space. Thus, at least from the point of

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view of a theorist, reasoning about topology should be the first objective in the development of spatial and (because of the procedural ingredient of approximation also) temporal reasoning formalisms. (We presently adopt a rather general point of view, dealing only with (generalizations of) the very basic structures of topology. Finer instruments of measurement like, *e.g.*, metrics would have to be incorporated in a more elaborate model.)

By means of the language underlying the system of Moss and Parikh one can speak about *subset frames*, *i.e.*, pairs $(X, \mathcal{O})$ where X is a non-empty set (of *points* or *states*) and $\mathcal{O}$ a set of subsets of X (the distinguished *system of opens* or *neighbourhoods*). Formulas are evaluated at *neighbourhood situations* consisting of a state and a neighbourhood thereof. The two modal connectives contained in the language, K and $\Box$, quantify across points *close* to the actual one and sets from $\mathcal{O}$ *refining* the current open, respectively. (One can think of K as a *knowledge operator* and $\Box$ as an *effort operator* assigned to some agent (being busy with approximating an object and acquiring knowledge, respectively).) Both modalities interact according to the topological structure being under discussion.

Now, it may happen that shrinking of opens proceeds in a *uniform* way. For example, computing a 0–1–stream $f : \mathbb{N} \longrightarrow \{0, 1\}$ yields, step by step, a bisection of the actual open (consisting of all streams sharing the output obtained already). A similar phenomenon appears whenever the measurement precision has to be increased repeatedly during an experiment (say by one decimal place each time). Thus, *uniform* knowledge acquisition (about the function f and the measured object, respectively) turns up behind these examples.

The idea of uniformity has a formal counterpart in topology. Let be given a set X. Then, a *uniform structure* on X is a filter $\mathcal{U}$ in the powerset of $X \times X$ that satisfies certain closure properties. In particular, every *entourage* (that is, every element of $\mathcal{U}$) contains the diagonal of X. Furthermore, $\mathcal{U}$ is closed under converse and a kind of squaring; see Bourbaki [3, Definition II.1.1]. Note that relations of this type are subject to *two-dimensional modal logic; cf* Marx and Venema [16, Definition 2.1.2]. We will refer to them later on, in the concluding section of this paper.

Subsequently we extend the Moss–Parikh framework of topological reasoning to the effect that we take the first step towards a *modal logic of uniform subset spaces* (as a basic tool for *uniform reasoning*). Unlike the logic of subset frames mentioned above the *square* structure of sets must be taken into account now. Actually, we assume that we are given a system of distinguished entourages all of which are real squares. (Well-known metric spaces provide examples where one can find a base of the uniformity consisting of squares.) The language forming the basis of this approach is introduced in Section 2. A corresponding logical system **U** can be obtained as a synthesis of the logic of subset spaces and the two-dimensional logic of squares. We present the accompanying completeness result in Section 3. Afterwards, in Section 4, we give reasons for decidability of the set of **U**–theorems. Finally, we are heading for possible extensions towards a modal logic of uniform structures. In this connection a *hybrid* version of **U** is studied in Section 5. Concluding the paper we indicate other extensions and discuss the limitations of our approach (among other things).

2 Syntax, semantics, axioms

In this section we define precisely the syntax and semantics of the language of structured subset spaces. Moreover, we list the axioms of a logic corresponding to this class of models.

Let PROP be a denumerable set of symbols called *proposition variables.* We use A, B, $\ldots$ to denote typical elements of PROP. We define the set WFF of *well-formed formulas* of a trimodal language over PROP by the generative rule

$$\alpha \ ::= \ A \mid \neg\alpha \mid \alpha\wedge\beta \mid L_{\mathrm{h}}\alpha \mid L_{\mathrm{v}}\alpha \mid \Diamond\alpha.$$

That is, our language contains three unary diamonds L_{h}, L_{v} and $\Diamond$; the latter is intended to model *shrinking* of sets, whereas both modalities L_{h} and L_{v} correspond to the square structure of the subsets we consider. The missing boolean connectives $\top, \bot, \vee, \rightarrow, \leftrightarrow$ are treated as abbreviations, as needed. The duals of the modal operators are written K_{h}, K_{v} and $\Box$, respectively. (Our use of the letters L and K needs justification since it differs from that in traditional modal logic. Actually, the choice of names L and K, which is to emphasize the connection of these operators to possibility and knowledge, respectively, is the one that is usual for the common modal logic of subset spaces, *cf* Dabrowski *et al.* [5].)

We give next meaning to formulas. For this purpose we have to define the relevant domains first. We let $\mathcal{P}(X)$ designate the powerset of a given set X.

DEFINITION 1 (Structured subset spaces) 1. Given a non-empty set X, a subset $U \subseteq X\times X$ is called a *square (with respect to X)* iff $U = Y\times Y$ for some $Y \subseteq X$; in this case we write $Y = \mathrm{pr}(U)$ (the *projection* of U).

2. A *structured subset frame* $(X, \mathcal{U})$ consists of a non-empty set X and a set $\mathcal{U}$ of squares with respect to X.

3. Let $(X, \mathcal{U})$ be a structured subset frame. The set of *entourage situations* of $(X, \mathcal{U})$ is

$$\mathcal{N} := \{x, y, U \mid (x, y) \in U \text{ and } U \in \mathcal{U}\}.$$

4. A *structured subset space* or *structured model*[1] is a triple $(X, \mathcal{U}, V)$, where $(X, \mathcal{U})$ is a structured subset frame and V a *(square) valuation*, *i.e.*, a mapping

$$V : \mathrm{PROP} \longrightarrow \mathcal{P}(X \times X).$$

We say that $\mathcal{M} := (X, \mathcal{U}, V)$ is *based on* $(X, \mathcal{U})$.

For a given structured subset space $\mathcal{M}$ we define now the relation of satisfaction, $\models_{\mathcal{M}}$, between entourage situations of the underlying frame and formulas in WFF.

DEFINITION 2 (Satisfaction and validity) Let $\mathcal{M} := (X, \mathcal{U}, V)$ be a structured subset space based on $\mathcal{S} := (X, \mathcal{U})$, and x, y, U an entourage situation of $\mathcal{S}$. Then

$$\begin{array}{lll}
x, y, U \models_{\mathcal{M}} A & :\Longleftrightarrow & (x, y) \in V(A) \\
x, y, U \models_{\mathcal{M}} \neg\alpha & :\Longleftrightarrow & x, y, U \not\models_{\mathcal{M}} \alpha \\
x, y, U \models_{\mathcal{M}} \alpha \wedge \beta & :\Longleftrightarrow & x, y, U \models_{\mathcal{M}} \alpha \text{ and } x, y, U \models_{\mathcal{M}} \beta \\
x, y, U \models_{\mathcal{M}} L_{\mathrm{h}}\alpha & :\Longleftrightarrow & z, y, U \models_{\mathcal{M}} \alpha \text{ for some } z \text{ such that } (z, y) \in U \\
x, y, U \models_{\mathcal{M}} L_{\mathrm{v}}\alpha & :\Longleftrightarrow & x, z, U \models_{\mathcal{M}} \alpha \text{ for some } z \text{ such that } (x, z) \in U \\
x, y, U \models_{\mathcal{M}} \Diamond\alpha & :\Longleftrightarrow & \left\{ \begin{array}{l} x, y, U' \models_{\mathcal{M}} \alpha \text{ for some } U' \in \mathcal{U} \\ \quad \text{such that } (x, y) \in U' \subseteq U, \end{array} \right.
\end{array}$$

for all $A \in \mathrm{PROP}$ and $\alpha, \beta \in \mathrm{WFF}$. In case $x, y, U \models_{\mathcal{M}} \alpha$ is true we say that α *holds in* $\mathcal{M}$ *at* the entourage situation x, y, U. The formula α is called *valid in* $\mathcal{M}$, iff it holds in $\mathcal{M}$ at every entourage situation. (Manner of writing: $\mathcal{M} \models \alpha$.) (Moreover, the notion of validity can be extended as usual to structured subset frames $\mathcal{S}$ by quantifying over all structured subset spaces based on $\mathcal{S}$. However, frame validity will not be considered in the paper.)

Having defined the language we ask for an axiomatization of the validities of the class of structured subset spaces. Let $\mathbf{K}_3$ designate the minimal trimodal system over $\{L_{\mathrm{h}}, L_{\mathrm{v}}, \Diamond\}$. We propose the following list of axioms:

1. All $\mathbf{K}_3$–schemata

[1] Throughout this paper we will use the first term although the second one is closer to the manner of speaking in modal logic; however, we follow Dabrowski *et al.* [5] once again here.

2. $\alpha \to (L_i\alpha \wedge K_i L_i \alpha)$
3. $L_i L_i \alpha \to L_i \alpha$
4. $L_{\mathrm{h}} L_{\mathrm{v}} \alpha \leftrightarrow L_{\mathrm{v}} L_{\mathrm{h}} \alpha$
5. $(\Diamond A \to A) \wedge (\Diamond \neg A \to \neg A)$
6. $\alpha \to \Diamond \alpha$
7. $\Diamond\Diamond\alpha \to \Diamond\alpha$
8. $\Diamond L_i \alpha \to L_i \Diamond \alpha$,

where $i \in \{\mathrm{h}, \mathrm{v}\}$, $A \in \mathrm{PROP}$, and $\alpha \in \mathrm{WFF}$.

Note that the product logic $\mathbf{S5} \times \mathbf{S5}$ is axiomatized by the schemata 1 to 4, in particular; *cf* Marx and Venema [16, Section 2.2]. Axiom 5 is due to the special semantics of the proposition variables, and $1, 6$ and 7 make the modality $\Diamond$ an **S4**–diamond. Finally, Axiom 8 describes the interaction of $\Diamond$ and the L–operators, corresponding to descending with respect to the set inclusion relation $\supseteq$.[2] Thus, for the basic system we consider in the present paper this axiom really reflects the underlying geometric ideas, as it forces shrinking of entourages.

Obviously, 'forgetting' the commutation axiom 4 and the subscripts to the L–modalities yields the well-known axiomatization of the usual modal logic of subset spaces; *cf* Dabrowski *et al.* [5].

3 Completeness

The topic of this section is to give a proof for the completeness part of Theorem 3 below.

Let **U** be the modal system obtained from the above axioms and the common derivation rules *modus ponens* and *necessitation with respect to each modality:*

$$\frac{\alpha \to \beta, \alpha}{\beta}$$

$$\frac{\alpha}{K_{\mathrm{h}}\alpha} \qquad \frac{\alpha}{K_{\mathrm{v}}\alpha} \qquad \frac{\alpha}{\Box\alpha},$$

for all $\alpha, \beta \in \mathrm{WFF}$. Then, we have following theorem.

THEOREM 3 (Soundness and completeness of **U**) *Let $\alpha \in \mathrm{WFF}$ be a formula. Then α is **U**–derivable, iff it is valid in every structured subset space.*

[2]It should be stressed that we do *not* have a commutation axiom like Schema 4 here. In particular, even disregarding Schema 5 the logic of structured subset spaces does not contain the Gabbay–Shehtman logic $[\mathbf{S4}, \mathbf{S5}, \mathbf{S5}]$, *cf* Gabbay and Shehtman [8, Section 3], but is 'two-dimensional' in essence.

While the soundness part of Theorem 3 is obvious, the main idea of proving completeness is to perform a particular step-by-step construction, similar as in Sec. 2.2 of the paper Dabrowski *et al.* [5]. Though step-by-step is brought out as a general method meanwhile, *cf* Blackburn *et al.* [1, 4.6], special arrangements have to be made here due to the interpretation of formulas in structured subset spaces.

Let $\mathcal{C}$ be the set of all maximal **U**–consistent set of formulas, and

$$\xrightarrow{L_\mathrm{h}}, \ \xrightarrow{L_\mathrm{v}}, \text{ and } \xrightarrow{\Diamond},$$

the distinguished accessibility relations on $\mathcal{C}$ induced by the modalities $L_\mathrm{h}, L_\mathrm{v}$ and $\Diamond$, respectively. Suppose that $\Gamma_0 \in \mathcal{C}$ is to be realized. We choose a denumerably infinite set of points, Y, and fix an element $x_0 \in Y$. We will construct inductively a sequence of quadruples (X_n, P_n, d_n, s_n) such that, for every $n \in \mathbb{N}$,

- $X_n \subseteq Y$ is a finite set containing x_0,
- P_n is a finite partial order containing a least element p_0; moreover, for every $p \in P_n$ the set $\{q \in P_n \mid q \leq p\}$ is linearly ordered (that is, $(P_n, \leq)$ is a finite tree with root p_0),
- $d_n : P_n \longrightarrow \mathcal{P}(X_n \times X_n)$ is a mapping satisfying
  - $d_n(p)$ is a square for every $p \in P_n$,
  - $d_n(p_0) = X_n \times X_n$, and
  - $p \leq q \iff d_n(p) \supseteq d_n(q)$ for all $p, q \in P_n$, and
- $s_n : (X_n \times X_n) \times P_n \longrightarrow \mathcal{C}$ is a partial function such that, whenever $x, y, z \in X_n$ and $p, q \in P_n$,
  - $s_n(x, y, p)$ is defined iff $(x, y) \in d_n(p)$; in this case it holds that
    * if $(z, y) \in d_n(p)$, then $s_n(x, y, p) \xrightarrow{L_\mathrm{h}} s_n(z, y, p)$,
    * if $(x, z) \in d_n(p)$, then $s_n(x, y, p) \xrightarrow{L_\mathrm{v}} s_n(x, z, p)$, and
    * if $q \leq p$, then $s_n(x, y, q) \xrightarrow{\Diamond} s_n(x, y, p)$,
  - $s_n(x_0, x_0, p_0) = \Gamma_0$.

The intermediate structures (X_n, P_n, d_n, s_n) represent *approximations* to the desired model of a special kind. To be more precise, the construction will ensure, in particular, that

- $X_n \subseteq X_{n+1}$,

- $(P_{n+1}, \leq)$ is an *end extension* of $(P_n, \leq)$,
- $d_{n+1}(p) \cap X_n = d_n(p)$, for every $p \in P_{n+1}$, and
- $s_{n+1} \mid_{(X_n \times X_n) \times P_n} = s_n$,

for all $n \in \mathbb{N}$. Furthermore, the following requirements will be satisfied:

- if $L_{\mathrm{h}}\beta \in s_n(x, y, p)$, then there are $k > n$ and $z \in X_k$ such that $(z, y) \in d_k(p)$ and $\beta \in s_k(z, y, p)$,
- if $L_{\mathrm{v}}\beta \in s_n(x, y, p)$, then there are $k > n$ and $z \in X_k$ such that $(x, z) \in d_k(p)$ and $\beta \in s_k(x, z, p)$, and
- if $\Diamond\beta \in s_n(x, y, p)$, then for some $k > n$ there is a $q \in P_k$ such that $q \geq p$, $(x, y) \in d_k(q)$ and $\beta \in s_k(x, y, q)$.

Let us assume for the moment that the construction has been carried out successfully, meeting all these conditions. Let (X, P, d, s) be the *limit* of the structures (X_n, P_n, d_n, s_n), *i.e.*,

- $X = \bigcup_{n \in \mathbb{N}} X_n$ and $P = \bigcup_{n \in \mathbb{N}} P_n$,
- $d(p) = \bigcup_{n \geq m} d_n(p)$, where m is the smallest number such that $p \in P_m$, and
- s is given by $s(x, y, p) := s_n(x, y, p)$, where n is the smallest number l such that $s_l(x, y, p)$ is defined ($x, y \in X$, and $p \in P$).

We define a square valuation V by

$$V(A) := \{(x, y) \in X \mid A \in s(x, y, p_0)\},$$

for all $A \in \mathrm{PROP}$. Then, $\mathcal{M} := (X, d(P), V)$ is a structured subset space. In the same way as in the paper Dabrowski *et al.* [5], Proposition 2.6 and Lemma 2.5, one can prove the following *Truth Lemma* by induction on the structure of formulas.

LEMMA 4 (Truth lemma) *For all formulas* $\beta \in \mathrm{WFF}$ *and entourage situations* $x, y, d(p)$ *of the frame* $(X, d(P))$ *we have that*

$$x, y, d(p) \models_{\mathcal{M}} \beta \quad \textit{iff} \quad \beta \in s(x, y, p).$$

Letting $\beta := \neg\alpha$ be contained in $\Gamma_0 = s_0(x_0, x_0, p_0)$, Theorem 3 follows immediately from that.

It remains to define (X_n, P_n, d_n, s_n), for all $n \in \mathbb{N}$. At this point the differences in the technical details become apparent between the proof contained in Dabrowski *et al.* [5] and the present one.

The case $n = 0$ is more or less obvious, thus omitted. If $n \geq 1$, then an existential formula contained in some of the sets $s_m(x, y, p) \in \mathcal{C}$, where $m < n$, is to be realized by the actual procedure in a way that respects the above requirements. For this purpose the order of the construction steps must be arranged carefully. This is done in the following way. First, we choose an enumeration μ_n of the set $Q^n_{L_\mathrm{h}} :=$

$$\left\{ (L_\mathrm{h}\beta, x, y, p) \in \mathrm{WFF} \times X_n \times X_n \times P_n \mid \left(\begin{array}{c} s_n(x, y, p) \text{ exists and} \\ L_\mathrm{h}\beta \in s_n(x, y, p) \end{array} \right) \right\}$$

for every $n \in \mathbb{N}$, and enumerations ν_n, κ_n of the corresponding sets $Q^n_{L_\mathrm{v}}$, $Q^n_\Diamond$ as well. Then, an appropriate 'global' enumeration λ is defined, along which the approximation of the desired model proceeds correctly. That is, λ schedules processing to quadruples $(L_\mathrm{h}\beta, x, y, p)$, $(L_\mathrm{v}\beta, x, y, p)$ and $(\Diamond\beta, x, y, p)$, respectively, in such a way that all of the sets $Q^n_{L_\mathrm{h}}$, $Q^n_{L_\mathrm{v}}$, and $Q^n_\Diamond$, are exhausted eventually. All this can be done in a fairly standard way; *cf* Dabrowski *et al.* [5, Section 2.2]. So, we will not regard these enumerations any more subsequently.

We treat the case '$L_\mathrm{h}\beta$' of the inductive construction somewhat more detailedly now. So, let (X_n, P_n, d_n, s_n) be already defined and assume that $L_\mathrm{h}\beta \in s_m(x, y, p)$ is going to be realized in step $n + 1$, where $m \leq n$. We choose $z \in Y \setminus X_n$ and let $X_{n+1} := X_n \cup \{z\}$. The partial order remains unaltered in this case, *i.e.*, $P_{n+1} := P_n$. Our next task is the definition of d_{n+1}. This mapping is given by

$$d_{n+1}(q) := \begin{cases} (\mathrm{pr}\,(d_n(q)) \cup \{z\}) \times (\mathrm{pr}\,(d_n(q)) \cup \{z\}) & \text{if } q \leq p \\ d_n(q) & \text{otherwise,} \end{cases}$$

for all $q \in P_{n+1}$.

The definition of s_{n+1} is a little more involved. Concerning the 'p–level', s_{n+1} has to remedy the defect caused by $L_\mathrm{h}\beta$, in the sense of, *e.g.*, Marx and Venema [16, p. 16 ff]. This can be done with the aid of the $\mathbf{S5} \times \mathbf{S5}$–axioms in a familiar way. But the new pairs having z as a component also occur at the 'q–level' for every $q \leq p$. Thus corresponding maximal $\mathbf{U}$–consistent sets must be assigned to them in a faithful way, *i.e.*, reflecting the enlarged square structure at level q on $\mathcal{C}$. Here is the point where Axiom 8 plays its part, enabling us to 'lift' $\xrightarrow{L_\mathrm{h}}$–arrows 'from right to left' with respect to $\leq$. All these facts are summed up in the following lemma.

LEMMA 5 (Crucial properties of the canonical model) *Let* m, x, y, z, p *as well as* (X_n, P_n, d_n, s_n) *and* $L_{\mathrm{h}}\beta$ *be as above. Then, for all* $q \leq p$ *and* $u \in \mathrm{pr}\,(d_n(q)) \cup \{z\}$ *there exist* $\Gamma^q_{(u,z)}, \Gamma^q_{(z,u)} \in \mathcal{C}$ *satisfying*

1. $\beta \in \Gamma^p_{(z,y)}$,
2. $s_m(x,y,q) \xrightarrow{L_{\mathrm{h}}} \Gamma^q_{(z,y)} \xrightarrow{L_{\mathrm{v}}} \Gamma^q_{(z,u)}$,
3. $s_m(x,y,q) \xrightarrow{L_{\mathrm{v}}} \Gamma^q_{(x,z)} \xrightarrow{L_{\mathrm{h}}} \Gamma^q_{(u,z)}$, *and*
4. *both* $\Gamma^q_{(z,u)} \xrightarrow{\Diamond} \Gamma^{q'}_{(z,u)}$ *and* $\Gamma^q_{(u,z)} \xrightarrow{\Diamond} \Gamma^{q'}_{(u,z)}$, *whenever* $q \leq q' \leq p$.

Proof. It suffices to prove the lemma in case $q = q'$ and q is the immediate $\leq$–predecessor of p. The general case is yielded by (backward) induction and transitivity of $\xrightarrow{\Diamond}$.

Due to our construction we have that $s_m(x,y,q) \xrightarrow{\Diamond} s_m(x,y,p)$. Because of $L_{\mathrm{h}}\beta \in s_m(x,y,p)$ there exists $\Gamma^p_{(z,y)} \in \mathcal{C}$ such that

$$\beta \in \Gamma^p_{(z,y)} \text{ and } s_m(x,y,p) \xrightarrow{L_{\mathrm{h}}} \Gamma^p_{(z,y)}.$$

We define $\Gamma^p_{(z,u)} := \Gamma^p_{(z,y)}$, for all $u \in \mathrm{pr}\,(d_n(q)) \cup \{z\}$. Then we get

$$s_m(x,y,p) \xrightarrow{L_{\mathrm{h}}} \Gamma^p_{(z,y)} \xrightarrow{L_{\mathrm{v}}} \Gamma^p_{(z,u)}.$$

Axiom 8 (for 'h') gives us now an element $\Gamma_1 \in \mathcal{C}$ such that

$$s_m(x,y,q) \xrightarrow{L_{\mathrm{h}}} \Gamma_1 \xrightarrow{\Diamond} \Gamma^p_{(z,y)},$$

in the same way as the cross axiom has an effect in usual modal logic of subset spaces; *cf* Dabrowski *et al.* [5, Proposition 2.2]. Next, Axiom 8 (for 'v') delivers an element $\Gamma_2 \in \mathcal{C}$ such that

$$\Gamma_1 \xrightarrow{L_{\mathrm{v}}} \Gamma_2 \xrightarrow{\Diamond} \Gamma^p_{(z,u)}.$$

We define, therefore, $\Gamma^q_{(z,y)} := \Gamma_1$ and $\Gamma^q_{(z,u)} := \Gamma_2$. This shows 5.2 and the first assertion of 5.4. The remaining parts of the proof are similar. ∎

Lemma 5 puts us in a position to define the mapping s_{n+1} :

$$s_{n+1}(u,v,q) := \begin{cases} s_n(u,v,q) & \text{if } u \neq z \text{ and } v \neq z \\ \Gamma^q_{(u,z)} & \text{if } u \in \mathrm{pr}\,(d_n(q)) \cup \{z\},\ v = z \text{ and } q \leq p \\ \Gamma^q_{(z,v)} & \text{if } v \in \mathrm{pr}\,(d_n(q)) \cup \{z\},\ u = z \text{ and } q \leq p \\ \text{undefined} & \text{otherwise,} \end{cases}$$

for all $u, v \in X_{n+1}$ and $q \in P_{n+1}$. It is easy to see that the above requirements on $(X_{n+1}, P_{n+1}, d_{n+1}, s_{n+1})$ are satisfied then.

The other modalities are treated similarly. The case of L_{v} is even completely analogous. In case $\Diamond\beta$ is to be processed, *i.e.*, $\Diamond\beta \in s_m(x, y, p)$, the definitions are as follows. We let

$$X_{n+1} := X_n \cup \{z\} \text{ and } P_{n+1} := P_n \cup \{r\},$$

where z and r are 'new' each. Moreover, we let $p \leq r$ in P_{n+1}. Then, the interesting parts of the mappings d_{n+1} and s_{n+1} are given by

- $d_{n+1}(r) := \{(z, z)\}$ and
- $d_{n+1}(q) := (\text{pr}\,(d_n(q)) \cup \{z\}) \times (\text{pr}\,(d_n(q)) \cup \{z\})$ for all $q \leq p$,

and

- $s_{n+1}(r) :=$ any point Γ of the canonical model satisfying

$$\beta \in \Gamma \text{ and } s_m(x, y, p) \stackrel{\Diamond}{\longrightarrow} \Gamma$$

- and, for all $q \leq p$,
  - $s_{n+1}(x', z, q) = s_n(x', y, q)$ for all $x' \in \text{pr}\,(d_n(q))$,
  - $s_{n+1}(z, y', q) = s_n(x, y', q)$ for all $y' \in \text{pr}\,(d_n(q))$,
  - $s_{n+1}(z, z, q) = s_n(x, y, q)$,

respectively. Again, the above requirements on $(X_{n+1}, P_{n+1}, d_{n+1}, s_{n+1})$ are satisfied. This finally yields the proof of completeness.

It should be mentioned that one of the referees of a preliminary version of this paper pointed a possible simplification of the above proof out to us, suggesting the use of game–theoretic arguments à la Gabbay *et al.* [7].

4 Decidability

We argue below that the set of **U**–theorems is decidable.[3] To this end we introduce a class of Kripke models with respect to which **U** is sound and complete, too, and satisfies the finite model property. We will, in particular, make use of the fact that subsets are structured as *two*-dimensional objects.

Let W be a non-empty set and R, R' two binary relations on W. We say that the pair (R, R') *satisfies the cross property,* iff

[3] This result should be compared with the well-known undecidability results for three-dimensional logics; see Gabbay *et al.* [7, Ch. 8] or Gabbay and Shehtman [8, Section 15].

for all $u, v, w \in W$ there exists $v' \in W$ such that, if $u\, R\, v\, R'\, w$, then $u\, R'\, v'\, R\, w$ (or, in other words, iff $R \circ R' \subseteq R' \circ R$).

Note that R and L_{h}, S and L_{v}, and T and $\Diamond$, respectively, are intended to correspond to each other in the following.

DEFINITION 6 (Kripke–**U**–model) Let $\mathfrak{M} := (W, R, S, T, V)$ be a trimodal model, where $R, S, T \subseteq W \times W$ are binary relations and V a valuation. Then $\mathfrak{M}$ is called a *Kripke–***U***–model*, iff

1. R, S are equivalence relations, and T is reflexive and transitive,
2. the pairs of relations (R, S), (S, R), (T, R) and (T, S) satisfy the cross property, and
3. for all $w, w' \in W$ and $A \in \mathrm{PROP}$: $w \in V(A) \iff w' \in V(A)$, whenever $w\, T\, w'$.

It turns out that the canonical model considered in the previous section is an example of a Kripke–**U**–model. Moreover, every structured subset space $\mathcal{M}$ induces a semantically equivalent Kripke–**U**–model $\mathfrak{M}$ as follows: the set W of points of $\mathfrak{M}$ equals the set of entourage situations of the subset frame underlying $\mathcal{M}$, and the accessibility relations on W are given by

$$\begin{array}{lcl} (x, y, U)\, R\, (x', y', U') & :\iff & y = y' \text{ and } U = U' \\ (x, y, U)\, S\, (x', y', U') & :\iff & x = x' \text{ and } U = U' \\ (x, y, U)\, T\, (x', y', U') & :\iff & x = x',\ y = y' \text{ and } U \supseteq U'. \end{array}$$

Finally, the valuation on $\mathfrak{M}$ is induced by the valuation on $\mathcal{M}$.

In this way we get with the aid of Theorem 3:

THEOREM 7 (Kripke completeness) *The system* **U** *is sound and complete with respect to the class of Kripke–***U***–models.*

We use *filtrations* for proving the finite model property of **U** with respect to Kripke–**U**–models. For a given $\alpha \in \mathrm{WFF}$, we define a filter set $\Sigma \subseteq \mathrm{WFF}$ as follows. We first join together the set $\mathrm{sf}(\alpha)$ of subformulas of α and the set $\{\neg\beta \mid \beta \in \mathrm{sf}(\alpha)\}$. Let $\Sigma^{\neg}$ denote the resulting set of formulas. Afterwards we form the boolean closure of $\Sigma^{\neg}$, in the following sense. We take the set Σ' of all finite disjunctions or conjunctions of pairwise distinct elements of $\Sigma^{\neg}$, and let then Σ^{bc} be the set of all finite conjunctions and disjunctions, respectively, of pairwise distinct elements of Σ'. Finally, we close under application of $O_i O'_j$, where $O, O' \in \{L, K\}$ and $i, j \in \{\mathrm{h}, \mathrm{v}\}$, and join together all these intermediate sets. Let Σ denote the resulting set of formulas. Note that Σ is finite and subformula closed. (Actually, 2^{2^n} is an upper bound for the cardinality of Σ (except for a constant), where n is the length of α.)

We consider now the *smallest* filtrations $\xrightarrow[\Sigma]{L_{\mathrm{h}}}$ and $\xrightarrow[\Sigma]{L_{\mathrm{v}}}$ of the relations $\xrightarrow{L_{\mathrm{h}}}$ and $\xrightarrow{L_{\mathrm{v}}}$, respectively, on the set $\mathcal{C}$ (introduced in the previous section); *cf* Blackburn *et al.* [1, p. 79]. The following property is crucial.

LEMMA 8 *Let* $[\Gamma], [\Gamma']$ *be the equivalence classes of* $\Gamma, \Gamma' \in \mathcal{C}$ *with respect to* Σ, *respectively (i.e.,* $[\Gamma] = \{\Delta \in \mathcal{C} \mid \Delta \cap \Sigma = \Gamma \cap \Sigma\}$, *and* $[\Gamma']$ *is given correspondingly). Moreover, let*

$$[\Gamma] \xrightarrow[\Sigma]{L_i} [\Gamma'] \,, \text{ where } i \in \{\mathrm{h}, \mathrm{v}\}.$$

Then, for all $\Delta \in [\Gamma]$ *there is a* $\Delta' \in [\Gamma']$ *such that* $\Delta \xrightarrow{L_i} \Delta'$.

Proof. Let $\gamma_{\Gamma'} := \bigwedge\limits_{\beta \in \Gamma' \cap \Sigma} \beta$. Regarding that

$$\Sigma = \Sigma^{\mathrm{bc}} \cup \left\{ O_i O'_j \beta \mid \beta \in \Sigma^{\mathrm{bc}},\, O, O' \in \{L, K\},\, i, j \in \{\mathrm{h}, \mathrm{v}\} \right\},$$

one can show that the implication

$$L_i \gamma_{\Gamma'} \in \Delta \text{ for some } \Delta \in [\Gamma] \implies L_i \gamma_{\Gamma'} \in \Delta \text{ for all } \Delta \in [\Gamma]$$

is valid; to this end one has to utilize the definition of Σ and the **S5**–properties of the operator L_i (among other things). Since

$$L_i \gamma_{\Gamma'} \in \Delta \text{ implies } \exists\, \Theta : \left(\gamma_{\Gamma'} \in \Theta \text{ and } \Delta \xrightarrow{L_i} \Theta \right)$$

and

$$\gamma_{\Gamma'} \in \Theta \text{ implies } [\Theta] = [\Gamma'] \,,$$

this suffices to prove the assertion of the lemma. ∎

A corresponding lemma for the one-dimensional case is due to Krommes [14]. It is decisive for the present (two-dimensional) case that every 'prefix' $\in \{L_{\mathrm{h}}, L_{\mathrm{v}}, K_{\mathrm{h}}, K_{\mathrm{v}}\}^*$ of length 3 has a provably equivalent prefix of length at most 2. (For example, $L_{\mathrm{h}} K_{\mathrm{v}} L_{\mathrm{h}}$ and $K_{\mathrm{v}} L_{\mathrm{h}}$ as well as $L_{\mathrm{v}} L_{\mathrm{h}} L_{\mathrm{v}}$ and $L_{\mathrm{v}} L_{\mathrm{h}}$ correspond to each other in this sense.) A 'prefix reduction' of the same kind is no longer possible in case of cubes of dimension > 2. (For example, $L_1 K_2 L_3 K_1$ cannot be reduced further.)[4]

[4]See Scherer [18] for an extensive study of knowledge prefixes.

It remains to choose a filtration of $\xrightarrow{\Diamond}$. We take the *transitive closure of the smallest filtration* of $\xrightarrow{\Diamond}$ for that; *cf* Chagrov and Zakharyaschev [4, p. 141 ff]. It is not hard to see that, in fact, a finite Kripke–**U**–model results. This gives us the desired finite model property.

THEOREM 9 (Finite model property of **U**) *For every formula α that is not derivable in* **U** *there is a finite Kripke–***U***–model falsifying α.*

Proof. The required properties of the filtration can be obtained as follows. Reflexivity of $\xrightarrow[\Sigma]{\Diamond}$ (the filtration of $\xrightarrow{\Diamond}$ chosen above), $\xrightarrow[\Sigma]{L_{\mathrm{h}}}$ and $\xrightarrow[\Sigma]{L_{\mathrm{v}}}$ is obvious. Symmetry of $\xrightarrow[\Sigma]{L_{\mathrm{h}}}$ and $\xrightarrow[\Sigma]{L_{\mathrm{v}}}$ is an easy consequence of the fact that we have taken the respective smallest filtration. Transitivity of $\xrightarrow[\Sigma]{\Diamond}$ is obvious, too.

We show transitivity of $\xrightarrow[\Sigma]{L_{\mathrm{h}}}$ with the aid of Lemma 8 now. Let $[\Gamma]\,,[\Gamma']$ and $[\Gamma'']$ be equivalence classes with respect to Σ, and assume that

$$[\Gamma] \xrightarrow[\Sigma]{L_{\mathrm{h}}} [\Gamma'] \text{ and } [\Gamma'] \xrightarrow[\Sigma]{L_{\mathrm{h}}} [\Gamma'']$$

holds. Then, due to the definition of the smallest filtration there are $\Delta \in [\Gamma]\,$, $\Delta', \tilde{\Delta}' \in [\Gamma']$ and $\Delta'' \in [\Gamma']$ such that

$$\Delta \xrightarrow{L_{\mathrm{h}}} \Delta' \text{ and } \tilde{\Delta}' \xrightarrow{L_{\mathrm{h}}} \Delta''.$$

Since Δ' and $\tilde{\Delta}'$ belong to the same class some $\tilde{\Delta}'' \in [\Gamma'']$ exists such that $\Delta' \xrightarrow{L_{\mathrm{h}}} \tilde{\Delta}''$, according to Lemma 8. We infer $\Delta \xrightarrow{L_{\mathrm{h}}} \tilde{\Delta}''$ from this, for $\xrightarrow{L_{\mathrm{h}}}$ is a transitive relation. Consequently, $[\Gamma] \xrightarrow[\Sigma]{L_{\mathrm{h}}} [\Gamma'']\,$, as desired.

Furthermore, transitivity of $\xrightarrow[\Sigma]{L_{\mathrm{v}}}$ as well as the various *cross properties,* can be proved similarly. Finally, Property 3 of Definition 6 can be established in a standard way. ■

Note that the size of the refuting Kripke–**U**–model is doubly-exponential in the length of α, due to the cardinality of Σ mentioned above.

The claimed decidability result follows readily from Theorem 9.

COROLLARY 10 (Decidability) *The set of* **U***–theorems is decidable.*

Some remarks on the complexity of the logic are contained in the final section of this paper.

5 Hybridization

As it stands, the system **U** is rather weakly expressive. In particular, only few of the necessary requirements for dealing with uniform structures are met. So, suitable extensions of the system should be considered in order to rectify that. And one should ask for generalizations besides.

For a start, the *hybrid* logic of a slightly more general class of structured subset spaces will be studied to some extent below. Since higher dimensions are involved and hybrid logic has not been developed into this direction up to now, this could be interesting *per se.* (As to a hybrid version of the usual logic of subset spaces and a discussion of its usefulness, *cf.* Heinemann [12]; see also Heinemann [13] for an application to reasoning about knowledge.)

We add a set NOM of *nominals* to the language of Section 2. The elements of NOM are denoted by $\mathfrak{e}, \mathfrak{f}, \mathfrak{g}, \ldots$ and $\mathfrak{e}', \mathfrak{f}', \mathfrak{g}', \ldots$, as needed.[5] Furthermore, a *satisfaction operator* $@_{\mathfrak{e}}$ is associated to each nominal $\mathfrak{e}$ (and added to the language as well).

The concept of structured subset spaces (see Definition 1) is modified as follows.

1. Let $X = X' \times X''$ be a non-empty product set. Then, a subset $U \subseteq X$ is called a *rectangle (with respect to* X*)* iff $U = Y \times Z$ for some $Y \subseteq X'$ and $Z \subseteq X''$.

2. A *structured subset frame* $(X, \mathcal{U})$ consists now of a non-empty product set X and a set $\mathcal{U}$ of rectangles with respect to X.

3. A *hybrid structured subset space* is yielded by adding a *hybridized valuation* V to a structured subset frame; *i.e.*, the denotation of a nominal with respect to V is a *single* entourage situation.

The following clauses have to be added with regard to satisfaction (see Definition 2):

$$
\begin{array}{lcl}
x, y, U \models_{\mathcal{M}} \mathfrak{e} & :\Longleftrightarrow & V(\mathfrak{e}) = x, y, U \\
x, y, U \models_{\mathcal{M}} @_{\mathfrak{e}}\alpha & :\Longleftrightarrow & V(\mathfrak{e}) \models_{\mathcal{M}} \alpha,
\end{array}
$$

where $\mathfrak{e} \in$ NOM and $\alpha \in$ WFF.

An easy inspection shows that all the axioms from the list contained in Section 2 are sound with respect to the smaller class of hybrid structured subset spaces as well.

But what is missing for completeness? — There are several matters we have to take into account. First of all, hybrid logic has to be integrated

[5]This should remind one of the fact that nominals are intended to name **e***ntourage situations.*

into our framework. To this end we adapt the corresponding Hilbert–style axiomatization given in Blackburn *et al.* [1, p. 438 ff] (see also Blackburn and Tzakova [2]), to the present case.

1. $@_{\mathfrak{e}}(\alpha \to \beta) \to (@_{\mathfrak{e}}\alpha \to @_{\mathfrak{e}}\beta)$
2. $@_{\mathfrak{e}}\alpha \leftrightarrow \neg @_{\mathfrak{e}} \neg \alpha$
3. $\mathfrak{e} \wedge \alpha \to @_{\mathfrak{e}}\alpha$
4. $@_{\mathfrak{e}}\mathfrak{e}$
5. $@_{\mathfrak{e}}\mathfrak{f} \wedge @_{\mathfrak{f}}\alpha \to @_{\mathfrak{e}}\alpha$
6. $@_{\mathfrak{e}}\mathfrak{f} \leftrightarrow @_{\mathfrak{f}}\mathfrak{e}$
7. $@_{\mathfrak{f}}@_{\mathfrak{e}}\alpha \leftrightarrow @_{\mathfrak{e}}\alpha$
8. $L_i @_{\mathfrak{e}}\alpha \to @_{\mathfrak{e}}\alpha$
9. $\Diamond @_{\mathfrak{e}}\alpha \to @_{\mathfrak{e}}\alpha$,

where $i \in \{\mathrm{h}, \mathrm{v}\}$, $\mathfrak{e}, \mathfrak{f} \in \mathrm{NOM}$ and $\alpha, \beta \in \mathrm{WFF}$. Note that the axiom schema called (*back*) in Blackburn *et al.* [1] occurs three times here (items 8 and 9 above) owing to the three modalities of the underlying system.

As to soundness, it can be seen easily again that Axioms 1 – 9 are valid in every hybrid structured subset space.

The hybrid rules we have to add are $@_{\mathfrak{e}}$*–necessitation* for every $\mathfrak{e} \in \mathrm{NOM}$, and the usual ones for *naming* and *pasting; cf.* Blackburn *et al.* [1, Section 7.3]. We do not write down these rules here, but mention that the latter also occurs in triplicate.

Giving this axiomatics of hybrid logic indicates already how we will try to obtain completeness, viz *canonically, i.e.*, à la Blackburn *et al.* [1], *loc cit.* (That is, with respect to a named and pasted model yielded by a distinguished maximal consistent set; we call this model 'canonical' below, for simplicity.) For this purpose we have to provide for *purity* of the axioms which go beyond $\mathbf{K}_3$. In particular, the schemata for structured subset spaces have to be re-formulated appropriately. Except for the axiom expressing persistency of the proposition variables (Schema 5 from the list contained in Section 2) this can obviously be done. That schema, 5, however, does not cause any problems on the canonical model at all.

Two points are left to be discussed. First, the general scenario of a hybrid subset space has to be arranged on the canonical model. *I.e.*, in particular, the denotation of every nominal must be unequivocally positioned there. The reader may consult Heinemann [12] in order to convince himself or herself that this can in fact be done with the aid of suitable hybrid axioms. (Clearly, some minor modifications caused by the additional dimension still must be incorporated.) We do not really go into detail concerning this.

We focus on the second point instead, viz establishing the required rectangle structure on the canonical model. Again we want suitable axioms to do this job. The following ones turn out to be right for that.

R1. $L_{\mathrm{h}}(\mathfrak{e} \wedge L_{\mathrm{v}}\mathfrak{g}) \wedge L_{\mathrm{h}}(\mathfrak{f} \wedge L_{\mathrm{v}}\mathfrak{g}) \rightarrow L_{\mathrm{h}}(\mathfrak{e} \wedge \mathfrak{f})$

R2. $L_{\mathrm{v}}(\mathfrak{e} \wedge L_{\mathrm{h}}\mathfrak{g}) \wedge L_{\mathrm{v}}(\mathfrak{f} \wedge L_{\mathrm{h}}\mathfrak{g}) \rightarrow L_{\mathrm{v}}(\mathfrak{e} \wedge \mathfrak{f})$

R3. $L_{\mathrm{h}}(\mathfrak{e} \wedge L_{\mathrm{v}}\mathfrak{g}) \wedge L_{\mathrm{v}}(\mathfrak{f} \wedge L_{\mathrm{h}}\mathfrak{g}) \wedge L_{\mathrm{h}}(\mathfrak{e}' \wedge L_{\mathrm{v}}\mathfrak{g}') \wedge L_{\mathrm{v}}(\mathfrak{f}' \wedge L_{\mathrm{h}}\mathfrak{g}') \wedge @_{\mathfrak{g}}\neg\mathfrak{g}' \rightarrow (@_{\mathfrak{e}}\neg\mathfrak{e}' \vee @_{\mathfrak{f}}\neg\mathfrak{f}')$,

where $\mathfrak{e}, \mathfrak{f}, \mathfrak{g}, \mathfrak{e}', \mathfrak{f}', \mathfrak{g}' \in \mathrm{NOM}$. Note that R1 – R3 are sound on hybrid structured subset spaces, too.

Let $\xrightarrow{L_{\mathrm{h}}}$ and $\xrightarrow{L_{\mathrm{v}}}$ designate the accessibility relations on the canonical model induced by the modalities L_{h} and L_{v}, respectively (as above). Furthermore, let x be an arbitrary point of that model, $[x]_{\mathrm{h}}$ the $\xrightarrow{L_{\mathrm{h}}}$–equivalence class and $[x]_{\mathrm{v}}$ the $\xrightarrow{L_{\mathrm{v}}}$–equivalence class of x. Finally, let

$$Q_x := \{y \mid (x,y) \in \xrightarrow{L_{\mathrm{h}}} \circ \xrightarrow{L_{\mathrm{v}}}\}.$$

Then it is claimed that $Q_x \cong [x]_{\mathrm{h}} \times [x]_{\mathrm{v}}$. A corresponding isomorphism can actually be defined as follows. According to Axiom R1 every element $y \in Q_x$ has a uniquely determined 'horizontal' component y_{h} (with respect to x), and according to Axiom R2 a uniquely determined 'vertical' component y_{v} as well. Thus the mapping

$$f : Q_x \longrightarrow [x]_{\mathrm{h}} \times [x]_{\mathrm{v}}, \ \text{ given by } \ y \mapsto (y_{\mathrm{h}}, y_{\mathrm{v}}) \ \text{ for all } y \in Q_x,$$

is well-defined. Injectivity of f is expressed by Axiom R3. Surjectivity follows from (the hybrid version of) the commutation axiom 4 from the list contained in Section 2, and from the fact that $\xrightarrow{L_{\mathrm{h}}}$ as well as $\xrightarrow{L_{\mathrm{v}}}$ are equivalence relations. Finally, it can easily be seen that f preserves relations in the sense that the property

$$\forall\, y, z \in Q_x : \Big(y \xrightarrow{L_{\mathrm{h}}} z \iff y_{\mathrm{h}} \xrightarrow{L_{\mathrm{h}}} z_{\mathrm{h}} \text{ and } y_{\mathrm{v}} = z_{\mathrm{v}}\Big)$$

holds for L_{h}, and an analogous property for L_{v}. — All in all we have shown that f is an isomorphism.

Summing up our exposition of hybridizing the logic of structured subset spaces we can now state:

THEOREM 11 (Hybrid completeness) *The hybrid system described above is sound and (canonically) complete with respect to the class of hybrid structured subset spaces.*

6 Concluding remarks

We showed above that the system **U** is sound and complete for the class of structured subset spaces, *i.e.*, spaces where the distinguished subsets are squares. Moreover, we proved that **U** satisfies the finite model property with respect to a suitable class of Kripke models and is decidable thus. (It should be mentioned here that we could not prove the finite model property with respect to structured subset spaces directly. Presumably this fmp does not hold; concerning a counter-example for the one-dimensional case see Dabrowski *et al.* [5, 1.3].)

Our decidability proof contained in Section 4 leads to a non-deterministic decision algorithm for satisfiability running in time $O\left(2^{2^n}\right)$ (because of the size of the refuting model estimated there). On the other hand, deciding **U**–satisfiability is clearly at least as hard as deciding satisfiability for the product logic $\mathbf{S5} \times \mathbf{S5}$. As is shown in Marx [15, Corollary 4.1] (see also Gabbay *et al.* [7, Theorem 5.24]), the latter problem is NEXPTIME–complete. Thus a gap appears between the upper and the lower bound. (Apart from few exceptions, *e.g., topological nexttime logic, cf* Heinemann [11], this seems to be the normal case for modal logics of subset spaces up to now.)

> *Question 1. What is the exact complexity of the* **U**–*satisfiability problem?*

Maybe it is possible to use a filter set which is structured in the same way as the filter set applied to the basic modal logic of subset spaces, in order to obtain the fmp for **U**; *cf* Dabrowski *et al.* [7, Section 2.3]. This would show that the lower bound is matched.

In Section 5 we proposed a hybrid variant of **U**. Other extensions to be taken into account may concern, for example,

- further modalities typical of multi–dimensional modal logic, *e.g.*, the *diagonal modality* $\iota\delta$ and the *converse modality* $\otimes$,
- further modalities increasing the expressive power of the language like the *global modality, cf* Blackburn *et al.* [1, Section 7.1] or
- higher dimensions, *i.e.*, spaces of n–tuples of states where $n \geq 3$.

In the latter case the situation with respect to effectivity properties is hopeless; *cf* the negative results concerning this in Gabbay *et al.* [7, Ch.

8], or Gabbay and Shehtman [8, Section 15]. The obstacle found in Section 4 of the present paper (impossible prefix reduction) is, therefore, in fact unsurmountable.

In contrast to this we conjecture that at least a completeness results holds in the first case. If the connectives $\iota\delta$ and $\otimes$ are included in the language, then a proof of completeness should be possible along the lines of Marx and Venema [16, Section 2.3]. A system of this kind is very appropriate to the present framework since it is closer than **U** to uniform structures. However, it is not clear from the outset whether or not such a system is decidable.

> *Question 2. Is the logic of structured subset spaces in the language including $\iota\delta$ and $\otimes$ decidable?*

Because of the squaring operation in uniform spaces it is also very desirable to have the *composition operator* to hand. Unfortunately this operator behaves badly from a computational point of view since we have neither completeness nor decidability then, actually; *cf* Marx and Venema [16, Section 2.4].

> *Question 3. How to modalize reasonably the squaring operation in uniform spaces?*

An answer to this question would yield the decisive breakthrough towards a modal basis for uniform reasoning. Maybe the modalities mentioned in the second of the above items will prove to be helpful in that. But it is not clear so far if the addition of some of those so-called *logical modalities* has any substantial impact on the modal logic of subset spaces at all.

> *Question 4. To what extent can logical modalities be useful to the modal logic of subset spaces?*

Each of the above questions indicates a future research duty.

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