

Comparative plausibility in neighbourhood models: axiom systems and sequent calculi

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Abstract

We introduce a family of comparative plausibility logics over neighbourhood models, generalising Lewis’ comparative plausibility operator over sphere models. We provide axiom systems for the logics, and prove their soundness and completeness with respect to the semantics. Then, we introduce two kinds of analytic proof systems for several logics in the family: a multi-premisses sequent calculus in the style of Lellmann and Pattinson, for which we prove cut admissibility, and a hypersequent calculus based on structured calculi for conditional logics by Girlando et al., tailored for countermodel construction over failed proof search. Our results constitute the first steps in the definition of a unified proof theoretical framework for logics equipped with a comparative plausibility operator.

Keywords: Comparative plausibility, neighbourhood semantics, sequent calculus, hypersequent calculus, countermodel construction.

1 Introduction

In the seminal work *Counterfactuals* [16], besides the well-known analysis of counterfactual sentences, David Lewis defined a notion of comparative plausibility which has then become a standard.² Specifically, Lewis introduced a *comparative plausibility operator* $A \preceq B$, read “ A is at least as plausible as B ”, which is evaluated on the plausibility ordering of worlds of a model.

Lewis’ notion of comparative plausibility is defined over *sphere models*. These are possible-world models in which every world x is endowed with a system of spheres $S(x)$, that is, a set of sets of worlds such that for every two sets in the class, one of the two is included in the other (if $\alpha, \beta \in S(x)$, then $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$). This property, known as *nesting*, determines a total ordering over the set of worlds belonging to a system of spheres, where worlds in the

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² Lewis [16] refers to \preceq as the operator for *comparative possibility*. In the literature, the same or similar operators also go under the names of *entrenchment* [15], *comparative similarity* [25] or *relative likelihood* [12]. Here we adopt the terminology of, e.g., [22].

inner spheres are taken to be more plausible than worlds in the outer spheres. Then, $A \preceq B$ is true at a world x if the innermost sphere in $S(x)$ containing a world which forces B also contains a world that forces A . The operator \preceq is interdefinable with Lewis' *conditional operator* $A > B$ expressing counterfactual sentences (formally, $A > B$ is equivalent to $(\perp \preceq A) \vee \neg((A \wedge \neg B) \preceq (A \wedge B))$).

Other than in Lewis' work, several operators expressing forms of similarity or closeness between states of affairs or concepts have been studied in the literature, and find applications in many areas of computer science and philosophy. In knowledge representation, Sheremet et al. developed in [25,26] the *logic of comparative concept similarity*, evaluated over distance models, which implements a description logic-like formalism for reasoning about similarity of concepts in ontologies. Refer to [1] for a Lewis-style semantics for this logic. Moreover, similarity operators can be used in deontic reasoning to express degrees of urgency of obligations [2] or, more recently, to express the preferred scenario an agent would choose in an ethical decision-making process [18]. In philosophical logic, a logic equipped with an operator to express *ceteris paribus* preference between states of affairs was introduced by Von Wright in [28], and formalised in [27]. Moreover, a logic expressing *ceteris paribus* preferences in a deontic setting was recently defined in [17].

A natural semantics to express generalized forms of Lewis' comparative plausibility is *preferential semantics*. Preferential models consist of a set of worlds equipped with an explicit preorder relation \leq_x for every world x , encoding similarity or preference among worlds. These models represent a generalisation of sphere models, where totality of the ordering is not assumed, and have been studied as a semantics for a family of conditional logic weaker than Lewis' counterfactual logic, called *Preferential Conditional Logics* [3,6], strongly related to non-monotonic logic P from [13]. In [12], Halpern proposes partially ordered preferential structures as a general framework to represent forms of preference or similarity.

We here propose a setting even more general than preferential semantics, by interpreting the comparative plausibility operator over *neighbourhood models* (Sec. 2). These possible-worlds models are endowed with a neighbourhood function which assigns to every world x a set of sets of worlds, $N(x)$, where nesting is not assumed. In this weaker setting, the truth condition for \preceq can be taken to express forms of similarity of closeness between states or concepts, which are not assumed to be totally ordered. Neighbourhood models were introduced to define a semantics for non-normal modal logics [24,20] and, among other applications, have been employed as a semantics for conditional logics [19,21,11].

We introduce axiom systems for the family of *logics of Comparative Plausibility in Neighbourhood models* (CPN logics), and prove their adequacy with respect to some relevant classes of neighbourhood models (Sec. 3). We then study the proof theory of CPN logics, by defining two kinds of proof systems for them. Our calculi are inspired from analytic proof systems for Lewis' conditional logics introduced in the literature.

We first present a *multi-premisses* sequent calculus in the style of Lellmann and Pattinson [15,14] (Sec. 4). The rules of these calculi display a number of premisses which depends on the number of comparative plausibility formulas occurring in the conclusion. The calculi for CPN logics represent simpler fragments of the calculi for Lewis' logics presented in [14]. We prove cut-admissibility for the multi-premisses calculi. While these calculi have strong proof-theoretical properties, they are not best suited for root-first proof search: due to the fact that the comparative plausibility rules are not invertible, a heavy use of backtracking is needed to construct derivations.

This motivates the introduction of a second family of proof systems, based on *hypersequents* (Sec. 5). The calculi are inspired from the structured calculi for Lewis' logics introduced in [22,8], which introduce an additional structural connective to Gentzen-style sequents representing \preceq -formulas. Following a strategy adopted e.g. in [4] in the context of non-normal modal logics, we further enrich the structure of sequents from [8] by introducing hypersequent-style calculi, and show that they simulate the multi-premisses calculi. Thanks to this richer structure we obtain invertibility of *all* the rules in the calculus, which we would not have using the sequent structure from [8], and a more direct construction of countermodels from branches of failed proof search trees. We conclude by discussing related works and further research directions (Sec. 6).

2 Neighbourhood semantics

For $Atm = \{p_0, p_1, p_2, \dots\}$ denumerable set of propositional variables, we consider the formulas of \mathcal{L} be defined by the BNF grammar $A ::= p \mid \perp \mid A \rightarrow A \mid A \preceq A$, where p is any element of Atm , and \preceq is the operator for comparative plausibility. We assume \top, \neg, \wedge, \vee to be defined as usual in terms of \perp, \rightarrow .

Definition 2.1 A *neighbourhood model* is a tuple $\mathcal{M} = \langle W, N, V \rangle$, where W is a non-empty set of worlds, V is a valuation function $Atm \rightarrow \mathcal{P}(W)$, and N is a function $W \rightarrow \mathcal{P}(\mathcal{P}(W))$, called *neighbourhood function*, satisfying the non-emptiness condition: for all $w \in W$: $\emptyset \notin N(w)$.³ For all $w \in W$ and $A \in \mathcal{L}$, the forcing relation $\mathcal{M}, w \Vdash A$ is defined inductively as follows:

$$\begin{aligned} \mathcal{M}, w \Vdash p & \quad \text{iff} \quad w \in V(p). \\ \mathcal{M}, w \not\Vdash \perp. \\ \mathcal{M}, w \Vdash B \rightarrow C & \quad \text{iff} \quad \text{if } \mathcal{M}, w \Vdash B, \text{ then } \mathcal{M}, w \Vdash C. \\ \mathcal{M}, w \Vdash B \preceq C & \quad \text{iff} \quad \text{for all } \alpha \in N(w), \text{ if there is } v \in \alpha \text{ s.t. } \mathcal{M}, v \Vdash C, \\ & \quad \text{then there is } u \in \alpha \text{ s.t. } \mathcal{M}, u \Vdash B. \end{aligned}$$

We say that A is *valid in a model* \mathcal{M} , written $\mathcal{M} \models A$, if $\mathcal{M}, w \Vdash A$ for all worlds w of \mathcal{M} , and it is *valid on a class of models* \mathcal{C} if $\mathcal{M} \models A$ for all $\mathcal{M} \in \mathcal{C}$.

In the following we simply write $w \Vdash A$ when \mathcal{M} is clear from the context. We shall also use $\alpha \Vdash^{\exists} A$ as an abbreviation for 'there is $w \in \alpha$ such that

³ Non-emptiness could be dropped as it has no impact on the satisfiability of \preceq -formulas [16]. We assume it as it allows for a clean formulation of the conditions for the extensions, and for uniformity with the neighbourhood semantics of conditional logics from [21,11].

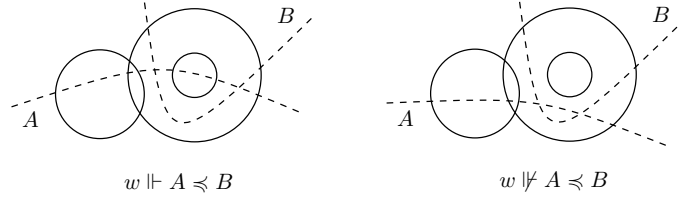


Fig. 1. Representation of the forcing clause for \preceq -formulas. Dashed lines represent the extensions of A and B .

$w \Vdash A$. Thus, we can rewrite the forcing clause of \preceq -formulas, graphically represented in Fig. 1, as follows:

$$w \Vdash B \preceq C \quad \text{iff} \quad \text{for all } \alpha \in N(w), \text{ if } \alpha \Vdash \exists C, \text{ then } \alpha \Vdash \exists B.$$

We observe that unary modalities can be defined on the basis of \preceq , namely $\Box_K A := \perp \preceq \neg A$, and $\Box_M A := \neg(\neg A \preceq \top)$, where \Box_K and \Box_M are the Box modalities of respectively logic K and non-normal logic M (cf. e.g [23]). Note also that sphere models can be recovered by adding the condition of *nesting* to the neighbourhood function: for all $\alpha, \beta \in N(w)$, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. Lewis considered in [16] several additional properties, which turn out to be of interest when formulated on the neighbourhood function. We consider here classes of neighbourhood models satisfying combinations of the following properties:

- N $N(w) \neq \emptyset$. (Normality)
- T There is $\alpha \in N(w)$ such that $w \in \alpha$. (Total reflexivity)
- W $N(w) \neq \emptyset$ and for all $\alpha \in N(w)$, $w \in \alpha$. (Weak centering)
- C $\{w\} \in N(w)$ and for all $\alpha \in N(w)$, $w \in \alpha$. (Centering)
- U If $v \in \alpha$ and $\alpha \in N(w)$, then $\bigcup N(v) = \bigcup N(w)$. (Uniformity)
- A If $v \in \alpha$ and $\alpha \in N(w)$, then $N(v) = N(w)$. (Absoluteness)

The condition of absoluteness can also be formulated as follows:

- A+ For all $v, w \in W$, $N(v) = N(w)$. (Strong Absoluteness)

Equivalence of A and A+ over formulas validity can be easily established using the same strategy described by Lewis [16, p. 122]. We will use A+ in Sec. 5.

Neighbourhood semantics can be used to express a variety of situations. By means of example, let $\alpha, \beta, \gamma, \dots$ in $N(w)$ represent sources of information available at w which are not arranged in any priority or reliability order. In this setting, $A \preceq B$ expresses that A is at least as plausible as B in that whenever w receives information B , it also receives information A . Then, model conditions represent natural assumptions about the information sources: by *normality*, every w has a source of information available, while according to *reflexivity* or *weak centering*, w belongs to some or all of the sources available to itself (e.g. online forums w must be registered at). Moreover, *uniformity* and *absoluteness* express kinds of information bubbles, since if v belongs to a source available to w , then w and v have access to the same sources of information.

cpr	$\frac{A \rightarrow B}{B \preccurlyeq A}$	tr	$(A \preccurlyeq B) \wedge (B \preccurlyeq C) \rightarrow (A \preccurlyeq C)$
n	$\neg(\perp \preccurlyeq \top)$	or	$(A \preccurlyeq B) \wedge (A \preccurlyeq C) \rightarrow (A \preccurlyeq B \vee C)$
t	$(\perp \preccurlyeq A) \rightarrow \neg A$	u-	$\neg(\perp \preccurlyeq A) \rightarrow (\perp \preccurlyeq (\perp \preccurlyeq A))$
w	$A \rightarrow (A \preccurlyeq \top)$	u	$(\perp \preccurlyeq A) \rightarrow (\perp \preccurlyeq \neg(\perp \preccurlyeq A))$
c	$(A \preccurlyeq \top) \rightarrow A$	a-	$(A \preccurlyeq B) \rightarrow (\perp \preccurlyeq \neg(A \preccurlyeq B))$
		a	$\neg(A \preccurlyeq B) \rightarrow (\perp \preccurlyeq (A \preccurlyeq B))$

Fig. 2. Axioms and rules for CPN logics.

3 Axiom systems for CPN logics

In this section we present the logics of Comparative Plausibility in Neighbourhood models (CPN logics in the following) corresponding to the classes of neighbourhood models introduced in Sec. 2. We propose axiom systems for CPN logics and show their soundness and completeness. Then, we compare CPN logics with Lewis' logics of comparative plausibility in sphere models.

Definition 3.1 CPN logics are defined by extending classical propositional logic (CPL) formulated in \mathcal{L} with the rules and axioms for \preccurlyeq from Fig. 2:

$$\begin{aligned} \mathbf{N}_{\preccurlyeq} &:= \text{CPL} \cup \{\text{tr, or, cpr}\} & \mathbf{NN}_{\preccurlyeq} &:= \mathbf{N}_{\preccurlyeq} \cup \{\mathbf{n}\} & \mathbf{NT}_{\preccurlyeq} &:= \mathbf{N}_{\preccurlyeq} \cup \{\mathbf{t}\} \\ \mathbf{NW}_{\preccurlyeq} &:= \mathbf{NT}_{\preccurlyeq} \cup \{\mathbf{w}\} & \mathbf{NC}_{\preccurlyeq} &:= \mathbf{NW}_{\preccurlyeq} \cup \{\mathbf{c}\} \end{aligned}$$

Moreover, for $\mathbf{L} \in \{\mathbf{N}_{\preccurlyeq}, \mathbf{NN}_{\preccurlyeq}, \mathbf{NT}_{\preccurlyeq}, \mathbf{NW}_{\preccurlyeq}, \mathbf{NC}_{\preccurlyeq}\}$, we define $\mathbf{LU}_{\preccurlyeq} := \mathbf{L} \cup \{\mathbf{u-}, \mathbf{u}\}$ and $\mathbf{LA}_{\preccurlyeq} := \mathbf{L} \cup \{\mathbf{a-}, \mathbf{a}\}$.

The logics generated by this definition are displayed in the lower layer of the lattice of systems in Fig. 3. The axioms of $\mathbf{N}_{\preccurlyeq}$ are those defined by Lewis in [16, Ch.6], while axioms for extensions of $\mathbf{N}_{\preccurlyeq}$ are reformulations of Lewis' axioms in terms of \preccurlyeq [15,8]. In the following, for every logic \mathbf{L} from Def. 3.1, we denote \mathbf{L}^* any extension of \mathbf{L} . As usual, we say that a formula A is *derivable* in $\mathbf{N}_{\preccurlyeq}^*$, written $\vdash_{\mathbf{N}_{\preccurlyeq}^*} A$, if there is a finite sequence of formulas ending with A where every formula is an axiom of $\mathbf{N}_{\preccurlyeq}^*$, or is obtained from previous formulas by *modus ponens* or *cpr*. Moreover, we say that A is *deducible* in $\mathbf{N}_{\preccurlyeq}^*$ from a set of formulas Φ if there is a finite set $\{B_1, \dots, B_n\} \subseteq \Phi$ such that $\vdash_{\mathbf{N}_{\preccurlyeq}^*} B_1 \wedge \dots \wedge B_n \rightarrow A$.

For each logic $\mathbf{N}_{\preccurlyeq}^*$, we call $\mathbf{N}_{\preccurlyeq}^*$ -*model* any neighbourhood model satisfying the conditions corresponding to the letters appearing beside \mathbf{N} in the name of the logic. Thus, $\mathbf{N}_{\preccurlyeq}$ -models denotes the class of all neighbourhood models, $\mathbf{NN}_{\preccurlyeq}$ -models the class of all models satisfying normality, and so on.

We show that each logic $\mathbf{N}_{\preccurlyeq}^*$ is characterised by the class of all $\mathbf{N}_{\preccurlyeq}^*$ -models. We first prove that the logics are sound with respect to the corresponding classes of models.

Theorem 3.2 (Soundness) *For every formula A , if A is derivable in $\mathbf{N}_{\preccurlyeq}^*$, then A is valid in all $\mathbf{N}_{\preccurlyeq}^*$ -models.*

Proof. We show that the modal axioms and rules of $\mathbf{N}_{\preccurlyeq}^*$ are valid (resp. sound) in the corresponding models, considering some relevant examples. (cpr) Assume $\mathcal{M} \models A \rightarrow B$, and $\alpha \in N(w)$, $\alpha \Vdash^{\exists} A$. Then $\alpha \Vdash^{\exists} B$, therefore $w \Vdash B \preccurlyeq A$. (tr) If $w \Vdash (A \preccurlyeq B) \wedge (B \preccurlyeq C)$, then for every $\alpha \in N(w)$, $\alpha \Vdash^{\exists} B$ implies

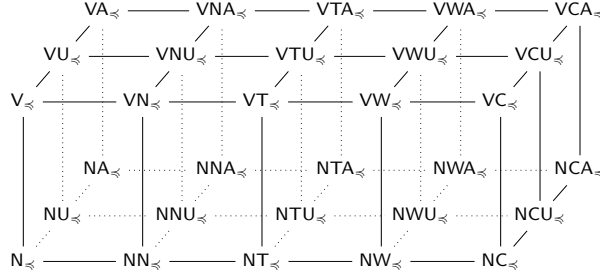


Fig. 3. The family of CPN logics. The system in the upper layer are Lewis' logics.

$\alpha \Vdash^{\exists} A$, and $\alpha \Vdash^{\exists} C$ implies $\alpha \Vdash^{\exists} B$, then $\alpha \Vdash^{\exists} C$ implies $\alpha \Vdash^{\exists} A$, therefore $w \Vdash A \preceq C$. (or) If $w \Vdash (A \preceq B) \wedge (A \preceq C)$, then for every $\alpha \in N(w)$, $\alpha \Vdash^{\exists} B$ implies $\alpha \Vdash^{\exists} A$, and $\alpha \Vdash^{\exists} C$ implies $\alpha \Vdash^{\exists} A$, then $\alpha \Vdash^{\exists} B \vee C$ implies $\alpha \Vdash^{\exists} A$, therefore $w \Vdash A \preceq B \vee C$. (n) By normality, for all $w \in W$ there is $\alpha \in N(w)$. Moreover, $\alpha \not\Vdash^{\exists} \perp$ and since $\alpha \neq \emptyset$, $\alpha \Vdash^{\exists} \top$, then $w \Vdash \neg(\perp \preceq \top)$. (t) Assume $w \Vdash \perp \preceq A$. Then for all $\alpha \in N(w)$, $\alpha \not\Vdash^{\exists} A$. Moreover by total reflexivity, there is $\beta \in N(w)$ such that $w \in \beta$. Then $w \not\Vdash A$, thus $w \Vdash \neg A$. (w) Assume $w \Vdash A$. By weak centering, $N(w) \neq \emptyset$, and $w \in \alpha$ for all $\alpha \in N(w)$. Then for all $\alpha \in N(w)$, $\alpha \Vdash^{\exists} A$, thus $w \Vdash \top \preceq A$. (c) Assume $w \Vdash A \preceq \top$. Then for all $\alpha \in N(w)$, $\alpha \Vdash^{\exists} A$. Moreover by centering, $\{w\} \in N(w)$, therefore $w \Vdash A$. \square

Using a canonical model construction inspired from [16], we shall now prove that \mathbf{N}_{\preceq}^* is complete with respect to the class of all \mathbf{N}_{\preceq}^* -models. As usual, for any logic \mathbf{L} and set of formulas Φ , we say that Φ is *L-consistent* if $\Phi \not\vdash_{\mathbf{L}} \perp$, and that it is *L-maximal consistent* (maxcons) if it is consistent and for every $B \notin \Phi$, $\Phi \cup \{B\} \vdash_{\mathbf{L}} \perp$. The proof of the following Lemma is standard.

Lemma 3.3 (a) If Φ is a *L-consistent* set of formulas, then there is a *L-maximal consistent* set Ψ such that $\Phi \subseteq \Psi$. (b) If Φ is a *L-maximal consistent* set, then for all $A, B \in \mathcal{L}$, (i) if $\Phi \vdash_{\mathbf{L}} A$, then $A \in \Phi$; (ii) $A \in \Phi$ if and only if $\neg A \notin \Phi$; (iii) if $A \vee B \in \Phi$, then $A \in \Phi$ or $B \in \Phi$.

We consider the following notion of *cut around*,⁴ and prove the subsequent lemma that will be needed in the following.

Definition 3.4 Let Φ be a maximal consistent set of formulas, and Σ be a set of formulas. We say that Σ is a *cut around* Φ if for all finite sets $\{B_1, \dots, B_n\} \subseteq \Sigma$ and all $A \notin \Sigma$, $(B_1 \vee \dots \vee B_n) \preceq A \notin \Phi$. Moreover, let $co\Sigma = \{\Psi \text{ maxcons} \mid \Psi \cap \Sigma = \emptyset\}$.

Lemma 3.5 If Σ is a *cut around* Φ for some maximal consistent set Φ , then for every formula A , $A \in \Sigma$ if and only if for all $\Psi \in co\Sigma$, $A \notin \Psi$.

Proof. If $A \in \Sigma$ and $\Psi \in co\Sigma$, then $\Psi \cap \Sigma = \emptyset$, thus $A \notin \Psi$. If instead $A \notin \Sigma$, then suppose by contradiction that $\{\neg B \mid B \in \Sigma\} \cup \{A\} \vdash_{\mathbf{L}} \perp$. Then there

⁴ This terminology comes from Lewis [16], but our definition is different from Lewis' one.

are formulas $B_1, \dots, B_n \in \Sigma$ such that $\vdash_{\mathbf{L}} \neg B_1 \wedge \dots \wedge \neg B_n \rightarrow \neg A$, thus $\vdash_{\mathbf{L}} A \rightarrow B_1 \vee \dots \vee B_n$, therefore $\vdash_{\mathbf{L}} (B_1 \vee \dots \vee B_n) \preceq A$. By closure under derivation, $(B_1 \vee \dots \vee B_n) \preceq A \in \Phi$, but by definition of cut around, $(B_1 \vee \dots \vee B_n) \preceq A \notin \Phi$. We conclude that $\{\neg B \mid B \in \Sigma\} \cup \{A\} \not\vdash_{\mathbf{L}} \perp$. Then by Lemma 3.3, there is $\Psi \in W_{\mathbf{L}}$ such that $\{\neg B \mid B \in \Sigma\} \cup \{A\} \subseteq \Psi$, therefore $\Psi \in \text{co}\Sigma$ and $A \in \Psi$. \square

From Lemma 3.5 it immediately follows that $\perp \in \Sigma$ for all Σ cut around Φ . We now define the canonical model.

Definition 3.6 For every CPN logic \mathbf{L} , the *canonical model* for \mathbf{L} is the tuple $M_{\mathbf{L}} = \langle W_{\mathbf{L}}, N_{\mathbf{L}}, V_{\mathbf{L}} \rangle$, where:

- $W_{\mathbf{L}}$ is the class of all \mathbf{L} -maximal consistent sets;
- for all $\Phi \in W_{\mathbf{L}}$, $N_{\mathbf{L}}(\Phi) = \{\text{co}\Sigma \mid \Sigma \text{ cut around } \Phi \text{ and } \text{co}\Sigma \neq \emptyset\}$;
- $V_{\mathbf{L}}(p) = \{\Phi \in W_{\mathbf{L}} \mid p \in \Phi\}$.

Lemma 3.7 (Truth lemma) *If \mathbf{L} is a CPN logic and $M_{\mathbf{L}}$ is the canonical model for \mathbf{L} , then for all $A \in \mathcal{L}$ and all $\Phi \in W_{\mathbf{L}}$, $\Phi \Vdash A$ if and only if $A \in \Phi$.*

Proof. By induction on the construction of A . For atomic and propositional formulas the proof is standard. We consider the case $A = B \preceq C$.

(\Rightarrow) Suppose $\Phi \Vdash B \preceq C$. Then for all $\alpha \in N_{\mathbf{L}}(\Phi)$, $\alpha \Vdash^{\exists} C$ implies $\alpha \Vdash^{\exists} B$. By definition, this means that for every Σ cut around Φ , $\text{co}\Sigma \Vdash^{\exists} C$ implies $\text{co}\Sigma \Vdash^{\exists} B$. Let $\Phi_{B \preceq} = \{D \mid B \preceq D \in \Phi\}$. Then since $B \preceq B \in \Phi$, $B \in \Phi_{B \preceq}$. Moreover, $\Phi_{B \preceq}$ is a cut around Φ : if $E_1, \dots, E_n \in \Phi_{B \preceq}$ and $F \notin \Phi_{B \preceq}$, then $B \preceq E_1, \dots, B \preceq E_n \in \Phi$, and $B \preceq F \notin \Phi$. Thus by axiom *or* and closure under derivation, $B \preceq (E_1 \vee \dots \vee E_n) \in \Phi$, whence by *tr*, $(E_1 \vee \dots \vee E_n) \preceq F \notin \Phi$. Now suppose by contradiction that $\{\neg D \mid D \in \Phi_{B \preceq}\} \cup \{C\} \not\vdash_{\mathbf{L}} \perp$. Then by Lemma 3.3 there is $\Psi \in W_{\mathbf{L}}$ such that $\{\neg D \mid D \in \Phi_{B \preceq}\} \cup \{C\} \subseteq \Psi$. We then have $C \in \Psi$ and $\Phi_{B \preceq} \cap \Psi = \emptyset$, which implies $\Psi \in \text{co}\Phi_{B \preceq}$. By i.h., $\Psi \Vdash C$, thus $\text{co}\Phi_{B \preceq} \Vdash^{\exists} C$, which implies $\text{co}\Phi_{B \preceq} \Vdash^{\exists} B$. This means that there is $\Omega \in \text{co}\Phi_{B \preceq}$ such that $\Omega \Vdash B$, therefore by i.h., $B \in \Omega$. Furthermore, by definition we have $\Omega \cap \Phi_{B \preceq} = \emptyset$, then $B \notin \Phi_{B \preceq}$, which contradicts $B \in \Phi_{B \preceq}$. Therefore $\{\neg D \mid D \in \Phi_{B \preceq}\} \cup \{C\} \vdash_{\mathbf{L}} \perp$. Then there are $D_1, \dots, D_n \in \Phi_{B \preceq}$ such that $\vdash_{\mathbf{L}} \neg D_1 \wedge \dots \wedge \neg D_n \rightarrow \neg C$, that is $\vdash_{\mathbf{L}} C \rightarrow D_1 \vee \dots \vee D_n$, whence by *cpr*, $(D_1 \vee \dots \vee D_n) \preceq C \in \Phi$. Moreover by definition of $\Phi_{B \preceq}$, $B \preceq D_1, \dots, B \preceq D_n \in \Phi$. Then by *or*, $B \preceq (D_1 \vee \dots \vee D_n) \in \Phi$, finally by *tr*, $B \preceq C \in \Phi$.

(\Leftarrow) Suppose $\Phi \not\Vdash B \preceq C$. Then there is $\alpha \in N_{\mathbf{L}}(\Phi)$ such that $\alpha \Vdash^{\exists} C$ and $\alpha \not\Vdash^{\exists} B$, i.e., there is a Σ cut around Φ with $\text{co}\Sigma \Vdash^{\exists} C$ and $\text{co}\Sigma \not\Vdash^{\exists} B$. By i.h. there is $\Psi \in \text{co}\Sigma$ such that $C \in \Psi$, and for all $\Omega \in \text{co}\Sigma$, $B \notin \Omega$. Then $C \notin \Sigma$, and from Lemma 3.5 it follows that $B \in \Sigma$. Then by definition $B \preceq C \notin \Phi$. \square

Lemma 3.8 (Model lemma) *The canonical model for \mathbf{N}_{\preceq}^* is a \mathbf{N}_{\preceq}^* -model.*

Proof. Non-emptiness is immediate. We consider the other conditions.

(NN_{\preceq}^*) For every $\Phi \in W_{\text{NN}_{\preceq}^*}$, $\neg(\perp \preceq \top) \in \Phi$, then by Lemma 3.7, $\Phi \Vdash \neg(\perp \preceq \top)$, thus there is $\alpha \in N_{\text{NN}_{\preceq}^*}(\Phi)$ such that $\alpha \Vdash^{\exists} \top$ and $\alpha \not\Vdash^{\exists} \perp$.

(NT_{\preceq}^*) For any $\Phi \in W_{\text{NT}_{\preceq}^*}$, let $\Sigma = \{A \mid \perp \preceq A \in \Phi\}$. Σ is a cut around Φ , since for all $B_1, \dots, B_n \in \Sigma$ and $C \notin \Sigma$, $\perp \preceq B_1, \dots, \perp \preceq B_n \in \Phi$ and

$\perp \preceq C \notin \Phi$, then by **or**, $\perp \preceq B_1 \vee \dots \vee B_n \in \Phi$, and by **tr**, $B_1 \vee \dots \vee B_n \preceq C \notin \Phi$. Moreover, for any $A \in \mathcal{L}$, if $A \in \Sigma$, then $\perp \preceq A \in \Phi$, thus by **t**, $\neg A \in \Phi$, whence $A \notin \Phi$. Thus $\Sigma \cap \Phi = \emptyset$, which implies $\Phi \in \text{co}\Sigma$, and since $\text{co}\Sigma \neq \emptyset$, $\text{co}\Sigma \in N_{\text{NT}\preceq}(\Phi)$.

(**NW** $_{\preceq}^*$) Since $\text{NW}_{\preceq} \vdash \mathbf{n}$, by item (**NN** $_{\preceq}$), $N_{\text{NW}\preceq}(\Phi) \neq \emptyset$ for all $\Phi \in W_{\text{NW}\preceq}$. Moreover let $\text{co}\Sigma \in N_{\text{NW}\preceq}(\Phi)$ for a Σ cut around Φ . Then there is $\Psi \in \text{co}\Sigma$. Since $\top \in \Psi$, by Lemma 3.5, $\top \notin \Sigma$, then for all $A \in \Sigma$, $A \preceq \top \notin \Phi$, thus by axiom **w**, $A \notin \Phi$. This means $\Sigma \cap \Phi = \emptyset$, therefore $\Phi \in \text{co}\Sigma$.

(**NC** $_{\preceq}^*$) Since axiom **w** belongs to **NC** $_{\preceq}$, by item (**NW** $_{\preceq}$), for all $\Phi \in W_{\text{NC}\preceq}$ and all $\alpha \in N_{\text{NC}\preceq}(\Phi)$, $\Phi \in \alpha$. Moreover, let $\Sigma = \{A \mid A \preceq \top \notin \Phi\}$. Then Σ is a cut around Φ : if $B_1, \dots, B_n \in \Sigma$ and $C \notin \Sigma$, then $B_1 \preceq \top, \dots, B_n \preceq \top \notin \Phi$ and $C \preceq \top \in \Phi$. By axiom **w**, $B_1 \notin \Phi, \dots, B_n \notin \Phi$, thus $B_1 \vee \dots \vee B_n \notin \Phi$, then by axiom **c**, $(B_1 \vee \dots \vee B_n) \preceq \top \notin \Phi$, therefore by **tr**, $(B_1 \vee \dots \vee B_n) \preceq C \notin \Phi$. Moreover, since $\top \preceq \top \in \Phi$, $\top \notin \Sigma$, by Lemma 3.5 there is $\Psi \in \text{co}\Sigma$, thus $\text{co}\Sigma \in N_{\text{NC}\preceq}(\Phi)$. Suppose $\Psi \neq \Phi$. Then there is $A \in \mathcal{L}$ such that $A \in \Psi$ and $A \notin \Phi$. Since $\Psi \in \text{co}\Sigma$, $A \notin \Sigma$, then $A \preceq \top \in \Phi$, thus by **c**, $A \in \Phi$, it follows $\Psi = \Phi$, therefore $\text{co}\Sigma = \{\Phi\}$.

(**NU** $_{\preceq}^*$) Suppose $\Psi \in \bigcup N_{\text{NU}\preceq}(\Phi)$. Then $\Psi \in \text{co}\Sigma$ for some Σ cut around Φ . We show that for all $A \in \mathcal{L}$, $\perp \preceq A \in \Phi$ iff $\perp \preceq A \in \Psi$. If $\perp \preceq A \in \Phi$, then by axiom **u**, $\perp \preceq \neg(\perp \preceq A) \in \Phi$, then by Def. 3.4, $\perp \notin \Sigma$ or $\neg(\perp \preceq A) \in \Sigma$. Since $\perp \in \Sigma$, we have $\neg(\perp \preceq A) \in \Sigma$, thus $\neg(\perp \preceq A) \notin \Psi$, then $\perp \preceq A \in \Psi$. If $\perp \preceq A \in \Psi$, then $\perp \preceq A \notin \Sigma$, thus $\perp \preceq (\perp \preceq A) \notin \Phi$, then by axiom **u-**, $\neg(\perp \preceq A) \notin \Phi$, therefore $\perp \preceq A \in \Phi$. Let $\Pi = \{A \mid \perp \preceq A \in \Phi\} = \{A \mid \perp \preceq A \in \Psi\}$. Then Π is a cut around Φ and Ψ : If $B_1, \dots, B_n \in \Pi$ and $C \notin \Pi$, then $\perp \preceq B_1, \dots, \perp \preceq B_n \in \Phi$ and $\perp \preceq C \notin \Phi$, thus by **or**, $\perp \preceq B_1 \vee \dots \vee B_n \in \Phi$, then by **tr**, $B_1 \vee \dots \vee B_n \preceq C \notin \Phi$. Moreover for all Ω cut around Φ or Ψ , $\Pi \subseteq \Omega$, therefore $\text{co}\Omega \subseteq \text{co}\Pi$. Then in particular $\text{co}\Sigma \subseteq \text{co}\Pi$, which implies $\Psi \in \text{co}\Pi$, thus $\text{co}\Pi \neq \emptyset$. It follows $\text{co}\Pi \in N_{\text{NU}\preceq}(\Phi)$ and $\text{co}\Pi \in N_{\text{NU}\preceq}(\Psi)$, therefore $\text{co}\Pi = \bigcup N_{\text{NU}\preceq}(\Phi) = \bigcup N_{\text{NU}\preceq}(\Psi)$.

(**NA** $_{\preceq}^*$) Suppose $\Psi \in \bigcup N_{\text{NA}\preceq}(\Phi)$. Then $\Psi \in \text{co}\Sigma$ for some Σ cut around Φ . We show that for all $A, B \in \mathcal{L}$, $A \preceq B \in \Phi$ iff $A \preceq B \in \Psi$. If $A \preceq B \in \Phi$, then by axiom **a-**, $\perp \preceq \neg(A \preceq B) \in \Phi$, then by Def. 3.4, $\perp \notin \Sigma$ or $\neg(A \preceq B) \in \Sigma$. Since $\perp \in \Sigma$, we have $\neg(A \preceq B) \in \Sigma$, thus $\neg(A \preceq B) \notin \Psi$, then $A \preceq B \in \Psi$. If $A \preceq B \in \Psi$, then $A \preceq B \notin \Sigma$, thus $\perp \preceq (A \preceq B) \notin \Phi$, then by axiom **a**, $\neg(A \preceq B) \notin \Phi$, therefore $A \preceq B \in \Phi$. It follows that for every Π , Π is a cut around Φ iff Π is a cut around Ψ , thus $\text{co}\Pi \in N_{\text{NA}\preceq}(\Phi)$ iff $\text{co}\Pi \in N_{\text{NA}\preceq}(\Psi)$, therefore $N_{\text{NA}\preceq}(\Phi) = N_{\text{NA}\preceq}(\Psi)$. \square

As a consequence of the previous lemmas we obtain the following result.

Theorem 3.9 (Completeness) *For every formula A , if A is valid in all \mathbf{N}_{\preceq}^* -models, then A is derivable in \mathbf{N}_{\preceq}^* .*

Proof. Suppose $\not\vdash_{\mathbf{N}_{\preceq}^*} A$. Then $\{\neg A\}$ is \mathbf{N}_{\preceq}^* -consistent, thus by Lemma 3.3 there is a \mathbf{N}_{\preceq}^* -maxcons set Φ such that $\neg A \in \Phi$. By definition, $\Phi \in W_{\mathbf{N}_{\preceq}^*}$, and by Lemma 3.7, $\mathcal{M}_{\mathbf{N}_{\preceq}^*}, \Phi \Vdash A$, moreover by Lemma 3.8, $\mathcal{M}_{\mathbf{N}_{\preceq}^*}$ is a \mathbf{N}_{\preceq}^* -model. \square

Let us now turn to the relationship between \mathbf{N}_{\preceq}^* and Lewis' logics of comparative plausibility over sphere models. Lewis [16] provides two equivalent axiomatisations of the minimal logic \mathbf{V}_{\preceq} , one of the two being $\mathbf{CPL} \cup \{\mathbf{cpr}, \mathbf{cpa}, \mathbf{tr}, \mathbf{co}\}$, where \mathbf{co} is the *connection axiom* $(A \preceq B) \vee (B \preceq A)$, and \mathbf{cpa} is $(A \preceq A \vee B) \vee (B \preceq A \vee B)$. We show that a further equivalent axiomatisation of \mathbf{V}_{\preceq} can be given by extending our minimal logic \mathbf{N}_{\preceq} with the axiom \mathbf{co} :

Proposition 3.10 *For all $A \in \mathcal{L}$, $\vdash_{\mathbf{V}_{\preceq}} A$ if and only if $\vdash_{\mathbf{N}_{\preceq} \cup \{\mathbf{co}\}} A$.*

Proof. Since $\mathbf{N}_{\preceq} \cup \{\mathbf{co}\}$ and $\mathbf{V}_{\preceq} = \mathbf{CPL} \cup \{\mathbf{cpr}, \mathbf{cpa}, \mathbf{tr}, \mathbf{co}\}$ differ only with respect to \mathbf{cpa} and \mathbf{or} , it suffices to show that (i) $\vdash_{\mathbf{N}_{\preceq} \cup \{\mathbf{co}\}} \mathbf{cpa}$ and (ii) $\vdash_{\mathbf{V}_{\preceq}} \mathbf{or}$. (i) From $(A \preceq B) \vee (B \preceq A)$, by \mathbf{cpr} we have $((A \preceq A) \wedge (A \preceq B)) \vee ((B \preceq A) \wedge (B \preceq B))$, then by \mathbf{or} , $(A \preceq A \vee B) \vee (B \preceq A \vee B)$. (ii) From $(B \preceq B \vee C) \vee (C \preceq B \vee C)$, by \mathbf{tr} we have $(A \preceq B) \wedge (A \preceq C) \rightarrow (A \preceq B \vee C) \vee (A \preceq B \vee C)$, thus $(A \preceq B) \wedge (A \preceq C) \rightarrow (A \preceq B \vee C)$. \square

Note also that the extensions of \mathbf{N}_{\preceq} are defined by the same axioms characterising the extensions of \mathbf{V}_{\preceq} . It follows that each Lewis' logic can be obtained from the corresponding CPN logic by adding the connection axiom \mathbf{co} . The relations among these systems are displayed in Fig. 3.

4 Multi-premisses sequent calculi for CPN logics

In this section we present Gentzen-style sequent calculi for the CPN logics \mathbf{N}_{\preceq} , \mathbf{NN}_{\preceq} , \mathbf{NT}_{\preceq} , \mathbf{NW}_{\preceq} , \mathbf{NC}_{\preceq} , \mathbf{NA}_{\preceq} , and \mathbf{NNA}_{\preceq} . From now on, let \mathbf{N}_{\preceq}^* denote any of these systems. For each logic we introduce a calculus $\mathbf{G.N}_{\preceq}^*$ defined on the basis of the sequent systems for Lewis' logics by Lellmann and Pattinson [14,15]. In these calculi, the rules have up to $m + n$ premisses, where m (resp. n) is the number of \preceq -formulas occurring in the antecedent (resp. consequent) of the conclusion. Calculi for \mathbf{N}_{\preceq}^* can be provided by restricting the calculi in [14,15] to rules with at most one \preceq -formula in the consequent ($n = 1$), thus obtaining simpler calculi, where each rule introduces at most $m + 1$ premisses.

As usual, we call *sequent* any pair $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite, possibly empty multisets of formulas of \mathcal{L} . $\Gamma \Rightarrow \Delta$ is interpreted as the formula $\bigwedge \Gamma \rightarrow \bigvee \Delta$.

The rules of the calculi $\mathbf{G.N}_{\preceq}^*$ can be found in Fig. 4. Each modal rule simultaneously analyses a number (at least one) of \preceq -formulas appearing in a sequent. The *principal formulas* of each rule \mathbf{R}_n are the n or $n + 1$ \preceq -formulas in the conclusion which get analysed in the premiss. In some rules the principal \preceq -formulas are copied into the premisses in order to ensure admissibility of contraction. We denote derivability in $\mathbf{G.N}_{\preceq}^*$ as $\mathbf{G.N}_{\preceq}^* \vdash \Gamma \Rightarrow \Delta$.

Theorem 4.1 (Soundness) *If $\mathbf{G.N}_{\preceq}^* \vdash \Gamma \Rightarrow \Delta$ then $\mathbf{N}_{\preceq}^* \vdash \bigwedge \Gamma \rightarrow \bigvee \Delta$.*

Proof. We show that for every rule R of $\mathbf{G.N}_{\preceq}^*$ with premisses $\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n$ and conclusion $\Gamma \Rightarrow \Delta$, the corresponding Hilbert-style rule with premisses $\bigwedge \Gamma_1 \rightarrow \bigvee \Delta_1, \dots, \bigwedge \Gamma_n \rightarrow \bigvee \Delta_n$ and conclusion $\bigwedge \Gamma \rightarrow \bigvee \Delta$ is derivable in \mathbf{N}_{\preceq}^* . The propositional cases are standard. We use \mathbf{D}_1^n as a shorthand for D_1, \dots, D_n .

$$\begin{array}{c}
\text{init} \frac{}{\Gamma, p \Rightarrow p, \Delta} \quad \perp_L \frac{}{\Gamma, \perp \Rightarrow \Delta} \quad \rightarrow_L \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \rightarrow B \Rightarrow \Delta} \quad \rightarrow_R \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta} \\
\text{CP}_n \frac{\{C_k \Rightarrow A, D_1, \dots, D_{k-1}\}_{1 \leq k \leq n} \quad B \Rightarrow A, D_1, \dots, D_n}{\Gamma, C_1 \preccurlyeq D_1, \dots, C_n \preccurlyeq D_n \Rightarrow A \preccurlyeq B, \Delta} \\
\text{N}_n \frac{\{C_k \Rightarrow D_1, \dots, D_{k-1}\}_{1 \leq k \leq n} \quad \Rightarrow D_1, \dots, D_n}{\Gamma, C_1 \preccurlyeq D_1, \dots, C_n \preccurlyeq D_n \Rightarrow \Delta} \\
\text{T}_n \frac{\{C_k \Rightarrow D_1, \dots, D_{k-1}\}_{1 \leq k \leq n} \quad \Gamma, C_1 \preccurlyeq D_1, \dots, C_n \preccurlyeq D_n \Rightarrow D_1, \dots, D_n, \Delta}{\Gamma, C_1 \preccurlyeq D_1, \dots, C_n \preccurlyeq D_n \Rightarrow \Delta} \\
\text{W}_n \frac{\{C_k \Rightarrow A, D_1, \dots, D_{k-1}\}_{1 \leq k \leq n} \quad \Gamma, C_1 \preccurlyeq D_1, \dots, C_n \preccurlyeq D_n \Rightarrow A, A \preccurlyeq B, D_1, \dots, D_n, \Delta}{\Gamma, C_1 \preccurlyeq D_1, \dots, C_n \preccurlyeq D_n \Rightarrow A \preccurlyeq B, \Delta} \\
\text{W}_0 \frac{\Gamma \Rightarrow A \preccurlyeq B, A, \Delta}{\Gamma \Rightarrow A \preccurlyeq B, \Delta} \quad \text{C}_0 \frac{\Gamma, A \preccurlyeq B, A \Rightarrow \Delta \quad \Gamma, A \preccurlyeq B \Rightarrow B, \Delta}{\Gamma, A \preccurlyeq B \Rightarrow \Delta} \\
\text{A}_n \frac{\{\Gamma^\preccurlyeq, \Sigma^\preccurlyeq, C_k \Rightarrow A \preccurlyeq B, A, D_1, \dots, D_{k-1}, \Delta^\preccurlyeq\}_{1 \leq k \leq n} \quad \Gamma^\preccurlyeq, \Sigma^\preccurlyeq, B \Rightarrow A \preccurlyeq B, A, D_1, \dots, D_n, \Delta^\preccurlyeq}{\Gamma, C_1 \preccurlyeq D_1, \dots, C_n \preccurlyeq D_n \Rightarrow A \preccurlyeq B, \Delta} \\
\text{N}_n^{\Delta} \frac{\{\Gamma^\preccurlyeq, \Sigma^\preccurlyeq, C_k \Rightarrow D_1, \dots, D_{k-1}, \Delta^\preccurlyeq\}_{1 \leq k \leq n} \quad \Gamma^\preccurlyeq, \Sigma^\preccurlyeq \Rightarrow D_1, \dots, D_n, \Delta^\preccurlyeq}{\Gamma, C_1 \preccurlyeq D_1, \dots, C_n \preccurlyeq D_n \Rightarrow \Delta}
\end{array}$$

where $\Sigma^\preccurlyeq = C_1 \preccurlyeq D_1, \dots, C_n \preccurlyeq D_n$, $\Gamma^\preccurlyeq = \{C' \preccurlyeq D' \mid C' \preccurlyeq D' \in \Gamma\}$, $\Delta^\preccurlyeq = \{C' \preccurlyeq D' \mid C' \preccurlyeq D' \in \Delta\}$

$$\begin{array}{l}
\text{G.N}_{\preccurlyeq} = \{\text{init}, \perp_L, \rightarrow_L, \rightarrow_R\} \cup \{\text{CP}_n \mid n \geq 0\} \\
\text{G.NN}_{\preccurlyeq} = \text{G.N}_{\preccurlyeq} \cup \{\text{N}_n \mid n \geq 1\} \\
\text{G.NW}_{\preccurlyeq} = \text{G.N}_{\preccurlyeq} \cup \{\text{W}_n \mid n \geq 0\} \cup \{\text{T}_n \mid n \geq 1\} \\
\text{G.NA}_{\preccurlyeq} = \{\text{init}, \perp_L, \rightarrow_L, \rightarrow_R\} \cup \{\text{A}_n \mid n \geq 0\} \\
\text{G.NT}_{\preccurlyeq} = \text{G.N}_{\preccurlyeq} \cup \{\text{T}_n \mid n \geq 1\} \\
\text{G.NC}_{\preccurlyeq} = \text{G.N}_{\preccurlyeq} \cup \{\text{W}_0, \text{C}_0\} \\
\text{G.NNA}_{\preccurlyeq} = \text{G.NA}_{\preccurlyeq} \cup \{\text{N}_n^{\Delta} \mid n \geq 1\}
\end{array}$$

Fig. 4. Gentzen-style calculi for CPN logics.

(CP_n) Suppose $\vdash C_1 \rightarrow A$, $\vdash C_2 \rightarrow A \vee D_1$, ..., $\vdash C_n \rightarrow A \vee \bigvee \mathbf{D}_1^{n-1}$ and $\vdash B \rightarrow A \vee \bigvee \mathbf{D}_1^n$. Then by cpr, we have $\vdash A \preccurlyeq C_1$, $\vdash A \vee D_1 \preccurlyeq C_2$, ..., $\vdash A \vee \bigvee \mathbf{D}_1^{n-1} \preccurlyeq C_n$ and $\vdash A \vee \bigvee \mathbf{D}_1^n \preccurlyeq B$. If $n = 0$, the conclusion immediately follows. From $\vdash C_1 \preccurlyeq D_1$ it follows by tr that $A \preccurlyeq D_1$. Thus, $\vdash (C_1 \preccurlyeq D_1) \rightarrow (A \preccurlyeq D_1)$. Since $\vdash A \preccurlyeq A$, by or, we have that $\vdash (C_1 \preccurlyeq D_1) \rightarrow (A \preccurlyeq A \vee D_1)$. By tr, $\vdash (C_1 \preccurlyeq D_1) \rightarrow (A \preccurlyeq C_2)$. From $\vdash C_2 \preccurlyeq D_2$ it follows by tr that $A \preccurlyeq C_2$. Thus, $\vdash (C_1 \preccurlyeq D_1) \wedge (C_2 \preccurlyeq D_2) \rightarrow (A \preccurlyeq D_2)$. By or applied to $\vdash A \preccurlyeq A$ and to $A \preccurlyeq D_1$, we have $\vdash (C_1 \preccurlyeq D_1) \wedge (C_2 \preccurlyeq D_2) \rightarrow (A \preccurlyeq A \vee D_1 \vee D_2)$, to which we apply tr twice and conclude $\vdash (C_1 \preccurlyeq D_1) \wedge (C_2 \preccurlyeq D_2) \wedge (C_3 \preccurlyeq D_3) \rightarrow (A \preccurlyeq D_3)$. We iterate the steps above until we obtain $\vdash \bigwedge_{i \leq n} (C_i \preccurlyeq D_i) \rightarrow (A \preccurlyeq D_n)$. Then, by applications of or to $A \preccurlyeq A$ and to $A \preccurlyeq D_1, \dots, A \preccurlyeq D_{n-1}$, we obtain $\vdash \bigwedge_{i \leq n} (C_i \preccurlyeq D_i) \rightarrow (A \preccurlyeq A \vee \bigvee \mathbf{D}_1^n)$. A final application of tr yields $\vdash \bigwedge_{i \leq n} (C_i \preccurlyeq D_i) \rightarrow (A \preccurlyeq B)$. Therefore $\vdash \bigwedge \Gamma \wedge \bigwedge_{i \leq n} (C_i \preccurlyeq D_i) \rightarrow (A \preccurlyeq B) \vee \bigvee \Delta$ for every Γ, Δ .

(N_n) Suppose $\vdash C_1 \rightarrow \perp$, $\vdash C_2 \rightarrow D_1$, ..., $\vdash C_n \rightarrow \bigvee \mathbf{D}_1^{n-1}$ and $\vdash \top \rightarrow \bigvee \mathbf{D}_1^n$. Then by cpr, $\vdash \perp \preccurlyeq C_1$, $\vdash D_1 \preccurlyeq C_2$, ..., $\vdash \bigvee \mathbf{D}_1^{n-1} \preccurlyeq C_n$ and $\vdash \bigvee \mathbf{D}_1^n \preccurlyeq \top$. Reasoning as in the case of CP_n, we conclude that $\vdash \bigwedge_{i \leq n} (C_i \preccurlyeq D_i) \rightarrow (\perp \preccurlyeq \top)$. By n, $\vdash \bigwedge_{i \leq n} (C_i \preccurlyeq D_i) \rightarrow \perp$, then $\vdash \bigwedge \Gamma \wedge \bigwedge_{i \leq n} (C_i \preccurlyeq D_i) \rightarrow \bigvee \Delta$ for all Γ, Δ .

(T_n) Suppose $\vdash C_1 \rightarrow \perp$, $\vdash C_2 \rightarrow D_1$, ..., $\vdash C_n \rightarrow \bigvee \mathbf{D}_1^{n-1}$ and $\vdash \bigwedge \Gamma \wedge$

$\bigwedge_{i \leq n} (C_i \preceq D_i) \rightarrow \bigvee \mathbf{D}_1^n \vee \bigvee \Delta$. Then by **cpr**, $\vdash \perp \preceq C_1, \vdash D_1 \preceq C_2, \dots, \vdash \bigvee \mathbf{D}_1^{n-1} \preceq C_n$. By applications of **tr** and **or**, we have that $\vdash \bigwedge_{i \leq n} (C_i \preceq D_i) \rightarrow (\perp \preceq D_1) \wedge \dots \wedge (\perp \preceq D_n)$. By **t**, $\vdash \bigwedge_{i \leq n} (C_i \preceq D_i) \rightarrow \neg D_1 \wedge \dots \wedge \neg D_n$. Then we have $\vdash \bigwedge \Gamma \wedge \bigwedge_{i \leq n} (C_i \preceq D_i) \rightarrow (\bigvee \mathbf{D}_1^n \vee \Delta) \wedge \neg D_1 \wedge \dots \wedge \neg D_n$, from which we conclude that $\vdash \bigwedge \Gamma \wedge \bigwedge_{i \leq n} (C_i \preceq D_i) \rightarrow \bigvee \Delta$.

(\mathbf{W}_n) Suppose $\vdash C_1 \rightarrow A, \vdash C_2 \rightarrow A \vee D_1, \dots, \vdash C_n \rightarrow A \vee \bigvee \mathbf{D}_1^{n-1}$ and $\vdash \bigwedge \Gamma \wedge \bigwedge_{i \leq n} (C_i \preceq D_i) \rightarrow A \vee \bigvee \mathbf{D}_1^n \vee A \preceq B \vee \bigvee \Delta$. Then by **cpr**, $\vdash A \preceq C_1, \vdash A \vee D_1 \preceq C_2, \dots, \vdash A, \bigvee \mathbf{D}_1^{n-1}$. Reasoning as in the case of \mathbf{CP}_n , we obtain proofs of the following: $\vdash (C_1 \preceq D_1) \rightarrow (A \preceq D_1), \dots, \vdash \bigwedge_{i \leq n} (C_i \preceq D_i) \rightarrow (A \preceq D_n)$. Moreover by **w**, $\vdash \bigwedge \Gamma \wedge \bigwedge_{i \leq n} (C_i \preceq D_i) \rightarrow (A \preceq \top) \vee \bigvee_{i \leq n} (D_i \preceq \top) \vee A \preceq B \vee \bigvee \Delta$. Thus, applying **tr** to $A \preceq D_1, \dots, A \preceq D_n$, we obtain $\vdash \bigwedge \Gamma \wedge \bigwedge_{i \leq n} (C_i \preceq D_i) \rightarrow (A \preceq \top) \vee A \preceq B \vee \bigvee \Delta$. Since by **cpr**, $\vdash \top \preceq B$ for every B , by **tr** we obtain $\vdash \bigwedge \Gamma \wedge \bigwedge_{i \leq n} (C_i \preceq D_i) \rightarrow (A \preceq B) \vee A \preceq B \vee \bigvee \Delta$.

(\mathbf{W}_0) If $\vdash \bigwedge \Gamma \rightarrow A \vee \bigvee \Delta$, then by **w**, $\vdash \bigwedge \Gamma \rightarrow (A \preceq \top) \vee \bigvee \Delta$, thus since by **cpr**, $\vdash \top \preceq B$ for every B , we have $\vdash \bigwedge \Gamma \rightarrow (A \preceq B) \vee \bigvee \Delta$.

(\mathbf{C}_0) Suppose $\vdash \bigwedge \Gamma \wedge A \rightarrow \bigvee \Delta$ and $\vdash \bigwedge \Gamma \rightarrow B \vee \bigvee \Delta$. Then by **w**, $\vdash \bigwedge \Gamma \rightarrow (B \preceq \top) \vee \bigvee \Delta$, thus by **tr**, $\vdash \bigwedge \Gamma \wedge (A \preceq B) \rightarrow (A \preceq \top) \vee \bigvee \Delta$. Then by **c**, $\vdash \bigwedge \Gamma \wedge (A \preceq B) \rightarrow A \vee \bigvee \Delta$, therefore $\vdash \bigwedge \Gamma \wedge (A \preceq B) \rightarrow \bigvee \Delta$.

(\mathbf{A}_n) Suppose $\vdash \bigwedge \Gamma^\preceq \wedge \bigwedge_{i \leq n} (C_i \preceq D_i) \wedge C_1 \rightarrow A \vee (A \preceq B) \vee \bigvee \Delta^\preceq$, $\vdash \bigwedge \Gamma^\preceq \wedge \bigwedge_{i \leq n} (C_i \preceq D_i) \wedge C_2 \rightarrow A \vee (A \preceq B) \vee D_1 \vee \bigvee \Delta^\preceq, \dots, \vdash \bigwedge \Gamma^\preceq \wedge \bigwedge_{i \leq n} (C_i \preceq D_i) \wedge C_n \rightarrow A \vee \bigvee \mathbf{D}_1^{n-1} \vee \bigvee \Delta^\preceq$ and $\vdash \bigwedge \Gamma^\preceq \wedge \bigwedge_{i \leq n} (C_i \preceq D_i) \wedge B \rightarrow A \vee \bigvee \mathbf{D}_1^n \vee \bigvee \Delta^\preceq$. Using the same strategy as in \mathbf{CP}_n , we prove that $\vdash \bigwedge_{i \leq n} (C_i \preceq D_i) \rightarrow (A \preceq B)$ follows from the simpler set of assumptions where we remove $\bigwedge \Gamma^\preceq, \bigwedge_{i \leq n} (C_i \preceq D_i), \bigvee \Delta^\preceq$ and $A \preceq B$. From this, we conclude $\vdash \bigwedge \Gamma \wedge \bigwedge_{i \leq n} (C_i \preceq D_i) \wedge \bigwedge_{i \leq n} (C_i \preceq D_i) \rightarrow (A \preceq B) \vee \bigvee \Delta$ for any Γ, Δ . (\mathbf{N}_n^A) is similar. \square

We now show that $\mathbf{G.N}_\preceq^*$ enjoy cut admissibility, where a rule is said to be (*height-preserving*) *admissible* if, whenever the premisses are derivable, also the conclusion is derivable (with a derivation of at most the same height). We start by considering the following auxiliary result.

Proposition 4.2 *The rules below are height-preserving admissible in $\mathbf{G.N}_\preceq^*$:*

$$\text{wk}_L \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \quad \text{wk}_R \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta} \quad \text{ctr}_L \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \quad \text{ctr}_R \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}$$

Proof. By induction on the height h of the derivation of the premiss of the rules. The cases of wk_L, wk_R and ctr_R are immediate. We show admissibility of ctr_L , for $h > 0$ and the last rule applied in the derivation being \mathbf{CP}_{n+1} . Let $C_i = C_{i+1}$ and $D_i = D_{i+1}$.

$$\text{CP}_n \frac{\{C_k \Rightarrow A, D_1, \dots, D_{k-1}\}_{k \leq n} \quad B \Rightarrow A, D_1, \dots, D_i, D_{i+1}, \dots, D_n}{G, C_1 \preceq D_1, \dots, C_i \preceq D_i, C_{i+1} \preceq D_{i+1}, \dots, C_n \preceq D_n \Rightarrow A \preceq B, \Delta} \\ \text{ctr}_L \frac{}{G, C_1 \preceq D_1, \dots, C_i \preceq D_i, C_{i+2} \preceq D_{i+2}, \dots, C_n \preceq D_n \Rightarrow A \preceq B, \Delta}$$

For each $j \geq i + 2$ apply contraction to the following sequent of smaller height:

$$\text{ctr}_L \frac{C_j \Rightarrow A, D_1, \dots, D_i, D_{i+1}, \dots, D_{j-1}}{C_j \Rightarrow A, D_1, \dots, D_i, D_{i+2}, \dots, D_{j-1}}$$

A final application of ctr_R on smaller height and CP_{n-1} yields the desired result:

$$\text{CP}_{n-1} \frac{\frac{\{C_k \Rightarrow A, D_1, \dots, D_{k-1}\}_{k \leq i} \quad \text{ctr}_L \frac{B \Rightarrow A, D_1, \dots, D_i, D_{i+1}, \dots, D_n}{B \Rightarrow A, D_1, \dots, D_i, \dots, D_n}}{\{C_j \Rightarrow A, D_1, \dots, D_i, D_{i+2}, \dots, D_{j-1}\}_{i+2 \leq j \leq n}}}{G, C_1 \leq D_1, \dots, C_i \leq D_i, C_{i+2} \leq D_{i+2}, \dots, C_n \leq D_n \Rightarrow A \leq B, \Delta} \quad \square$$

Theorem 4.3 *The cut rule is admissible in $\mathbf{G.N}_{\leq}^*$, where A is the cut formula:*

$$\text{cut} \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma', A \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Proof. By induction on lexicographically ordered pairs (c, h) , where c is the complexity of the cut formula (i.e., the number of binary connectives or modalities occurring in it), and h is the sum of the heights of the derivations of the premisses of cut . We distinguish cases according to whether the cut formula is principal in the last rules applied in the derivation of the premisses of cut .

If the cut formula is not principal in the last rule application of the derivation of one of the two premisses of cut , then the conclusion of cut is standardly obtained by i.h. on h . Suppose the cut formula is principal in the last rule application of the derivations of both premisses of cut .

- Both premisses of cut are derived by CP_n :

$$\text{CP}_n \frac{\frac{\{C_k \Rightarrow A_i, D_1, \dots, D_{k-1}\}_{k \leq n} \quad B_i \Rightarrow A_i, D_1, \dots, D_n}{\Gamma, C_1 \leq D_1, \dots, C_n \leq D_n \Rightarrow A_i \leq B_i, \Delta} \quad \text{CP}_n \frac{\{A_\ell \Rightarrow E, B_1, \dots, B_{\ell-1}\}_{\ell \leq m} \quad F \Rightarrow E, B_1, \dots, B_i, \dots, B_m}{\Gamma', A_1 \leq B_1, \dots, A_i \leq B_i, \dots, A_m \leq B_m \Rightarrow E \leq F, \Delta'}}{\frac{\Gamma, \Gamma', A_1 \leq B_1, \dots, A_{i-1} \leq B_{i-1}, C_1 \leq D_1, \dots, C_n \leq D_n, A_{i+1} \leq B_{i+1}, \dots, A_m \leq B_m \Rightarrow E \leq F, \Delta, \Delta'}}{\Gamma, \Gamma', A_1 \leq B_1, \dots, A_{i-1} \leq B_{i-1}, C_1 \leq D_1, \dots, C_n \leq D_n, A_{i+1} \leq B_{i+1}, \dots, A_m \leq B_m \Rightarrow E \leq F, \Delta, \Delta'}}$$

The derivation is converted as follows: First, for every $k \leq n$ we obtain the following derivation, by induction on c :

$$\text{cut} \frac{C_k \Rightarrow A_i, D_1, \dots, D_{k-1} \quad A_i \Rightarrow E, B_1, \dots, B_{i-1}}{C_k \Rightarrow E, B_1, \dots, B_{i-1}, D_1, \dots, D_{k-1}}$$

Always by induction on c , for every $1 \leq \ell \leq n - i$ we obtain the following, where the double line denotes several applications of ctr_R :

$$\text{cut} \frac{A_{i+\ell} \Rightarrow E, B_1, \dots, B_i, \dots, B_{i+\ell-1} \quad \text{cut} \frac{B_i \Rightarrow A_i, D_1, \dots, D_n \quad A_i \Rightarrow E, B_1, \dots, B_{i-1}}{B_i \Rightarrow E, D_1, \dots, D_n, B_1, \dots, B_{i-1}}}{\frac{A_{i+\ell} \Rightarrow E, E, B_1, \dots, B_{i-1}, B_1, \dots, B_{i-1}, D_1, \dots, D_n, B_{i+1}, \dots, B_{i+\ell-1}}{A_{i+\ell} \Rightarrow E, B_1, \dots, B_{i-1}, D_1, \dots, D_n, B_{i+1}, \dots, B_{i+\ell-1}}}}$$

By induction on c we construct the following derivation \mathcal{S} :

$$\text{ctr}_R \frac{F \Rightarrow E, B_1, \dots, B_i, \dots, B_m \quad \text{cut} \frac{B_i \Rightarrow A_i, D_1, \dots, D_n \quad A_i \Rightarrow E, B_1, \dots, B_{i-1}}{B_i \Rightarrow E, B_1, \dots, B_{i-1}, D_1, \dots, D_n}}{F \Rightarrow E, E, B_1, \dots, B_{i-1}, B_1, \dots, B_{i-1}, D_1, \dots, D_n, B_{i+1}, \dots, B_m}}{\text{ctr}_R \frac{F \Rightarrow E, B_1, \dots, B_{i-1}, D_1, \dots, D_n, B_{i+1}, \dots, B_m}}{F \Rightarrow E, B_1, \dots, B_{i-1}, D_1, \dots, D_n, B_{i+1}, \dots, B_m}}$$

A final application of CP_{n-1} yields a derivation of the conclusion of cut :

$$\text{CP}_{n-1} \frac{\frac{\{A_\ell \Rightarrow E, B_1, \dots, B_{\ell-1}\}_{\ell < i}}{\{C_k \Rightarrow E, B_1, \dots, B_{i-1}, D_1, \dots, D_{k-1}\}_{k \leq n}} \quad \mathcal{S}}{\Gamma, \Gamma', A_1 \preceq B_1, \dots, A_{i-1} \preceq B_{i-1}, C_1 \preceq D_1, \dots, C_n \preceq D_n, A_{i+1} \preceq B_{i+1}, \dots, A_m \preceq B_m \Rightarrow E \preceq F, \Delta, \Delta'}$$

- The cut formula is principal in W_n and CP_m :

$$W_m \frac{\frac{\Gamma, C_1 \preceq D_1, \dots, C_n \preceq D_n \Rightarrow \Delta, A_i \preceq B_i, A_i, D_1, \dots, D_n}{\text{cut}} \quad \frac{\{C_k \Rightarrow A_i, D_1, \dots, D_{k-1}\}_{1 < k \leq n}}{\text{CP}_m} \quad \frac{\{A_\ell \Rightarrow E, B_1, \dots, B_{\ell-1}\}_{\ell \leq m} \quad F \Rightarrow E, B_1, \dots, B_i, \dots, B_m}{\Gamma', A_1 \preceq B_1, \dots, A_i \preceq B_i, \dots, A_m \preceq B_m \Rightarrow \Delta', E \preceq F}}{\Gamma, \Gamma', A_1 \preceq B_1, \dots, A_{i-1} \preceq B_{i-1}, C_1 \preceq D_1, \dots, C_n \preceq D_n, A_{i+1} \preceq B_{i+1}, \dots, A_m \preceq B_m \Rightarrow \Delta, \Delta', E \preceq F}$$

We first perform a cut on smaller h between the premiss of W_m and the rightmost premiss of cut, obtaining sequent $\Sigma = \Gamma, \Gamma', A_1 \preceq B_1, \dots, A_{i-1} \preceq B_{i-1}, A_i, C_1 \preceq D_1, \dots, C_n \preceq D_n, A_{i+1} \preceq B_{i+1}, \dots, A_m \preceq B_m \Rightarrow \Delta, \Delta', E \preceq F, D_1, \dots, D_n$.

Next, we perform a cut by induction on c on Σ and on $A_i \Rightarrow E, B_1, \dots, B_{i-1}$. This yields a derivation of the sequent $\Sigma' = \Gamma, \Gamma', A_1 \preceq B_1, \dots, A_{i-1} \preceq B_{i-1}, C_1 \preceq D_1, \dots, C_n \preceq D_n, A_{i+1} \preceq B_{i+1}, \dots, A_m \preceq B_m \Rightarrow \Delta, \Delta', E \preceq F, D_1, \dots, D_n, E, B_1, \dots, B_{i-1}$. Then, for $1 < k \leq n$, we generate the following derivation, by induction on c :

$$\text{cut} \frac{C_k \Rightarrow A_i, D_1, \dots, D_{k-1} \quad A_i \Rightarrow E, B_1, \dots, B_{i-1}}{C_k \Rightarrow E, B_1, \dots, B_{i-1}, D_1, \dots, D_{k-1}}$$

A final application of W_j , where $j = n + (i - 1)$, yields the desired conclusion:

$$W_j \frac{\frac{\{A_\ell \Rightarrow E, B_1, \dots, B_{\ell-1}\}_{\ell \leq i}}{\{C_k \Rightarrow E, B_1, \dots, B_{i-1}, D_1, \dots, D_{k-1}\}_{k \leq n}} \quad \Sigma'}{\Gamma, \Gamma', A_1 \preceq B_1, \dots, A_{i-1} \preceq B_{i-1}, C_1 \preceq D_1, \dots, C_n \preceq D_n, A_{i+1} \preceq B_{i+1}, \dots, A_m \preceq B_m \Rightarrow \Delta, \Delta', E \preceq F}$$

- The cut formula is principal in W_0 and CP_m :

$$W_0 \frac{\frac{\Gamma \Rightarrow \Delta, A_i \preceq B_i, A_i}{\text{cut}} \quad \frac{\{A_\ell \Rightarrow E, B_1, \dots, B_{\ell-1}\}_{\ell \leq m} \quad F \Rightarrow E, B_1, \dots, B_i, \dots, B_m}{\text{CP}_m} \quad \frac{\Gamma', A_1 \preceq B_1, \dots, A_i \preceq B_i, \dots, A_m \preceq B_m \Rightarrow \Delta', E \preceq F}}{\Gamma, \Gamma', A_1 \preceq B_1, \dots, A_{i-1} \preceq B_{i-1}, A_{i+1} \preceq B_{i+1}, \dots, A_m \preceq B_m \Rightarrow \Delta, \Delta', E \preceq F}$$

We first perform a cut by induction on h on the premiss of W_0 and on the rightmost premiss of cut, obtaining sequent $\Sigma = \Gamma, \Gamma', A_1 \preceq B_1, \dots, A_{i-1} \preceq B_{i-1}, A_{i+1} \preceq B_{i+1}, \dots, A_m \preceq B_m \Rightarrow \Delta, \Delta', E \preceq F, A_i$. Then, applying cut to Σ and $A_i \Rightarrow E, B_1, \dots, B_{i-1}$ we obtain $\Sigma' = \Gamma, \Gamma', A_1 \preceq B_1, \dots, A_{i-1} \preceq B_{i-1}, A_{i+1} \preceq B_{i+1}, \dots, A_m \preceq B_m \Rightarrow \Delta, \Delta', E \preceq F, E, B_1, \dots, B_{i-1}$. Let Γ^* denote the premiss of Σ' and $\Delta^* = \Delta, \Delta'$. We now construct the following derivation, containing $i - 1$ applications of C_0 :

$$C_0 \frac{\frac{\frac{A_i \Rightarrow E}{A_1, \Gamma^* \Rightarrow \Delta^*, E \preceq F, E}}{\text{wk}} \quad \frac{\frac{\frac{A_{i-2} \Rightarrow E, B_1, \dots, B_{i-2}}{\text{wk}} \quad \frac{\frac{A_{i-1} \Rightarrow E, B_1, \dots, B_{i-2}}{\text{wk}} \quad \frac{A_{i-1}, \Gamma^* \Rightarrow \Delta^*, E \preceq F, E, B_1, \dots, B_{i-2}}{\text{C}_0} \quad \Sigma'}{\Gamma^* \Rightarrow \Delta^*, E, B_1, \dots, B_{i-2}}}{A_{i-2}, \Gamma^* \Rightarrow \Delta^*, E \preceq F, E, B_1, \dots, B_{i-3}}}{\text{C}_0} \quad \frac{\Gamma^* \Rightarrow \Delta^*, E \preceq F, E, B_1, \dots, B_{i-3}}{\text{wk}}}{\Gamma^* \Rightarrow \Delta^*, E \preceq F, E, B_1, \dots, B_{i-2}}}{\text{C}_0} \quad \frac{\Gamma^* \Rightarrow \Delta^*, E \preceq F, E, B_1}{\text{wk}}}{\Gamma^* \Rightarrow \Delta^*, E \preceq F, E, B_1}}{\Gamma^* \Rightarrow \Delta^*, E \preceq F, E} \quad \frac{\Gamma^* \Rightarrow \Delta^*, E \preceq F, E}{W_0} \quad \frac{\Gamma^* \Rightarrow \Delta^*, E \preceq F, E}{\Gamma^* \Rightarrow \Delta^*, E \preceq F}$$

$$\begin{array}{c}
\text{CP}_2 \frac{A \Rightarrow A \quad A \Rightarrow A, B \quad \vee_L \frac{B \Rightarrow A, B, C \quad C \Rightarrow A, B, C}{B \vee C \Rightarrow A, B, C}}{A \preceq B, A \preceq C \Rightarrow A \preceq B \vee C} \\
\wedge_L \frac{(A \preceq B) \wedge (A \preceq C) \Rightarrow A \preceq B \vee C}{\Rightarrow (A \preceq B) \wedge (A \preceq C) \rightarrow (A \preceq B \vee C)} \\
\rightarrow_R \frac{}{\Rightarrow (A \preceq B) \wedge (A \preceq C) \rightarrow (A \preceq B \vee C)} \\
\\
\text{CP}_2 \frac{A \Rightarrow A \quad B, A \Rightarrow B \quad C \Rightarrow A, B, C}{A \preceq B, B \preceq C \Rightarrow A \preceq C} \\
\wedge_L \frac{(A \preceq B) \wedge (B \preceq C) \Rightarrow A \preceq C}{\Rightarrow (A \preceq B) \wedge (B \preceq C) \rightarrow (A \preceq C)} \\
\rightarrow_R \frac{}{\Rightarrow (A \preceq B) \wedge (B \preceq C) \rightarrow (A \preceq C)} \\
\\
\text{cut} \frac{\Rightarrow A \rightarrow B \quad \rightarrow_L \frac{A \Rightarrow A, B \quad A, B \Rightarrow B}{A, A \rightarrow B \Rightarrow B}}{A \Rightarrow B} \\
\text{CP}_0 \frac{A \Rightarrow B}{\Rightarrow B \preceq A}
\end{array}$$

Fig. 5. Derivations of the axioms and rules of \mathbf{N}_{\preceq} in $\mathbf{G.N}_{\preceq}$.

The remaining cases are: $\text{CP}_n + \text{W}_m$, which is proved similarly as $\text{W}_m + \text{CP}_n$; $\text{C}_0 + \text{CP}_n$, similar to $\text{CP}_n + \text{W}_0$; $\text{C}_0 + \text{W}_0$, which is immediate, $\text{CP}_n + \text{N}_n$, which is proven in the same way as $\text{CP}_n + \text{CP}_m$, $\text{CP}_n + \text{T}_m$, which is proven as $\text{CP}_n + \text{W}_m$, and the cases for absoluteness, which are proven as their counterpart without absoluteness. \square

Thanks to cut-admissibility, we obtain cut-free completeness of the calculi, by deriving the axioms and inference rules of \mathbf{N}_{\preceq}^* in $\mathbf{G.N}_{\preceq}^*$.

Corollary 4.4 (Completeness) *If $\mathbf{N}_{\preceq}^* \vdash \bigwedge \Gamma \rightarrow \bigvee \Delta$ then $\mathbf{G.N}_{\preceq}^* \vdash \Gamma \Rightarrow \Delta$.*

Proof. Derivations of the \mathbf{N}_{\preceq} axioms in $\mathbf{G.N}_{\preceq}$ are displayed in Fig. 5. The derivations employ standard propositional rules for \wedge and \vee , which can be defined in $\mathbf{G.N}_{\preceq}^*$. The derivations of the axioms for extensions are straightforward. *Modus ponens* is simulated using cut in the usual way. \square

Termination of root-first proof search in $\mathbf{G.N}_{\preceq}^*$ can be easily proved by observing that non redundant rule applications strictly decrease the complexity of formulas. However, $\mathbf{G.N}_{\preceq}^*$ are not suited for root-first proof search: the comparative plausibility rules are *not invertible*, meaning that derivability of the conclusion does not imply derivability of the premiss(es) of the rule. As a consequence, backtrack points are generated when constructing root-first a derivation. Next section introduces proof systems having only invertible rules.

5 Hypersequent calculi for CPN logics

In this section we present hypersequent calculi $\mathbf{H.N}_{\preceq}^*$ for the same family of CPN logics treated in Sec. 4, namely \mathbf{N}_{\preceq} , \mathbf{NN}_{\preceq} , \mathbf{NT}_{\preceq} , \mathbf{NW}_{\preceq} , \mathbf{NC}_{\preceq} , \mathbf{NA}_{\preceq} and \mathbf{NNA}_{\preceq} , always denoted by \mathbf{N}_{\preceq}^* . Disregarding the hypersequent structure, the calculi $\mathbf{H.N}_{\preceq}^*$ are fragments of the sequent calculi for Lewis' logics by Olivetti and Pozzato [22] and Girlando et al. [8], the difference being that we do not assume the communication rule *com*. The basic components of the calculi $\mathbf{H.N}_{\preceq}^*$ are Gentzen-style sequents to which is added the following *block structure* from [22], representing \preceq -formulas in the right-hand side of sequents.

Definition 5.1 A *block* is a structure $[\Sigma \triangleleft A]$, where Σ is a multiset of formulas and A is a formula. A *sequent with blocks* is a pair $\Gamma \Rightarrow \Delta$, where Γ is a

$\frac{\text{init}}{\mathcal{G} \mid \Gamma, p \Rightarrow p, \Delta}$	$\frac{\perp_L}{\mathcal{G} \mid \Gamma, \perp \Rightarrow \Delta}$	$\frac{\rightarrow_L}{\frac{\mathcal{G} \mid \Gamma, A \rightarrow B, B \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, A \rightarrow B \Rightarrow \Delta, A}{\mathcal{G} \mid \Gamma, A \rightarrow B \Rightarrow \Delta}}$
$\frac{\approx_L}{\frac{\mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow \Delta, [B, \Sigma \triangleleft C] \quad \mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \triangleleft C], [\Sigma \triangleleft A]}{\mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \triangleleft C]}}$		
$\frac{\rightarrow_R}{\frac{\mathcal{G} \mid \Gamma, A \rightarrow B, A \Rightarrow \Delta, B}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \rightarrow B}}$	$\frac{\approx_R}{\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \preccurlyeq B, [A \triangleleft B]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \preccurlyeq B}}$	$\frac{\text{jp}}{\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A] \mid A \Rightarrow \Sigma}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A]}}$
$\frac{\text{N}}{\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\perp \triangleleft \top]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}}$	$\frac{\text{T}}{\frac{\mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow \Delta, B \quad \mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow \Delta, [\perp \triangleleft A]}{\mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow \Delta}}$	
$\frac{\text{W}}{\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A], \Sigma}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A]}}$	$\frac{\text{C}}{\frac{\mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow \Delta, B \quad \mathcal{G} \mid \Gamma, A \preccurlyeq B, A \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow \Delta}}$	
$\frac{\text{A}_L}{\frac{\mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow \Delta \mid \Omega, A \preccurlyeq B \Rightarrow \Theta}{\mathcal{G} \mid \Gamma, A \preccurlyeq B \Rightarrow \Delta \mid \Omega \Rightarrow \Theta}}$	$\frac{\text{A}_R}{\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \preccurlyeq B \mid \Omega \Rightarrow \Theta, A \preccurlyeq B}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A \preccurlyeq B \mid \Omega \Rightarrow \Theta}}$	

$\text{H.N}_{\preccurlyeq} = \{\text{init}, \perp_L, \rightarrow_L, \rightarrow_R\} \cup \{\approx_L, \approx_R, \text{jp}\}$	$\text{H.NW}_{\preccurlyeq} = \text{H.N}_{\preccurlyeq} \cup \{\text{T}, \text{W}\}$
$\text{H.NN}_{\preccurlyeq} = \text{H.N}_{\preccurlyeq} \cup \{\text{N}\}$	$\text{H.NC}_{\preccurlyeq} = \text{H.N}_{\preccurlyeq} \cup \{\text{W}, \text{C}\}$
$\text{H.NT}_{\preccurlyeq} = \text{H.N}_{\preccurlyeq} \cup \{\text{T}\}$	$\text{H.NA}_{\preccurlyeq}^* = \text{H.N}_{\preccurlyeq}^* \cup \{\text{A}_L, \text{A}_R\}$

Fig. 6. Rules of hypersequent calculi $\text{H.N}_{\preccurlyeq}^*$.

multiset of formulas, and Δ is a multiset of formulas and blocks. Sequents are interpreted in \mathcal{L} as follows (where Δ' does not contain blocks):

$$i(\Gamma \Rightarrow \Delta', [\Sigma_1 \triangleleft C_1], \dots, [\Sigma_k \triangleleft C_k]) = \bigwedge \Gamma \rightarrow \bigvee \Delta' \vee (\bigvee \Sigma_1 \preccurlyeq C_1) \vee \dots \vee (\bigvee \Sigma_k \preccurlyeq C_k).$$

A *hypersequent* \mathcal{H} is a finite multiset of sequents with blocks $\Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$, where $\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n$ are called the *components* of \mathcal{H} . We say that a hypersequent is *valid in a model* \mathcal{M} if it has a component $\Gamma_k \Rightarrow \Delta_k$ such that $\mathcal{M} \models i(\Gamma_k \Rightarrow \Delta_k)$.

While hypersequents do not have a formula interpretation, sequents with blocks are interpreted as formulas of \mathcal{L} , in a way different from [22,8]. Specifically, for us $[B_1, \dots, B_n \triangleleft A]$ is interpreted as $(B_1 \vee \dots \vee B_n) \preccurlyeq A$, while in [22,8] it corresponds to $(B_1 \preccurlyeq A) \vee \dots \vee (B_n \preccurlyeq A)$. These two interpretations are equivalent in $\mathbf{V}_{\preccurlyeq}$ but are not equivalent in $\mathbf{N}_{\preccurlyeq}$.

The calculi $\text{H.N}_{\preccurlyeq}^*$ are defined in Fig. 6. The rules are *cumulative*, meaning that each rule has the principal formula copied in the premisses. Differently from the calculi $\text{G.N}_{\preccurlyeq}^*$ in the previous section, $\text{H.N}_{\preccurlyeq}^*$ have separate left and right rules for \preccurlyeq , and all rules have a fixed number of premisses.

We point out that the hypersequent structure is not necessary to define sequent calculi with blocks for CPN logics. Moreover, it can be checked that a hypersequent is derivable if and only if one of its components is derivable. Following the strategy from [4], we chose to employ a hypersequential structure to obtain invertibility of *all* the rules of the calculi, there including the *jp* rule, which was not invertible in [8]. Together with their cumulative formulation, invertibility of the rules allows to directly construct countermodels from

failed proof search, without the need of backtracking or inserting any additional computation. Moreover, differently from [8], the countermodel construction modularly extends to logics with *absoluteness*. The rules for absoluteness are inspired from [9], and correspond to condition A+ from Sec. 2. Soundness of the rules is proved as follows. Let $\text{H.N}_{\preceq}^* \vdash A$ denote derivability of A in H.N_{\preceq}^* .

Theorem 5.2 (Soundness) *For every formula A , if A is derivable in H.N_{\preceq}^* , then A is valid in all N_{\preceq}^* -models.*

Proof. For every rule R of H.N_{\preceq}^* , we show that if the premisses of R are valid in a N_{\preceq}^* -model \mathcal{M} , then the conclusion is also valid in \mathcal{M} . We only consider some relevant examples of modal rules. (\preceq_L) Suppose $\mathcal{M} \models \mathcal{G} \mid \Gamma, A \preceq B \Rightarrow \Delta, [B, \Sigma \triangleleft C]$ and $\mathcal{M} \models \mathcal{G} \mid \Gamma, A \preceq B \Rightarrow \Delta, [\Sigma \triangleleft C], [\Sigma \triangleleft A]$. If $\mathcal{M} \models \mathcal{G}$ we are done. Otherwise $\mathcal{M} \models \bigwedge \Gamma \wedge (A \preceq B) \rightarrow (B \vee \bigvee \Sigma \preceq C) \vee \bigvee \Delta$ and $\mathcal{M} \models \bigwedge \Gamma \wedge (A \preceq B) \rightarrow (\bigvee \Sigma \preceq C) \vee (\bigvee \Sigma \preceq A) \vee \bigvee \Delta$. Then by **tr**, $\mathcal{M} \models \bigwedge \Gamma \wedge (A \preceq B) \rightarrow (\bigvee \Sigma \preceq C) \vee (\bigvee \Sigma \preceq B) \vee \bigvee \Delta$, and by **cpr** and **or**, $\mathcal{M} \models \bigwedge \Gamma \wedge (A \preceq B) \rightarrow (\bigvee \Sigma \preceq C) \vee (\bigvee \Sigma \preceq B \vee \bigvee \Sigma) \vee \bigvee \Delta$, therefore by **tr**, $\mathcal{M} \models \bigwedge \Gamma \wedge (A \preceq B) \rightarrow (\bigvee \Sigma \preceq C) \vee (\bigvee \Sigma \preceq C) \vee \bigvee \Delta$, thus $\mathcal{M} \models \bigwedge \Gamma \wedge (A \preceq B) \rightarrow (\bigvee \Sigma \preceq C) \vee \bigvee \Delta$. It follows $\mathcal{M} \models \mathcal{G} \mid \Gamma, A \preceq B \Rightarrow \Delta, [\Sigma \triangleleft C]$. (**jp**) Suppose that $\mathcal{M} \models \mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A] \mid A \Rightarrow \Sigma$. If $\mathcal{M} \models \mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A]$ we are done, otherwise $\mathcal{M} \models A \rightarrow \bigvee \Sigma$. Then by **cpr**, $\mathcal{M} \models \bigvee \Sigma \preceq A$, therefore $\mathcal{M} \models \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A]$. (**T**) Suppose that $\mathcal{M} \models \mathcal{G} \mid \Gamma, A \preceq B \Rightarrow \Delta, B$ and $\mathcal{M} \models \mathcal{G} \mid \Gamma, A \preceq B \Rightarrow \Delta, [\perp \triangleleft A]$. If $\mathcal{M} \models \mathcal{G}$ we are done, otherwise $\mathcal{M} \models \bigwedge \Gamma \wedge (A \preceq B) \rightarrow \bigvee \Delta \vee (B \wedge (\perp \preceq A))$. Then by **tr**, $\mathcal{M} \models \bigwedge \Gamma \wedge (A \preceq B) \rightarrow \bigvee \Delta \vee (B \wedge (\perp \preceq B))$, and by **t**, $\mathcal{M} \models \bigwedge \Gamma \wedge (A \preceq B) \rightarrow \bigvee \Delta \vee (B \wedge \neg B)$, therefore $\mathcal{M} \models \bigwedge \Gamma \wedge (A \preceq B) \rightarrow \bigvee \Delta$. \square

The calculi H.N_{\preceq}^* enjoy admissibility of the following structural properties:

Lemma 5.3 *It holds that all the rules of H.N_{\preceq}^* are height-preserving invertible, and that the following rules of weakening and contraction are height-preserving admissible in H.N_{\preceq}^* , where A in wk_R or ctr_R can be a formula or a block.*

$$\begin{array}{c} \text{wk}_L \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid A, \Gamma \Rightarrow \Delta} \quad \text{wk}_R \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A} \quad \text{wk}_C \frac{\mathcal{G}}{\mathcal{G} \mid \mathcal{H}} \quad \text{wk}_B \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma \triangleleft A]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma, B \triangleleft A]} \\ \text{ctr}_L \frac{\mathcal{G} \mid A, A, \Gamma \Rightarrow \Delta}{\mathcal{G} \mid A, \Gamma \Rightarrow \Delta} \quad \text{ctr}_R \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A, A}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, A} \quad \text{ctr}_C \frac{\mathcal{G} \mid \mathcal{H} \mid \mathcal{H}}{\mathcal{G} \mid \mathcal{H}} \quad \text{ctr}_B \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma, B, B \triangleleft A]}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, [\Sigma, B \triangleleft A]} \end{array}$$

Proof. Height-preserving admissibility of weakening can be standardly proved by induction on the height of the derivation. Invertibility of all the rules of H.N_{\preceq}^* immediately follows. For instance, the premiss of the **jp** rule can be derived from the conclusion of **jp** using wk_C . Admissibility of contraction also follows by standard induction on the height of derivations. \square

Concerning completeness, a proof can be given by showing that the derivations in the calculi G.N_{\preceq}^* can be simulated in H.N_{\preceq}^* .

Theorem 5.4 (Simulation) *For A \mathcal{L} formula, if $\text{G.N}_{\preceq}^* \vdash A$ then $\text{H.N}_{\preceq}^* \vdash A$.*

Proof. We show that the rules of $\mathbf{G.N}_{\approx}^*$ can be stepwise simulated by the rules of $\mathbf{H.N}_{\approx}^*$. Then the proof of the claim is similar to the one given in [8]. Let $\Sigma_n = C_1 \approx D_1, \dots, C_n \approx D_n$ and $\mathbf{D}_1^n = D_1, \dots, D_n$. Here follows the translation of \mathbf{CP}_n .

$$\begin{array}{c} \frac{B \Rightarrow A, \mathbf{D}_1^n}{\text{jp} \frac{\Gamma, \Sigma_n \Rightarrow [A, \mathbf{D}_1^n \triangleleft B], \Delta \mid B \Rightarrow A, \mathbf{D}_1^n}{\Gamma, \Sigma_n \Rightarrow [A, \mathbf{D}_1^n \triangleleft B], \Delta}} \quad \frac{C_n \Rightarrow A, \mathbf{D}_1^{n-1}}{\text{wk} \frac{\Gamma, \Sigma_n \Rightarrow [A, \mathbf{D}_1^{n-1} \triangleleft C_n], \Delta \mid C_n \Rightarrow A, \mathbf{D}_1^{n-1}}{\Gamma, \Sigma_n \Rightarrow [A, \mathbf{D}_1^{n-1} \triangleleft C_n], \Delta}} \\ \approx_L \frac{\Gamma, \Sigma_n \Rightarrow [A, \mathbf{D}_{n-1}^n B], \Delta}{\Gamma, \Sigma_n \Rightarrow [A, D_1, D_2 \triangleleft B], \Delta} \quad \frac{C_1 \Rightarrow A}{\text{wk} \frac{\Gamma, \Sigma_n \Rightarrow [A \triangleleft C_1], \Delta \mid C_1 \Rightarrow A}{\Gamma, \Sigma_n \Rightarrow [A \triangleleft C_1], \Delta}} \\ \vdots \\ \frac{\Gamma, \Sigma_n \Rightarrow [A, D_1 \triangleleft B], \Delta}{\text{jp} \frac{\Gamma, \Sigma_n \Rightarrow [A, D_1 \triangleleft B], \Delta}{\Gamma, \Sigma_n \Rightarrow [A \triangleleft B], \Delta}} \quad \frac{\Gamma, \Sigma_n \Rightarrow [A \triangleleft B], \Delta}{\text{jp} \frac{\Gamma, \Sigma_n \Rightarrow [A \triangleleft B], \Delta}{\Gamma, \Sigma_n \Rightarrow A \triangleleft B, \Delta}} \\ \approx_R \frac{\Gamma, \Sigma_n \Rightarrow [A \triangleleft B], \Delta}{\Gamma, \Sigma_n \Rightarrow A \triangleleft B, \Delta} \end{array}$$

Rule \mathbf{N}_n is derived in a similar way, by replacing \approx_R with \mathbf{N} and removing occurrences of A and B in the derivation above. For \mathbf{A}_n and $\mathbf{N}^{\mathbf{A}}_n$ each jp is followed by applications of \mathbf{A}_L and \mathbf{A}_R . To derive rule \mathbf{W}_n , replace the upper leftmost occurrence of jp with rule \mathbf{W} . Rule \mathbf{W}_0 is immediately derivable using jp and \mathbf{W} , and rule \mathbf{C}_0 using \mathbf{C} . The case of rule \mathbf{T}_n is more complex. We start with the following derivation.

$$\begin{array}{c} \frac{C_1 \Rightarrow}{\text{wk}_B, \text{wk} \frac{\Gamma, \Sigma_n \Rightarrow \Delta, \mathbf{D}_2^n, [\perp \triangleleft C_1] \mid C_1 \Rightarrow \perp}{\Gamma, \Sigma_n \Rightarrow \Delta, \mathbf{D}_2^n, [\perp \triangleleft C_1]}} \\ \text{T} \frac{\Gamma, \Sigma_n \Rightarrow \Delta, \mathbf{D}_2^n, D_1}{\Gamma, \Sigma_n \Rightarrow \Delta, \mathbf{D}_2^n} \\ \vdots \\ \text{T} \frac{\Gamma, \Sigma_n \Rightarrow \Delta, D_n, D_{n-1}}{\Gamma, \Sigma_n \Rightarrow \Delta, D_n} \quad \Gamma, \Sigma_n \Rightarrow \Delta, [\perp \triangleleft C_n] \\ \text{T} \frac{\Gamma, \Sigma_n \Rightarrow \Delta, D_n}{\Gamma, \Sigma_n \Rightarrow \Delta} \end{array}$$

The leftmost sequent is premiss $\Gamma, \Sigma_n \Rightarrow \Delta, \mathbf{D}_1^n$ of \mathbf{T}_n . We now construct from the remaining premisses of \mathbf{T}_n , that is, $\{C_k \Rightarrow D_1, \dots, D_{k-1}\}$, for $k \leq n$, derivations of sequents $\Gamma, \Sigma_n \Rightarrow \Delta, \mathbf{D}, [\perp \triangleleft C_k]$, for $1 < k \leq n$, where $\mathbf{D} = D_n, \dots, D_{k+1}$ if $k < n$, and is empty otherwise. In applications of the jp rule, we omit specifying the leftmost component of the hypersequents.

$$\begin{array}{c} \frac{C_k \Rightarrow \mathbf{D}_1^{k-1}}{\text{wk}_B, \text{wk} \frac{\dots \mid C_k \Rightarrow \mathbf{D}_1^{k-1}, \perp}{\Gamma, \Sigma_n \Rightarrow \Delta, \mathbf{D}, [\mathbf{D}_1^{k-1}, \perp \triangleleft C_k]}} \quad \frac{C_{k-1} \Rightarrow \mathbf{D}_1^{k-2}}{\text{wk}_C, \text{wk} \frac{\dots \mid C_{k-1} \Rightarrow \mathbf{D}_1^{k-2}, \perp}{\Gamma, \Sigma_n \Rightarrow \Delta, \mathbf{D}, [\mathbf{D}_1^{k-2}, \perp \triangleleft C_k], [\mathbf{D}_1^{k-2}, \perp \triangleleft C_{k-1}]}} \\ \approx_L \frac{\Gamma, \Sigma_n \Rightarrow \Delta, \mathbf{D}, [\mathbf{D}_1^{k-2}, \perp \triangleleft C_k]}{\Gamma, \Sigma_n \Rightarrow \Delta, \mathbf{D}, [\mathbf{D}_1^{k-2}, \perp \triangleleft C_k]} \\ \vdots \\ \frac{\Gamma, \Sigma_n \Rightarrow \Delta, \mathbf{D}, [D_1, D_2, \perp \triangleleft C_k]}{\approx_L \frac{\Gamma, \Sigma_n \Rightarrow \Delta, \mathbf{D}, [D_1, D_2, \perp \triangleleft C_k]}{\Gamma, \Sigma_n \Rightarrow \Delta, \mathbf{D}, [D_1, D_2, \perp \triangleleft C_k]}} \quad \frac{C_2 \Rightarrow D_1}{\text{wk}_C, \text{wk} \frac{\dots \mid C_2 \Rightarrow D_1, \perp}{\Gamma, \Sigma_n \Rightarrow \Delta, \mathbf{D}, [D_1, \perp \triangleleft C_k], [D_1, \perp \triangleleft C_2]}} \\ \approx_L \frac{\Gamma, \Sigma_n \Rightarrow \Delta, \mathbf{D}, [D_1, \perp \triangleleft C_k]}{\approx_L \frac{\Gamma, \Sigma_n \Rightarrow \Delta, \mathbf{D}, [D_1, \perp \triangleleft C_k]}{\Gamma, \Sigma_n \Rightarrow \Delta, \mathbf{D}, [\perp \triangleleft C_k]}} \end{array}$$

The rightmost premiss of the lower occurrence of \approx_L , not shown, is sequent $\Gamma, \Sigma_n \Rightarrow \Delta, \mathbf{D}, [\perp \triangleleft C_k], [\perp \triangleleft C_1]$, which is derivable by jp from premiss $C_1 \Rightarrow \square$.

Since $\mathbf{G.N}_{\approx}^*$ are complete with respect to \mathbf{N}_{\approx}^* , this simulation entails that $\mathbf{H.N}_{\approx}^*$ are also complete. Here we present in more detail an alternative com-

pleteness proof based on the semantics. In particular, we define a terminating bottom-up proof-search strategy in $\mathbf{H.N}_{\preceq}^*$, and show that whenever the strategy fails, one can directly extract a countermodel of the root formula/hypersequent. The strategy is based on the following notion of saturation.

Definition 5.5 Let $\mathcal{H} = \Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$ be a hypersequent occurring in proof for \mathcal{H}' in $\mathbf{H.N}_{\preceq}^*$. The *saturation conditions* associated to each application of a rule of $\mathbf{H.N}_{\preceq}^*$ are as follows: (init) $\Gamma_k \cap \Delta_k = \emptyset$. (\perp_L) $\perp \notin \Gamma_k$. (\rightarrow_L) If $A \rightarrow B \in \Gamma_k$, then $A \in \Delta_k$ or $B \in \Gamma_k$. (\rightarrow_R) If $A \rightarrow B \in \Delta_k$, then $A \in \Gamma_k$ and $B \in \Delta_k$. (\preceq_L) If $A \preceq B \in \Gamma_k$ and $[\Sigma \triangleleft C] \in \Delta_k$, then $B \in \Sigma$ or there is $[\Pi \triangleleft A] \in \Delta_k$ such that $set(\Sigma) \subseteq set(\Pi)$. (\preceq_R) If $A \preceq B \in \Delta_k$, then there is $[\Sigma \triangleleft B] \in \Delta_k$ such that $A \in \Sigma$. (jp) If $[\Sigma \triangleleft A] \in \Delta_k$, then there is $\Gamma_j \Rightarrow \Delta_j \in \mathcal{H}$ such that $A \in \Gamma_j$ and $set(\Sigma) \subseteq \Delta_j$. (N) There is $[\Sigma \triangleleft \top] \in \Delta_k$ such that $\perp \in \Sigma$. (T) If $A \preceq B \in \Gamma_k$, then $B \in \Delta_k$ or there is $[\Sigma, \perp \triangleleft A] \in \Delta_k$. (W) If $[\Sigma \triangleleft A] \in \Delta_k$, then $set(\Sigma) \subseteq \Delta_k$. (C) If $A \preceq B \in \Gamma_k$, then $B \in \Delta_k$ or $A \in \Gamma_k$. (A_L) If $A \preceq B \in \Gamma_k$, then for all $\Gamma_j \Rightarrow \Delta_j \in \mathcal{H}$, $A \preceq B \in \Gamma_j$. (A_R) If $A \preceq B \in \Delta_k$, then for all $\Gamma_j \Rightarrow \Delta_j \in \mathcal{H}$, $A \preceq B \in \Delta_j$. We say that \mathcal{H} is *saturated with respect to an application of a rule R* if it satisfies the saturation condition (R) for that particular rule application, and it is *saturated with respect to $\mathbf{H.N}_{\preceq}^*$* if it is saturated with respect to all possible applications of any rule of $\mathbf{H.N}_{\preceq}^*$.

The strategy consists simply in applying the rules backward until no additional rule application is possible respecting the following two conditions: (i) no rule can be applied to an initial hypersequent; (ii) the application of a rule is not allowed if the hypersequent is already saturated with respect to that specific rule application. The conditions (i) and (ii) ensure that proof-search terminates for every hypersequent \mathcal{H} .

Proposition 5.6 *Proof-search for \mathcal{H} in $\mathbf{H.N}_{\preceq}^*$ in accordance with the strategy always terminates after a finite number of steps.*

Proof. Let \mathcal{P} be a proof of \mathcal{H} constructed according to the strategy. Then all formulas occurring in \mathcal{P} (both inside and outside blocks) are subformulas of formulas of \mathcal{H} or they are \perp or \top , so they are finitely many. Moreover, the saturation conditions prevent duplications of the same formulas (both inside and outside blocks) and the same blocks. It follows that all hypersequents occurring in \mathcal{P} have a finite length, moreover every branch of \mathcal{P} contains only finitely many hypersequents. \square

If the strategy succeeds, then it constructs a derivation of the root hypersequent \mathcal{H} . Otherwise, a saturated hypersequent will occur in the leaf of a branch. We now prove that the proof-search strategy is complete, showing that whenever the strategy fails, from every saturated hypersequent one can directly construct a countermodel for \mathcal{H} .

Proposition 5.7 (Countermodel construction) *Let $\mathcal{H} = \Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_k \Rightarrow \Delta_k$ be a saturated hypersequent occurring in a proof search tree for \mathcal{H}_0*

in $\mathbf{H.N}_{\approx}^*$ built in accordance with the strategy. For Σ multiset of formulas, let $\Sigma^\Delta = \{n \mid \Gamma_n \Rightarrow \Delta_n \in \mathcal{H} \text{ and } \text{set}(\Sigma) \subseteq \Delta_n\}$. We define $\mathcal{M} = \langle W, N, V \rangle$:

- $W = \{n \mid \Gamma_n \Rightarrow \Delta_n \in \mathcal{H}\}$.
- For every $n \in W$, $N(n) = \{\Sigma^\Delta \mid \text{there is } A \text{ such that } [\Sigma \triangleleft A] \in \Delta_n\}$.
- For every $p \in \text{Atm}$, $V(p) = \{n \in W \mid p \in \Gamma_n\}$.

Then for all $n \in W$, (i) if $A \in \Gamma_n$, then $n \Vdash A$; (ii) if $A \in \Delta_n$, then $n \not\Vdash A$; and (iii) if $[\Sigma \triangleleft A] \in \Delta_n$, then $n \not\Vdash \bigvee \Sigma \preceq A$. Moreover \mathcal{M} is a \mathbf{N}_{\approx}^* -model.

Proof. The claims (i), (ii) and (iii) are proved simultaneously by induction on the following notion of complexity of formulas and blocks: $c(p) = c(\perp) = 1$, $c(A \rightarrow B) = c(A \preceq B) = c(A) + c(B) + 1$, $c([B_1, \dots, B_n \triangleleft A]) = c(B_1) + \dots + c(B_n) + c(A)$. For $A = p, \perp, B \rightarrow C$ the proof is routine. We consider the case $A = B \preceq C$. ($B \preceq C \in \Gamma_n$) Suppose $\alpha \in N(n)$. By definition, $\alpha = \Sigma^\Delta$ for some Σ such that there is $[\Sigma \triangleleft D] \in \Delta_n$. Then by saturation of \preceq_L , $C \in \Sigma$ or there is Π such that $\text{set}(\Sigma) \subseteq \text{set}(\Pi)$ and $[\Pi \triangleleft B] \in \Delta_n$. In the first case, for every $m \in \Sigma^\Delta$, $C \in \Delta_m$, then by i.h., $m \not\Vdash C$. Therefore $\Sigma^\Delta \not\Vdash^{\exists} C$. In the second case, by saturation of jp there is $m \in W$ such that $B \in \Gamma_m$ and $\text{set}(\Pi) \subseteq \Delta_m$, thus $\text{set}(\Sigma) \subseteq \Delta_m$. Then by i.h., $m \Vdash B$, and by definition $m \in \Sigma^\Delta$. Therefore $\Sigma^\Delta \Vdash^{\exists} B$. It follows $n \Vdash B \preceq C$. ($B \preceq C \in \Delta_n$) By saturation of \preceq_R , there is $[\Sigma \triangleleft C] \in \Delta_n$ such that $B \in \Sigma$. Then by definition, $\Sigma^\Delta \in N(n)$, and by i.h., $m \not\Vdash B$ for every $m \in \Sigma^\Delta$, that is $\Sigma^\Delta \not\Vdash^{\exists} B$. Moreover, by saturation of jp there is $m \in W$ such that $\text{set}(\Sigma) \subseteq \Delta_m$ and $C \in \Gamma_m$. Then by i.h., $m \Vdash C$, and by definition $m \in \Sigma^\Delta$, thus $\Sigma^\Delta \Vdash^{\exists} C$. Therefore $n \not\Vdash B \preceq C$. ($[\Sigma \triangleleft A] \in \Delta_n$) Analogous to the previous item, considering that by i.h. $m \not\Vdash B$ for all $m \in \Sigma^\Delta$ and $B \in \Sigma$, that is $\Sigma^\Delta \not\Vdash^{\exists} \bigvee \Sigma$.

We now show that \mathcal{M} satisfies the conditions of \mathbf{N}_{\approx}^* -models. (Non-emptiness) If $\alpha \in N(n)$, then $\alpha = \Sigma^\Delta$ for some Σ such that there is $[\Sigma \triangleleft A] \in \Delta_n$. Then by saturation of jp , there is $m \in W$ such that $A \in \Gamma_m$ and $\text{set}(\Sigma) \subseteq \Delta_m$, thus $m \in \Sigma^\Delta$. (Normality) By saturation of \mathbf{N} , there is $[\Sigma, \perp \triangleleft \top] \in \Delta_n$, thus $(\Sigma, \perp)^\Delta \in N(n)$, that is $N(n) \neq \emptyset$. (Total reflexivity) We modify the definition of the neighbourhood function as follows. For all $n \in W$, let $\mathcal{O}(n) = \bigcup N(n) \cup \{n\}$. Then, define $N^\top(n) = N(n) \cup \mathcal{O}(n)$. We show that the claim (i) above still holds ((ii) and (iii) are proved as before): Suppose $B \preceq C \in \Delta_n$. As before we can prove that $\Sigma^\Delta \not\Vdash^{\exists} C$ or $\Sigma^\Delta \Vdash^{\exists} B$ for all $[\Sigma \triangleleft A] \in \Delta_n$. Here we show that the same holds for $\mathcal{O}(n)$. If there is $\alpha \in N(n)$ such that $\alpha \Vdash^{\exists} B$, then $\mathcal{O}(n) \Vdash^{\exists} B$. If instead there is no $\alpha \in N(n)$ such that $\alpha \Vdash^{\exists} B$, then $\alpha \not\Vdash^{\exists} C$ for all $\alpha \in N(n)$, that is $\bigcup N(n) \not\Vdash^{\exists} C$. Assume by contradiction that $\mathcal{O}(n) \Vdash^{\exists} C$. Then $n \Vdash C$. Moreover by saturation of \top , $C \in \Delta_n$ or there is $[\Pi \triangleleft B] \in \Delta_n$ such that $\perp \in \Pi$. If $C \in \Delta_n$, then by i.h., $n \not\Vdash C$, contradicting $n \Vdash C$. If $[\Pi \triangleleft B] \in \Delta_n$, then by saturation of jp there is $m \in W$ such that $B \in \Gamma_m$ and $\text{set}(\Pi) \subseteq \Delta_m$. Then by i.h., $m \Vdash B$, moreover $\Pi^\Delta \in N(n)$ and $m \in \Pi^\Delta$, thus $\Pi^\Delta \Vdash^{\exists} B$, against the hypothesis. Therefore $\mathcal{O}(n) \not\Vdash^{\exists} C$. (Weak centering) If $\alpha \in N(n)$, then $\alpha = \Sigma^\Delta$ for some Σ such that there is $[\Sigma \triangleleft A] \in \Delta_n$. Then by saturation of \mathbf{W} , $\text{set}(\Sigma) \subseteq \Delta_n$, thus

$n \in \Sigma^\Delta$. (Centering) We modify the definition of the neighbourhood function as $N^C(n) = N(n) \cup \{\{n\}\}$. We show that (i) still holds ((ii) and (iii) are as before): Suppose $B \preceq C \in \Delta_n$. As before we can prove that $\Sigma^\Delta \not\Vdash^\exists C$ or $\Sigma^\Delta \Vdash^\exists B$ for all $[\Sigma \triangleleft A] \in \Delta_n$. Here we show that the same holds for $\{n\}$. By saturation of \mathbf{C} , $C \in \Delta_n$ or $B \in \Gamma_n$, thus by i.h., $m \Vdash B$ or $m \not\Vdash C$, therefore $\{n\}^\Delta \Vdash^\exists B$ or $\{n\} \not\Vdash^\exists C$. (Strong absoluteness) We modify the definition of N as $N^A(n) = \{\Sigma^\Delta \mid \text{there are } m \in W \text{ and } A \text{ such that } [\Sigma \triangleleft A] \in \Delta_m\}$. We show that (i) still holds ((ii) and (iii) are as before). Suppose $B \preceq C \in \Gamma_n$ and $\alpha \in N(n)$. Then $\alpha = \Sigma^\Delta$ for some $[\Sigma \triangleleft D] \in \Delta_m$ for some $m \in W$. By saturation of \mathbf{A}_L , $B \preceq C \in \Gamma_m$, then by saturation of \preceq_L , $C \in \Sigma$ or there is Π such that $\text{set}(\Sigma) \subseteq \text{set}(\Pi)$ and $[\Pi \triangleleft B] \in \Delta_m$. In the first case, $\Sigma^\Delta \not\Vdash^\exists C$. In the second case, by saturation of \mathbf{jp} there is $k \in W$ such that $B \in \Gamma_k$ and $\text{set}(\Pi) \subseteq \Delta_k$, thus $\text{set}(\Sigma) \subseteq \Delta_m$, therefore $\Sigma^\Delta \Vdash^\exists B$. \square

Note that, since all rules are cumulative, the claims (i) and (ii) of Prop. 5.7 also hold for the root hypersequent \mathcal{H}_0 , thus \mathcal{M} is a countermodel of \mathcal{H}_0 . Moreover, since every proof built in accordance with the strategy either provides a derivation of the root hypersequent, or contains a saturated hypersequent, this result entails a constructive proof of the completeness of $\mathbf{H.N}_{\preceq}^*$.

Theorem 5.8 (Semantic completeness) *For every hypersequent \mathcal{H} , if \mathcal{H} is valid in all \mathbf{N}_{\preceq}^* -models, then \mathcal{H} is derivable in $\mathbf{H.N}_{\preceq}^*$.*

Here follows an example of the countermodel construction.

Example 5.9 We show that axiom \mathbf{co} is not derivable in \mathbf{N}_{\preceq} . Here follows a failed proof of $\Rightarrow (p \preceq q) \vee (q \preceq p)$ in $\mathbf{H.N}_{\preceq}$, where \mathcal{H} is saturated, and \vee_R is admissible from the rules of \mathbf{N}_{\preceq} :

$$\begin{array}{c} \mathcal{H} : \Rightarrow [q \triangleleft p], [p \triangleleft q], p \preceq q, q \preceq p, (p \preceq q) \vee (q \preceq p) \mid q \Rightarrow p \mid p \Rightarrow q \\ \text{jp}(\times 2) \frac{}{\Rightarrow [q \triangleleft p], [p \triangleleft q], p \preceq q, q \preceq p, (p \preceq q) \vee (q \preceq p)} \\ \preceq_R(\times 2) \frac{}{\Rightarrow p \preceq q, q \preceq p, (p \preceq q) \vee (q \preceq p)} \\ \vee_R \frac{}{\Rightarrow (p \preceq q) \vee (q \preceq p)} \end{array}$$

We consider the following enumeration of the components of the saturated hypersequent \mathcal{H} : 1: $\Rightarrow [q \triangleleft p], [p \triangleleft q], p \preceq q, q \preceq p, (p \preceq q) \vee (q \preceq p)$; 2: $q \Rightarrow p$; and 3: $p \Rightarrow q$. Then, following the construction of Prop. 5.7 we obtain the following countermodel $\mathcal{M} = \langle W, N, V \rangle$: $W = \{1, 2, 3\}$. $N(1) = \{p^\Delta, q^\Delta\} = \{\{2\}, \{3\}\}$, and $N(2) = N(3) = \emptyset$. $V(p) = \{3\}$ and $V(q) = \{2\}$. Then we have $p^\Delta \Vdash^\exists q$ and $p^\Delta \not\Vdash^\exists p$, thus $1 \not\Vdash p \preceq q$, moreover $q^\Delta \Vdash^\exists p$ and $q^\Delta \not\Vdash^\exists q$, thus $1 \not\Vdash q \preceq p$. Therefore $1 \not\Vdash (p \preceq q) \vee (q \preceq p)$.

6 Conclusions

We introduced CPN logics, which are a generalisation of Lewis' logics of comparative plausibility defined over neighbourhood rather than sphere models. As a difference with sphere models, neighbourhoods need not to be nested, allowing to express more general notions of comparative plausibility. From a

proof-theoretic viewpoint, CPN logics are captured by suitable restrictions of sequent calculi for Lewis' logics: they coincide to restrictions of calculi from [14,15] to a single principal \Leftarrow -formula in the right-hand side of sequents, and the single-component formulation of their hypersequent calculi corresponds to the structured calculi from [22,8] without the communication rule.

Overall, CPN logics represent a general theory of comparative plausibility with well-understood proof theory and semantics. Differently from stronger logics expressing comparative plausibility, CPN logics allow to model preference or similarity in situations where no priority order is assumed between states of affairs or concepts. Moreover, CPN logics are an expressive framework, encompassing Lewis' logics [16], which are obtained by adding nesting to CPN logics. In future work we plan to investigate the relations between CPN logics and other well-known comparative plausibility logics introduced in the literature, most notably Halpern's comparative plausibility logics defined over preferential structures [12]. We conjecture that Halpern's logics could be obtained by adding the property of *closure under non-empty intersections* to neighbourhood models, which is required to prove equivalence between neighbourhood and preferential structures. Moreover, we wish to relate our systems with the logic of comparative obligation introduced by Brown [2]. Brown's operator is defined on a kind of neighbourhood models containing a function $\mathcal{R} : W \rightarrow \mathcal{P}(\mathcal{P}(W))$, representing a degree of urgency of obligation.

Furthermore, CPN logics parallel the preferential conditional logics studied in [3]. These logics generalise Lewis' counterfactual logics, and admit a neighbourhood semantics, introduced in [11]. Interestingly, while comparative plausibility and conditional entailment are interdefinable in sphere models, the two operators are not interdefinable in neighbourhood semantics, giving rise to two independent theories. While in [11] a proof-theoretical analysis of the conditional operator in neighbourhood semantics is proposed, this work explores the behaviour of the comparative plausibility operator in neighbourhood structures. Moreover, having lost the interdefinability between \Leftarrow and $>$, we wish to study whether alternative and meaningful notions of conditional entailment can be defined in terms of comparative plausibility. We also intend to study applications of CPN logics, possibly related to the analysis of information sources.

Concerning the proof theory for CPN logics, we wish to analyse the complexity of the logics based on the decision procedure induced by the multi-premisses and the hypersequent calculi. Moreover, we plan to automate the proof search and countermodel construction of the hypersequent calculi within a theorem prover, along the lines of what done in [5,10,7]. We will also investigate extensions of the hypersequent calculi to CPN logics with uniformity, possibly adapting the approach proposed in [9] for Lewis' logics to our setting, as well as with other semantic conditions, aiming at developing a uniform proof-theoretic account of CPN logics.

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