Wijesekera-style constructive modal logics

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Abstract

We define a family of propositional constructive modal logics corresponding each to a different classical modal system. The logics are defined in the style of Wijesekera's constructive modal logic [38], and are both proof-theoretically and semantically motivated. On the one hand, they correspond to the single-succedent restriction of standard sequent calculi for classical modal logics. On the other hand, they are obtained by incorporating the hereditariness of intuitionistic Kripke models into the classical satisfaction clauses for modal formulas. We show that, for the considered classical logics, the proof-theoretical and the semantical approach return the same constructive systems.

Keywords: Constructive modal logic, intuitionistic modal logic, sequent calculus, neighbourhood semantics.

1 Introduction

Constructive or intuitionistic modal logics are extensions of intuitionistic logic with modalities $\Box$ and $\Diamond$. The motivations for the study of modalities with an intuitionistic basis are manifold, but they can be schematically classified into two kinds. On the one hand, from a theoretical perspective, it comes natural to combine intuitionistic and modal logic [35], considering in particular that both of them can be semantically arranged in terms of possible world models. In addition, the rejection of classical equivalences can allow for a finer analysis of the modalities. On the other hand, intuitionistic or constructive modal logics can be motivated by specific applications in computer science, such as type-theoretic interpretations, verification, and knowledge representation.

A peculiar feature of intuitionistic modal logics is that, similarly to the intuitionistic connectives, $\Box$ and $\Diamond$ are not interdefinable. This allows for the definition of systems in which $\Box$ and $\Diamond$ satisfy distinct principles. At the same time, it makes possible to define different intuitionistic or constructive counterparts of the same classical logic, as it is testified by the several intuitionistic versions of classical K which have been proposed in the literature (see [35] for a survey).

Intuitionistic modal logics have been formulated as monomodal (with only $\Box$ or only $\Diamond$) or bimodal (with both $\Box$ and $\Diamond$) systems. Considering logics including both modalities, two intuitionistic versions of K have been mostly considered: so-called Intuitionistic K (IK) and Constructive K (CK). The first system was introduced by Fischer Servi [13], Ploktin and Stirling [33], and Ewald [12], and can be defined as the set of formulas whose standard translation is derivable in first-order intuitionistic logic [35,40]. The second system, which is weaker that IK, was introduced by Bellin, de Paiva and Ritter [6], and was motivated by type-theoretic interpretations of the modalities and categorical semantics, but also by contextual reasoning [26,28]. Both the semantics [13,33,12,35,5,26,2,9,29,1] and the proof theory [3,36,14,19,25,24,23,6,4,27,10] of IK and CK have been extensively investigated, in particular significant consideration has been devoted to their extensions with standard modal axioms D, T, B, 4, and 5, so that entire families of intuitionistic and constructive modal logics are now available in the literature.

An additional bimodal constructive version of K was proposed by Wijesekera [38]. Wijesekera's logic aimed at representing reasoning with partial information about the states of concurrent transition systems, and was introduced as a modal extension of first-order intuitionistic logic. If we restrict our attention to its propositional fragment, Wijesekera's logic is intermediate between CK and IK. In particular, Wijesekera's logic (we call it WK) can be defined by extending (any axiomatisation of) intuitionistic propositional logic (IPL) with the following modal axioms and rules: [1]

$$nec\,\frac{A}{\Box A} \qquad \begin{array}{ll} K_\Box & \Box(A \supset B) \supset (\Box A \supset \Box B) \\ K_\Diamond & \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) \end{array} \qquad N_\Diamond \quad \neg\Diamond\bot$$

Then CK can be obtained by dropping $\neg\Diamond\bot$, whereas IK is obtained by extending WK with the axioms $\Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$ and $(\Diamond A \supset \Box B) \supset \Box(A \supset B)$. The interest of WK is not limited to its intended interpretation: this logic also exhibits an elegant relation with classical K, both from a semantical and from a proof-theoretical perspective. We now illustrate this relation.

Semantics for intuitionistic modal logics are typically defined by combining intuitionistic Kripke models and possible-world models for modal logics. A crucial requirement is that the resulting models must preserve the hereditary property of intuitionistic models, meaning that if a formula is true in a world w, then it is true also in all worlds reachable from w through the intuitionistic order $\leq$. Such a requirement can be fulfilled essentially in two ways. First, one can establish suitable combinations between $\leq$ and the modal relation $\mathcal{R}$, as it is done for instance in the semantics for IK. Alternatively, one can build the hereditariness into the satisfaction clauses for modal formulas by requiring that the standard clauses hold for all $\leq$-successors. This is the strategy adopted by Wijesekera [38] who presents models with two relations $\leq$ and $\mathcal{R}$ without any specific combination between them, where the modalities are interpreted in the following way: [2]

[1] Wijesekera [38] includes also the axiom $\Box A \wedge \Diamond(A \supset B) \supset \Diamond B$ which is derivable from the others.

[2] The semantics of CK is similar but it also requires 'fallible' worlds satisfying $\bot$ (cf. [26]).

$\mathcal{M}, w \Vdash \Box A$ iff for all $v \geq w$, for all u, if $v\mathcal{R}u$, then $\mathcal{M}, u \Vdash A$.
$\mathcal{M}, w \Vdash \Diamond A$ iff for all $v \geq w$, there is u such that $v\mathcal{R}u$ and $\mathcal{M}, u \Vdash A$.

Wijesekera [38] also provides a sequent calculus for WK, which is defined by extending a suitable calculus for IPL with the following modal rules (where $|\Gamma| \geq 0$ and $0 \leq |\Delta| \leq 1$):

$$\mathsf{K}^{\mathsf{i}}_{\Box}\ \frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A} \qquad \mathsf{K}^{\mathsf{i}}_{\Diamond}\ \frac{\Gamma, A \Rightarrow \Delta}{\Box\Gamma, \Diamond A \Rightarrow \Diamond\Delta}$$

Gentzen [15] showed that, given a suitable sequent calculus for classical logic, its restriction to single-succedent sequents (i.e., sequents with at most one formula in the consequent) provides a sequent calculus for intuitionistic logic. Interestingly, Wijesekera's logic can be seen as the system obtained by restricting to single-succedent sequents a standard sequent calculus for classical K (formulated with explicit $\Box$ and $\Diamond$), [3] so that this correspondence is preserved at the modal level. We then observe that WK displays a clear and elegant relation with classical K, both semantically and proof-theoretically:

- semantically, WK is obtained simply by incorporating hereditariness into the modal satisfaction clauses of K;
- proof-theoretically, it is obtained by restricting a standard sequent calculus for K to single-succedent sequents.

Despite its interest, Wijesekera's logic has received significantly less consideration than CK and IK. In particular, while alternative semantics and proof systems for WK have been studied [39,18,9,10], no systematic investigation of Wijesekera-style systems has been carried out so far.

Filling this gap is precisely the aim of this paper: we define a family of Wijesekera-style logics corresponding each to a different classical modal logic (for lack of a better name we call them *W-logics*), adopting as a guideline for the definition of these systems the semantical and proof-theoretical relation between WK and K just described. In particular, in Sec. 2 we present standard sequent calculi and semantics for a family of classical modal logics. Then we define constructive counterparts of these logics by (i) restricting the calculi to single-succedent sequents (Sec. 3), and (ii) expressing the classical satisfaction clauses for modal formulas over intuitionistic Kripke models, building hereditariness into these conditions (Sec. 4). The main contribution of this paper consists in showing that, despite being mutually independent, for a wide family of classical modal logics the semantical and the proof-theoretical approach return exactly the same constructive systems.

2 Preliminaries on classical modal logics

Let $\mathcal{L}$ be a propositional modal language based on a set *Atm* of countably many propositional variables $p_1, p_2, p_3, ...$; the *well-formed formulas* of $\mathcal{L}$ are

[3] Wijesekera [38] considers a multi-succedent calculus for IPL, however an equivalent calculus can be given by adding Wijesekera's modal rules to a single-succedent calculus (cf. [9] and Sec. 3 in this paper).

$nec\ \dfrac{A}{\Box A}$	$K_\Box$ $\quad \Box(A \supset B) \supset (\Box A \supset \Box B)$	$T_\Box$ $\quad \Box A \supset A$
	$K_\Diamond$ $\quad \Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$	$T_\Diamond$ $\quad A \supset \Diamond A$
$mon_\Box\ \dfrac{A \supset B}{\Box A \supset \Box B}$	$C_\Box$ $\quad \Box A \wedge \Box B \supset \Box(A \wedge B)$	D $\quad \Box A \supset \Diamond A$
	$C_\Diamond$ $\quad \Diamond(A \vee B) \supset \Diamond A \vee \Diamond B$	$P_\Box$ $\quad \neg\Box\bot$
$mon_\Diamond\ \dfrac{A \supset B}{\Diamond A \supset \Diamond B}$	$N_\Box$ $\quad \Box\top$	$P_\Diamond$ $\quad \Diamond\top$
	$N_\Diamond$ $\quad \neg\Diamond\bot$	
$dual$ $\quad \Box A \supset\subset \neg\Diamond\neg A$	$dual_\wedge$ $\quad \neg(\Box A \wedge \Diamond\neg A)$	$dual_\vee$ $\quad \Box A \vee \Diamond\neg A$

Fig. 1. Modal axioms and rules.

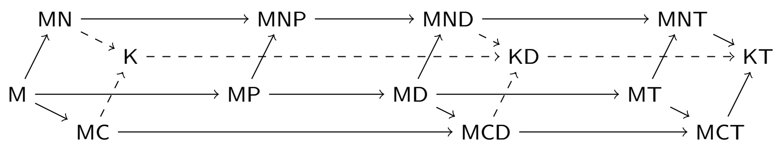

Fig. 2. Dyagram of classical modal logics.

generated by the following grammar, where p_i is any element of Atm:

$$A ::= p_i \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid \Box A \mid \Diamond A.$$

We also define $\top := \bot \supset \bot$, $\neg A := A \supset \bot$, and $A \supset\subset B := (A \supset B) \wedge (B \supset A)$.

We aim at enriching the family of Wijesekera-style propositional modal logics by defining constructive counterparts of well-known classical modal logics. We consider the following classical systems, which are defined in the language $\mathcal{L}$ extending (any axiomatisation of) classical propositional logic (CPL) with the following modal axioms and rules from Fig. 1:

M := $dual$ + $mon_\Box$	MNP := MN + $P_\Box$	MT := M + $T_\Box$
MN := M + $N_\Box$	MD := M + D	MNT := MN + $T_\Box$
MC := M + $C_\Box$	MND := MN + D	MCT := MC + $T_\Box$
K := M + $N_\Box$ + $C_\Box$	MCD := MC + D	KT := K + $T_\Box$
MP := M + $P_\Box$	KD := K + D	

The considered axiomatisation of K is equivalent to the more standard one with nec and $K_\Box$ (cf. e.g. [8]). The above list contains logics stronger than K as well as weaker (i.e., non-normal) systems. Note that given the duality between $\Box$ and $\Diamond$, the above systems can be equivalently defined by replacing $mon_\Box$, $N_\Box$, $C_\Box$, $T_\Box$, and $P_\Box$, with their $\Diamond$-versions $mon_\Diamond$, $N_\Diamond$, $C_\Diamond$, $T_\Diamond$, $P_\Diamond$ (Fig. 1). The relations among the classical systems are displayed in Fig. 2 (MCP and KP are not considered in the list as they coincide with MCD and KD).

We will define constructive counterparts of classical modal logics by restricting suitable sequent calculi for the classical systems. We consider to this purpose the calculi for classical modal logics defined by the rules in Fig. 3. As usual, we call *sequent* any pair $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite, possibly empty multisets of formulas of $\mathcal{L}$. A sequent $\Gamma \Rightarrow \Delta$ is interpreted as a formula of $\mathcal{L}$ as $\bigwedge \Gamma \supset \bigvee \Delta$ if Γ is non-empty, and it is interpreted as $\bigvee \Delta$ if Γ is empty, where $\bigvee \emptyset$ is interpreted as $\bot$. For every multiset $\Gamma = A_1, ..., A_n$, we denote

Propositional rules $\quad$ init $\Gamma, p \Rightarrow p, \Delta$ $\quad$ $\bot_L$ $\Gamma, \bot \Rightarrow \Delta$

$$\supset_L \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \quad \supset_R \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta} \quad \wedge_R \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

$$\wedge_L \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \quad \vee_R \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta} \quad \vee_L \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}$$

Modal rules

$$\mathsf{M}_\Box \frac{A \Rightarrow B}{\Gamma, \Box A \Rightarrow \Box B, \Delta} \quad \mathsf{M}_\Diamond \frac{A \Rightarrow B}{\Gamma, \Diamond A \Rightarrow \Diamond B, \Delta} \quad \mathsf{C}_\Box \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma', \Box\Gamma, \Box A \Rightarrow \Box B, \Diamond\Delta, \Delta'}$$

$$\wedge\text{-dual}_\mathsf{M} \frac{A, B \Rightarrow}{\Gamma, \Box A, \Diamond B \Rightarrow \Delta} \quad \mathsf{C}_\Diamond \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma', \Box\Gamma, \Diamond A \Rightarrow \Diamond B, \Diamond\Delta, \Delta'} \quad \mathsf{D} \frac{A \Rightarrow B}{\Gamma, \Box A \Rightarrow \Diamond B, \Delta}$$

$$\vee\text{-dual}_\mathsf{M} \frac{\Rightarrow A, B}{\Gamma \Rightarrow \Box A, \Diamond B, \Delta} \quad \wedge\text{-dual}_\mathsf{C} \frac{\Gamma, A, B \Rightarrow}{\Gamma', \Box\Gamma, \Box A, \Diamond B \Rightarrow \Delta} \quad \mathsf{D}_\Box \frac{A, B \Rightarrow}{\Gamma, \Box A, \Box B \Rightarrow \Delta}$$

$$\vee\text{-dual}_\mathsf{C} \frac{\Rightarrow A, B, \Delta}{\Gamma \Rightarrow \Box A, \Diamond B, \Diamond\Delta, \Delta'} \quad \mathsf{K}_\Box \frac{\Gamma \Rightarrow A, \Delta}{\Gamma', \Box\Gamma \Rightarrow \Box A, \Diamond\Delta, \Delta'} \quad \mathsf{D}_\Diamond \frac{\Rightarrow A, B}{\Gamma \Rightarrow \Diamond A, \Diamond B, \Delta}$$

$$\mathsf{K}_\Diamond \frac{\Gamma, A \Rightarrow \Delta}{\Gamma', \Box\Gamma, \Diamond A \Rightarrow \Diamond\Delta, \Delta'} \quad \mathsf{N}_\Box \frac{\Rightarrow A}{\Gamma \Rightarrow \Box A, \Delta} \quad \mathsf{N}_\Diamond \frac{A \Rightarrow}{\Gamma, \Diamond A \Rightarrow \Delta} \quad \mathsf{P}_\Box \frac{A \Rightarrow}{\Gamma, \Box A \Rightarrow \Delta}$$

$$\mathsf{P}_\Diamond \frac{\Rightarrow A}{\Gamma \Rightarrow \Diamond A, \Delta} \quad \mathsf{T}_\Box \frac{\Gamma, \Box A, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} \quad \mathsf{T}_\Diamond \frac{\Gamma \Rightarrow A, \Diamond A, \Delta}{\Gamma \Rightarrow \Diamond A, \Delta} \quad \mathsf{CD} \frac{\Gamma \Rightarrow \Delta}{\Gamma', \Box\Gamma \Rightarrow \Diamond\Delta, \Delta'}$$

Fig. 3. Sequent rules for classical modal logics (where $|\Gamma|, |\Gamma'|, |\Delta|, |\Delta'| \geq 0$).

with $\Box\Gamma$ and $\Diamond\Gamma$ the multisets $\Box A_1, ..., \Box A_n$ and $\Diamond A_1, ..., \Diamond A_n$, respectively. We consider G3-style calculi with all structural rules admissible (cf. [37, Ch. 3]). Moreover, we consider a formulation of the calculi in which both $\Box$ and $\Diamond$ occur explicitly, this formulation will be needed to handle the constructive systems, where the modalities are not interdefinable (for a sequent calculus with explicit $\Box$ and $\Diamond$ see e.g. [37, Ch. 9], for sequent calculi for non-normal modal logics see [20,21]). For each logic L, the corresponding calculus S.L contains the propositional rules and the following modal rules:

S.M := $\mathsf{M}_\Box$ + $\mathsf{M}_\Diamond$ + $\wedge$-dual$_\mathsf{M}$ + $\vee$-dual$_\mathsf{M}$
S.MN := S.M + $\mathsf{N}_\Box$ + $\mathsf{N}_\Diamond$
S.MC := $\mathsf{C}_\Box$ + $\mathsf{C}_\Diamond$ + $\wedge$-dual$_\mathsf{C}$ + $\vee$-dual$_\mathsf{C}$
S.K := $\mathsf{K}_\Box$ + $\mathsf{K}_\Diamond$
S.MP := S.M + $\mathsf{P}_\Box$ + $\mathsf{P}_\Diamond$
S.MNP := S.MN + $\mathsf{P}_\Box$ + $\mathsf{P}_\Diamond$

S.MD := S.M + D + $\mathsf{D}_\Box$ + $\mathsf{D}_\Diamond$ + $\mathsf{P}_\Box$ + $\mathsf{P}_\Diamond$
S.MND := S.MN + D + $\mathsf{D}_\Box$ + $\mathsf{D}_\Diamond$ + $\mathsf{P}_\Box$ + $\mathsf{P}_\Diamond$
S.MCD := S.MC + CD
S.KD := S.K + CD
S.MT := S.M + $\mathsf{T}_\Box$ + $\mathsf{T}_\Diamond$
S.MNT := S.MN + $\mathsf{T}_\Box$ + $\mathsf{T}_\Diamond$
S.MCT := S.MC + $\mathsf{T}_\Box$ + $\mathsf{T}_\Diamond$
S.KT := S.K + $\mathsf{T}_\Box$ + $\mathsf{T}_\Diamond$

Each calculus contains two duality rules $\wedge$-dual and $\vee$-dual (in S.K and its extensions they are obtained as the particular cases of $\mathsf{K}_\Diamond$ and $\mathsf{K}_\Box$ with $|\Delta| = \emptyset$, respectively $|\Gamma| = \emptyset$). The duality rules allow one to derive the Hilbert-style rules

$$Rdual_\wedge \frac{\neg(A \wedge B)}{\neg(\Box A \wedge \Diamond B)} \qquad Rdual_\vee \frac{A \vee B}{\Box A \vee \Diamond B}$$

which are classically equivalent to $dual_\wedge$ and $dual_\vee$ (Fig. 1), and taken together are equivalent to $dual$. The rules $\mathsf{C}_\Box$ and $\mathsf{C}_\Diamond$ can be seen as the generalisation of $\mathsf{M}_\Box$ and $\mathsf{M}_\Diamond$ to n-principal formulas in the antecedent, respectively in the consequent. Differently from $\mathsf{M}_\Box$, the rule $\mathsf{C}_\Box$ involves also $\Diamond$-formulas, similarly $\mathsf{C}_\Diamond$ involves also $\Box$-formulas, this is needed in order to preserve the admissibility of cut in the calculus. Note also that $\mathsf{C}_\Box$ and $\mathsf{C}_\Diamond$ are distinct from $\mathsf{K}_\Box$ and $\mathsf{K}_\Diamond$, since $\mathsf{C}_\Box$ and $\mathsf{C}_\Diamond$ are applicable only to sequents with non-empty antecedent, respectively non-empty consequent, while this is not required for $\mathsf{K}_\Box$ and $\mathsf{K}_\Diamond$. Finally, the calculi S.MD and S.MND contain also the rules $\mathsf{P}_\Box$ and $\mathsf{P}_\Diamond$, this is needed to ensure admissibility of contraction [11,31], and is consistent with the fact that the axioms $P_\Box$ and $P_\Diamond$ are derivable in MD. Each calculus S.L is a calculus for the corresponding logic L in the following sense:

Theorem 2.1 *For every considered classical modal logic* L*,* $\mathsf{S.L} \vdash \Gamma \Rightarrow \Delta$ *if and only if* $\mathsf{L} \vdash \bigwedge \Gamma \supset \bigvee \Delta$.

We now move to the semantics. Since non-normal logics do not have a (simple) relational semantics, [4] we consider a neighbourhood semantics that uniformly covers all considered systems.

Definition 2.2 A *neighbourhood model* is a tuple $\mathcal{M} = \langle \mathcal{W}, \mathcal{N}, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set of worlds, $\mathcal{N}$ is a function $\mathcal{P}(\mathcal{W}) \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}))$, called neighbourhood function, and $\mathcal{V}$ is a valuation function $Atm \longrightarrow \mathcal{P}(\mathcal{W})$. The forcing relation $\mathcal{M}, w \Vdash A$ is inductively defined as follows:

$\mathcal{M}, w \Vdash p$	iff	$w \in \mathcal{V}(p)$.
$\mathcal{M}, w \nVdash \bot$.		
$\mathcal{M}, w \Vdash B \wedge C$	iff	$\mathcal{M}, w \Vdash B$ and $\mathcal{M}, w \Vdash C$.
$\mathcal{M}, w \Vdash B \vee C$	iff	$\mathcal{M}, w \Vdash B$ or $\mathcal{M}, w \Vdash C$.
$\mathcal{M}, w \Vdash B \supset C$	iff	$\mathcal{M}, w \nVdash B$ or $\mathcal{M}, w \Vdash C$.
$\mathcal{M}, w \Vdash \Box B$	iff	there is $\alpha \in \mathcal{N}(w)$ s.t. for all $v \in \alpha$, $\mathcal{M}, v \Vdash B$.
$\mathcal{M}, w \Vdash \Diamond B$	iff	for all $\alpha \in \mathcal{N}(w)$, there is $v \in \alpha$ s.t. $\mathcal{M}, v \Vdash B$.

We consider the following properties on neighbourhood models:

(C) If $\alpha, \beta \in \mathcal{N}(w)$, then $\alpha \cap \beta \in \mathcal{N}(w)$. (N) $\mathcal{N}(w) \neq \emptyset$.
(D) If $\alpha, \beta \in \mathcal{N}(w)$, then $\alpha \cap \beta \neq \emptyset$. (P) $\emptyset \notin \mathcal{N}(w)$.
(T) If $\alpha \in \mathcal{N}(w)$, then $w \in \alpha$.

We say that $\mathcal{M}$ is a model for a logic L if it satisfies the condition (X) for every modal axiom $X_\Box$ of L (among $C_\Box$, $N_\Box$, $T_\Box$, D, $P_\Box$). As usual, we say that a formula A is valid in a model $\mathcal{M}$, written $\mathcal{M} \models A$, if $\mathcal{M}, w \Vdash A$ for every world w of $\mathcal{M}$.

In the following we simple write $w \Vdash A$ when $\mathcal{M}$ is clear from the context. We also use the following abbreviations:

$\alpha \Vdash^\forall A :=$ for all $w \in \alpha$, $w \Vdash A$; $\alpha \Vdash^\exists A :=$ there is $w \in \alpha$ s.t. $w \Vdash A$.

[4] Cf. [7] for multi-relational semantics for non-normal modal logics, and [34,11] for relational semantics with "non-normal" worlds for the logics containing $C_\Box$ but not $N_\Box$.

Propositional rules $\quad \mathsf{init}^{\mathsf{i}}\ \Gamma, p \Rightarrow p \qquad \bot_{\mathsf{L}}^{\mathsf{i}}\ \Gamma, \bot \Rightarrow \Delta$

$$\supset_{\mathsf{L}}^{\mathsf{i}} \frac{\Gamma, A \supset B \Rightarrow A \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \qquad \supset_{\mathsf{R}}^{\mathsf{i}} \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B} \qquad \wedge_{\mathsf{R}}^{\mathsf{i}} \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\wedge_{\mathsf{L}}^{\mathsf{i}} \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad \vee_{\mathsf{R}}^{\mathsf{i}} \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\ (i \in \{1,2\}) \qquad \vee_{\mathsf{L}}^{\mathsf{i}} \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}$$

Modal rules

$$\mathsf{M}_{\Box}^{\mathsf{i}} \frac{\Gamma, A \Rightarrow B}{\Gamma, \Box A \Rightarrow \Box B} \qquad \mathsf{M}_{\Diamond}^{\mathsf{i}} \frac{\Gamma, A \Rightarrow B}{\Gamma, \Diamond A \Rightarrow \Diamond B} \qquad \wedge\text{-}\mathsf{dual}_{\mathsf{M}}^{\mathsf{i}} \frac{A, B \Rightarrow}{\Gamma, \Box A, \Diamond B \Rightarrow \Delta} \qquad \mathsf{N}_{\Box}^{\mathsf{i}} \frac{\Rightarrow A}{\Gamma \Rightarrow \Box A}$$

$$\mathsf{C}_{\Box}^{\mathsf{i}} \frac{\Gamma, A \Rightarrow B}{\Gamma', \Box\Gamma, \Box A \Rightarrow \Box B} \qquad \mathsf{C}_{\Diamond}^{\mathsf{i}} \frac{\Gamma, A \Rightarrow B}{\Gamma', \Box\Gamma, \Diamond A \Rightarrow \Diamond B} \qquad \wedge\text{-}\mathsf{dual}_{\mathsf{C}}^{\mathsf{i}} \frac{\Gamma, A, B \Rightarrow}{\Gamma', \Box\Gamma, \Box A, \Diamond B \Rightarrow \Delta}$$

$$\mathsf{N}_{\Diamond}^{\mathsf{i}} \frac{A \Rightarrow}{\Gamma, \Diamond A \Rightarrow \Delta} \qquad \mathsf{K}_{\Box}^{\mathsf{i}} \frac{\Gamma \Rightarrow A}{\Gamma', \Box\Gamma \Rightarrow \Box A} \qquad \mathsf{K}_{\Diamond}^{\mathsf{i}} \frac{\Gamma, A \Rightarrow B}{\Gamma', \Box\Gamma, \Diamond A \Rightarrow \Diamond B} \qquad \mathsf{T}_{\Diamond}^{\mathsf{i}} \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \Diamond A}$$

$$\wedge\text{-}\mathsf{dual}_{\mathsf{K}}^{\mathsf{i}} \frac{\Gamma, A \Rightarrow}{\Gamma', \Box\Gamma, \Diamond A \Rightarrow \Delta} \qquad \mathsf{T}_{\Box}^{\mathsf{i}} \frac{\Gamma, \Box A, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} \qquad \mathsf{P}_{\Box}^{\mathsf{i}} \frac{A \Rightarrow}{\Gamma, \Box A \Rightarrow \Delta} \qquad \mathsf{P}_{\Diamond}^{\mathsf{i}} \frac{\Rightarrow A}{\Gamma \Rightarrow \Diamond A}$$

$$\mathsf{D}^{\mathsf{i}} \frac{A \Rightarrow B}{\Gamma, \Box A \Rightarrow \Diamond B} \qquad \mathsf{D}_{\Box}^{\mathsf{i}} \frac{A, B \Rightarrow}{\Gamma, \Box A, \Box B \Rightarrow \Delta} \qquad \mathsf{CD}^{\mathsf{i}} \frac{\Gamma \Rightarrow A}{\Gamma', \Box\Gamma \Rightarrow \Diamond A} \qquad \mathsf{CD}_{\Box}^{\mathsf{i}} \frac{\Gamma \Rightarrow}{\Gamma', \Box\Gamma \Rightarrow \Delta}$$

Fig. 4. Sequent rules for W-logics (where $|\Gamma|, |\Gamma'| \geq 0$, and $0 \leq |\Delta| \leq 1$).

Using these abbreviations, the satisfaction clauses for modal formulas can be equivalently written as

$$w \Vdash \Box B \quad \text{iff} \quad \text{there is } \alpha \in \mathcal{N}(w) \text{ such that } \alpha \Vdash^{\forall} B.$$
$$w \Vdash \Diamond B \quad \text{iff} \quad \text{for all } \alpha \in \mathcal{N}(w),\ \alpha \Vdash^{\exists} B.$$

The following holds (cf. e.g. [8,32]).

Theorem 2.3 *For every considered classical modal logic* L*,* $\mathsf{L} \vdash A$ *if and only if* $\mathcal{M} \models A$ *for all neighbourhood models* $\mathcal{M}$ *for* L*.*

3 Single-succedent calculi and W-logics

We now define a family of Wijesekera-style constructive modal logics corresponding to the classical logics considered in Sec. 2. In particular, we firstly define constructive modal calculi by restricting the classical calculi from Sec. 2 to single-succedent sequents, and study their structural properties. Then we define equivalent axiomatic systems, and prove some fundamental properties of them.

The rules obtained by restricting sequents to at most one formula in the consequent are displayed in Fig. 4. This restriction modifies the classical modal rules in two ways: first, the rules $\vee$-$\mathsf{dual}_{\mathsf{M}}$, $\vee$-$\mathsf{dual}_{\mathsf{R}}$, and $\mathsf{D}_{\Diamond}$ are dropped because they require at least two formulas in the consequent of sequents. Second, the right context is deleted from all rules with a principal formula in the consequent of sequents, namely $\mathsf{T}_{\Diamond}$, $\mathsf{C}_{\Box}$, $\mathsf{C}_{\Diamond}$, and $\mathsf{K}_{\Box}$. Note in particular that $\Diamond\Delta$ is removed from $\mathsf{C}_{\Box}$, $\mathsf{C}_{\Diamond}$, and $\mathsf{K}_{\Box}$, moreover $\Diamond A$ is removed from the premiss of $\mathsf{T}_{\Diamond}^{\mathsf{i}}$ (the copy of $\Diamond A$ into the premiss of $\mathsf{T}_{\Diamond}$ is needed in the classical calculus in order to ensure admissibility of right contraction, which is not expressible in the intuitionistic calculus). Note also that $\mathsf{K}_\Diamond$ and CD are split into two rules, respectively $\mathsf{K}^{\mathsf{i}}_\Diamond$ and $\wedge$-$\mathsf{dual}^{\mathsf{i}}_{\mathsf{K}}$, and CD^{i} and $\mathsf{CD}^{\mathsf{i}}_\Box$, which correspond to the cases in which the consequent of the premiss of $\mathsf{K}_\Diamond$ or CD is or is not empty. Finally, note that $\mathsf{C}^{\mathsf{i}}_\Box$ and $\mathsf{K}^{\mathsf{i}}_\Box$ become equivalent. Concerning the propositional rules, $\supset_{\mathsf{L}}$ is modified as usual by copying the principal implication into the left premiss in order to ensure admissibility of contraction [37], and $\vee_{\mathsf{R}}$ is replaced by its single-succedent version. All other rules remain unchanged. The resulting calculi $\mathsf{S.WL}$ are defined by extending the set of intuitionistic propositional rules with the following modal rules:

$\mathsf{S.WM} := \mathsf{M}^{\mathsf{i}}_\Box + \mathsf{M}^{\mathsf{i}}_\Diamond + \wedge\text{-}\mathsf{dual}^{\mathsf{i}}_{\mathsf{M}}$	$\mathsf{S.WMP} := \mathsf{S.WM} + \mathsf{P}^{\mathsf{i}}_\Box + \mathsf{P}^{\mathsf{i}}_\Diamond$
$\mathsf{S.WMN} := \mathsf{S.WM} + \mathsf{N}^{\mathsf{i}}_\Box + \mathsf{N}^{\mathsf{i}}_\Diamond$	$\mathsf{S.WMNP} := \mathsf{S.WMN} + \mathsf{P}^{\mathsf{i}}_\Box + \mathsf{P}^{\mathsf{i}}_\Diamond$
$\mathsf{S.WMC} := \mathsf{C}^{\mathsf{i}}_\Box + \mathsf{C}^{\mathsf{i}}_\Diamond + \wedge\text{-}\mathsf{dual}^{\mathsf{i}}_{\mathsf{C}}$	
$\mathsf{S.WK} := \mathsf{K}^{\mathsf{i}}_\Box + \mathsf{K}^{\mathsf{i}}_\Diamond + \wedge\text{-}\mathsf{dual}^{\mathsf{i}}_{\mathsf{K}}$	
$\mathsf{S.WMD} := \mathsf{S.WM} + \mathsf{D}^{\mathsf{i}} + \mathsf{D}^{\mathsf{i}}_\Box + \mathsf{P}^{\mathsf{i}}_\Box + \mathsf{P}^{\mathsf{i}}_\Diamond$	$\mathsf{S.WMT} := \mathsf{S.WM} + \mathsf{T}^{\mathsf{i}}_\Box + \mathsf{T}^{\mathsf{i}}_\Diamond$
$\mathsf{S.WMND} := \mathsf{S.WMN} + \mathsf{D}^{\mathsf{i}} + \mathsf{D}^{\mathsf{i}}_\Box + \mathsf{P}^{\mathsf{i}}_\Box + \mathsf{P}^{\mathsf{i}}_\Diamond$	$\mathsf{S.WMNT} := \mathsf{S.WMN} + \mathsf{T}^{\mathsf{i}}_\Box + \mathsf{T}^{\mathsf{i}}_\Diamond$
$\mathsf{S.WMCD} := \mathsf{S.WMC} + \mathsf{CD}^{\mathsf{i}} + \mathsf{CD}^{\mathsf{i}}_\Box$	$\mathsf{S.WMCT} := \mathsf{S.WMC} + \mathsf{T}^{\mathsf{i}}_\Box + \mathsf{T}^{\mathsf{i}}_\Diamond$
$\mathsf{S.WKD} := \mathsf{S.WK} + \mathsf{CD}^{\mathsf{i}} + \mathsf{CD}^{\mathsf{i}}_\Box$	$\mathsf{S.WKT} := \mathsf{S.WK} + \mathsf{T}^{\mathsf{i}}_\Box + \mathsf{T}^{\mathsf{i}}_\Diamond$

Note that the modal rules of $\mathsf{S.WK}$ coincide with those of Wijesekera [38] (except that they have side context in the conclusion in order to embed weakening in their application). $\mathsf{S.WK}$ coincides with the calculus $\mathsf{G.CCDL}^{\mathsf{p}}$ for WK proposed in [9].

From the point of view of the derivable principles, we observe two main consequences of the restriction of the calculi to single-succedent sequents. First, the rule $Rdual_\vee$ is no longer derivable in the calculi. This is due to the absence of $\vee$-$\mathsf{dual}_{\mathsf{M}}$ and $\vee$-$\mathsf{dual}_{\mathsf{C}}$, and the elimination of $\Diamond$-formulas from the conclusion of $\mathsf{K}^{\mathsf{i}}_\Box$. Second, $C_\Diamond$ is not derivable in $\mathsf{S.WMC}$, $\mathsf{S.WK}$ and their extensions, this is due to the restriction of $\mathsf{C}^{\mathsf{i}}_\Diamond$ and $\mathsf{K}^{\mathsf{i}}_\Diamond$ to only one $\Diamond$-formula in the right-hand side of the conclusion. By contrast, all other modal principles from Fig. 1 are still derivable in the corresponding calculi (cf. derivations in Fig. 5).

In the following, we denote with $\mathsf{S.W}^*$ any constructive calculus defined above. As usual, we say that a rule is *admissible* in $\mathsf{S.W}^*$ if whenever the premisses are derivable, the conclusion is also derivable, and that a single-premiss rule is *height-preserving admissible* if whenever the premiss is derivable, the conclusion is derivable with a derivation of at most the same height. We now prove that the calculi $\mathsf{S.W}^*$ enjoy admissibility of structural rules and cut, then we present equivalent axiomatic systems.

Proposition 3.1 (Admissibility of structural rules) *The following rules are height-preserving admissible in* $\mathsf{S.W}^*$*:*

$$\mathsf{Lwk}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \mathsf{Rwk}\ \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A} \qquad \mathsf{ctr}\ \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ .$$

Proof. Height-preserving admissibility of Lwk, Rwk, and ctr is proved by induction on the height of the derivation of their premiss, taking into account the last rule applied in the derivation. For Lwk and Rwk the proof is straight-

$$\dfrac{\dfrac{\dfrac{\dfrac{A, A \supset \bot \Rightarrow A \qquad A, \bot \Rightarrow}{A, A \supset \bot \Rightarrow}\supset^{\mathsf{i}}_{\mathsf{L}}}{\Box A, \Diamond(A \supset \bot) \Rightarrow \bot}\wedge\text{-}\mathsf{dual}^{\mathsf{i}}_{\mathsf{M}}}{\Box A \wedge \Diamond(A \supset \bot) \Rightarrow \bot}\wedge^{\mathsf{i}}_{\mathsf{L}}}{\Rightarrow \Box A \wedge \Diamond(A \supset \bot) \supset \bot}\supset^{\mathsf{i}}_{\mathsf{R}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{A, B \Rightarrow A \qquad A, B \Rightarrow B}{A, B \Rightarrow A \wedge B}\wedge^{\mathsf{i}}_{\mathsf{R}}}{\Box A, \Box B \Rightarrow \Box(A \wedge B)}\mathsf{C}^{\mathsf{i}}_{\Box}}{\Box A \wedge \Box B \Rightarrow \Box(A \wedge B)}\wedge^{\mathsf{i}}_{\mathsf{L}}}{\Rightarrow \Box A \wedge \Box B \supset \Box(A \wedge B)}\supset^{\mathsf{i}}_{\mathsf{R}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \supset B, A \Rightarrow A \qquad B, A \Rightarrow B}{A \supset B, A \Rightarrow B}\supset^{\mathsf{i}}_{\mathsf{L}}}{\Box(A \supset B), \Box A \Rightarrow \Box B}\mathsf{C}^{\mathsf{i}}_{\Box}}{\Box(A \supset B) \Rightarrow \Box A \supset \Box B}\supset^{\mathsf{i}}_{\mathsf{R}}}{\Rightarrow \Box(A \supset B) \supset (\Box A \supset \Box B)}\supset^{\mathsf{i}}_{\mathsf{R}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{A \supset B, A \Rightarrow A \qquad B, A \Rightarrow B}{A \supset B, A \Rightarrow B}\supset^{\mathsf{i}}_{\mathsf{L}}}{\Box(A \supset B), \Diamond A \Rightarrow \Diamond B}\mathsf{C}^{\mathsf{i}}_{\Diamond}}{\Box(A \supset B) \Rightarrow \Diamond A \supset \Diamond B}\supset^{\mathsf{i}}_{\mathsf{R}}}{\Rightarrow \Box(A \supset B) \supset (\Diamond A \supset \Diamond B)}\supset^{\mathsf{i}}_{\mathsf{R}}$$

$$\dfrac{\dfrac{\bot \Rightarrow \bot}{\Rightarrow \bot \supset \bot}\supset^{\mathsf{i}}_{\mathsf{R}}}{\Rightarrow \Box(\bot \supset \bot)}\mathsf{N}^{\mathsf{i}}_{\Box} \qquad \dfrac{\dfrac{\bot \Rightarrow}{\Diamond\bot \Rightarrow \bot}\mathsf{N}^{\mathsf{i}}_{\Diamond}}{\Rightarrow \Diamond\bot \supset \bot}\supset^{\mathsf{i}}_{\mathsf{R}} \qquad \dfrac{\dfrac{\Box A, A \Rightarrow A}{\Box A \Rightarrow A}\mathsf{T}^{\mathsf{i}}_{\Box}}{\Rightarrow \Box A \supset A}\supset^{\mathsf{i}}_{\mathsf{R}} \qquad \dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow \Diamond A}\mathsf{T}^{\mathsf{i}}_{\Diamond}}{\Rightarrow A \supset \Diamond A}\supset^{\mathsf{i}}_{\mathsf{R}}$$

$$\dfrac{\dfrac{A \Rightarrow A}{\Box A \Rightarrow \Diamond A}\mathsf{D}^{\mathsf{i}}}{\Rightarrow \Box A \supset \Diamond A}\supset^{\mathsf{i}}_{\mathsf{R}} \qquad \dfrac{\dfrac{\bot \Rightarrow}{\Box\bot \Rightarrow \bot}\mathsf{P}^{\mathsf{i}}_{\Box}}{\Rightarrow \Box\bot \supset \bot}\supset^{\mathsf{i}}_{\mathsf{R}} \qquad \dfrac{\dfrac{\bot \Rightarrow \bot}{\Rightarrow \bot \supset \bot}\supset^{\mathsf{i}}_{\mathsf{R}}}{\Rightarrow \Diamond(\bot \supset \bot)}\mathsf{P}^{\mathsf{i}}_{\Diamond}$$

Fig. 5. Derivations of the modal axioms.

forward, we consider some examples for ctr involving the modal rules. The derivations on the left are converted into the derivations on the right, which include applications of ctr which are height-preserving admissible by i.h.:

$$\mathsf{D}^{\mathsf{i}}_{\Box}\,\dfrac{A, A \Rightarrow}{\Gamma, \Box A, \Box A \Rightarrow \Delta} \quad \rightsquigarrow \quad \dfrac{\dfrac{A, A \Rightarrow}{A \Rightarrow}\mathsf{ctr}}{\Gamma, \Box A \Rightarrow \Delta}\mathsf{P}^{\mathsf{i}}_{\Box}$$

$$\mathsf{T}^{\mathsf{i}}_{\Box}\,\dfrac{\Gamma, \Box A, \Box A, A \Rightarrow \Delta}{\Gamma, \Box A, \Box A \Rightarrow \Delta} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, \Box A, \Box A, A \Rightarrow \Delta}{\Gamma, \Box A, A \Rightarrow \Delta}\mathsf{ctr}}{\Gamma, \Box A \Rightarrow \Delta}\mathsf{T}^{\mathsf{i}}_{\Box}$$

$$\mathsf{C}^{\mathsf{i}}_{\Diamond}\,\dfrac{\Gamma, A, A, B \Rightarrow C}{\Gamma', \Box\Gamma, \Box A, \Box A, \Diamond B \Rightarrow \Diamond C} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A, A, B \Rightarrow C}{\Gamma, A, B \Rightarrow C}\mathsf{ctr}}{\Gamma', \Box\Gamma, \Box A, \Diamond B \Rightarrow \Diamond C}\mathsf{C}^{\mathsf{i}}_{\Diamond}$$

$$\mathsf{C}^{\mathsf{i}}_{\Diamond}\,\dfrac{\Gamma, A, B \Rightarrow C}{\Gamma', \Box A, \Box\Gamma, \Box A, \Diamond B \Rightarrow \Diamond C} \quad \rightsquigarrow \quad \dfrac{\Gamma, A, B \Rightarrow C}{\Gamma', \Box\Gamma, \Box A, \Diamond B \Rightarrow \Diamond C}\mathsf{C}^{\mathsf{i}}_{\Diamond}$$

In the last example, one occurrence of $\Box A$ is introduced by the first application of $\mathsf{C}^{\mathsf{i}}_{\Diamond}$ as part of the side context, while this is not the case in the second application of $\mathsf{C}^{\mathsf{i}}_{\Diamond}$. □

Theorem 3.2 (Cut admissibility) *The following cut rule is admissible in* $\mathsf{S.W}^*$*:*

$$\mathsf{cut}\,\dfrac{\Gamma \Rightarrow A \qquad \Gamma', A \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta}\ .$$

Proof. By induction on lexicographically ordered pairs (c, h), where c is the complexity of the cut formula (i.e., the number of binary connectives or modalities occurring in it), and $h = h_1 + h_2$, called cut height, is the sum of the heights of the derivations of the premisses of cut. As usual, we distinguish some cases according to whether the cut formula is or not principal in the last rules applied in the derivation of the premisses of cut. For the cases where the last rules applied in the derivation of the premisses of cut are propositional we refer to [37, Ch. 4]. Here we only show a few most relevant cases involving modal rules, the other cases are similar.

(i) The cut formula is not principal in the last rule application in the derivation of the left premiss of cut. We consider the following two examples, where the derivation on the left is converted into the derivation on the right:

$$\dfrac{\dfrac{A, B \Rightarrow}{\Gamma, \Box A, \Diamond B \Rightarrow C}\,\wedge\text{-dual}^{\mathsf{i}}_{\mathsf{M}} \qquad \Gamma', C \Rightarrow \Delta}{\Gamma, \Gamma', \Box A, \Diamond B \Rightarrow \Delta}\,\mathsf{cut} \quad \rightsquigarrow \quad \dfrac{A, B \Rightarrow}{\Gamma, \Gamma', \Box A, \Diamond B \Rightarrow \Delta}\,\wedge\text{-dual}^{\mathsf{i}}_{\mathsf{M}}$$

$$\dfrac{\dfrac{\Gamma, \Box A, A \Rightarrow B}{\Gamma, \Box A \Rightarrow B}\,\mathsf{T}^{\mathsf{i}}_{\Box} \qquad \Gamma', B \Rightarrow \Delta}{\Gamma, \Gamma', \Box A \Rightarrow \Delta}\,\mathsf{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, \Box A, A \Rightarrow B \qquad \Gamma', B \Rightarrow \Delta}{\Gamma, \Gamma', \Box A, A \Rightarrow \Delta}\,\mathsf{cut}}{\Gamma, \Gamma', \Box A \Rightarrow \Delta}\,\mathsf{T}^{\mathsf{i}}_{\Box}$$

(ii) The cut formula is not principal in the last rule application in the derivation of the right premiss of cut. We consider the following example:

$$\dfrac{\Gamma \Rightarrow A \qquad \dfrac{\Gamma'', B \Rightarrow C}{\Gamma', A, \Box\Gamma'', \Diamond B \Rightarrow \Diamond C}\,\mathsf{C}^{\mathsf{i}}_{\Diamond}}{\Gamma, \Gamma', \Box\Gamma'', \Diamond B \Rightarrow \Diamond C}\,\mathsf{cut} \quad \rightsquigarrow \quad \dfrac{\Gamma'', B \Rightarrow C}{\Gamma, \Gamma', \Box\Gamma'', \Diamond B \Rightarrow \Diamond C}\,\mathsf{C}^{\mathsf{i}}_{\Diamond}$$

(iii) The cut formula is principal in the last rule application in the derivations of both premisses of cut. We consider the following three examples, where R^* denotes multiple applications of the rule R:

($\mathsf{C}^{\mathsf{i}}_{\Diamond}$; $\wedge\text{-dual}^{\mathsf{i}}_{\mathsf{C}}$)

$$\dfrac{\dfrac{\Gamma, A \Rightarrow B}{\Gamma', \Box\Gamma, \Diamond A \Rightarrow \Diamond B}\,\mathsf{C}^{\mathsf{i}}_{\Diamond} \qquad \dfrac{\Gamma'', C, B \Rightarrow}{\Gamma''', \Box\Gamma'', \Box C, \Diamond B \Rightarrow \Delta}\,\wedge\text{-dual}^{\mathsf{i}}_{\mathsf{C}}}{\Gamma', \Gamma''', \Box\Gamma, \Box\Gamma'', \Box C, \Diamond A \Rightarrow \Delta}\,\mathsf{cut}$$

$$\wr$$

$$\dfrac{\dfrac{\Gamma, A \Rightarrow B \qquad \Gamma'', C, B \Rightarrow}{\Gamma, \Gamma'', C, A \Rightarrow}\,\mathsf{cut}}{\Gamma', \Gamma''', \Box\Gamma, \Box\Gamma'', \Box C, \Diamond A \Rightarrow \Delta}\,\wedge\text{-dual}^{\mathsf{i}}_{\mathsf{C}}$$

($\mathsf{T}^{\mathsf{i}}_{\Diamond}$; $\mathsf{C}^{\mathsf{i}}_{\Diamond}$)

$$\dfrac{\dfrac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \Diamond A}\,\mathsf{T}^{\mathsf{i}}_{\Diamond} \qquad \dfrac{\Gamma', A \Rightarrow B}{\Gamma'', \Box\Gamma', \Diamond A \Rightarrow \Diamond B}\,\mathsf{C}^{\mathsf{i}}_{\Diamond}}{\Gamma, \Gamma'', \Box\Gamma' \Rightarrow \Diamond B}\,\mathsf{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow A \qquad \Gamma', A \Rightarrow B}{\Gamma, \Gamma' \Rightarrow B}\,\mathsf{cut}}{\Gamma, \Gamma' \Rightarrow \Diamond B}\,\mathsf{T}^{\mathsf{i}}_{\Diamond}}{\Gamma, \Box\Gamma' \Rightarrow \Diamond B}\,{\mathsf{T}^{\mathsf{i}}_{\Box}}^{*}}{\Gamma, \Gamma'', \Box\Gamma' \Rightarrow \Diamond B}\,\mathsf{Lwk}^{*}$$

($\mathsf{N}^{\mathsf{i}}_{\Box}$; D^{i})

$$\dfrac{\dfrac{\Rightarrow A}{\Gamma \Rightarrow \Box A}\,\mathsf{N}^{\mathsf{i}}_{\Box} \qquad \dfrac{A \Rightarrow B}{\Gamma', \Box A \Rightarrow \Diamond B}\,\mathsf{D}^{\mathsf{i}}}{\Gamma, \Gamma' \Rightarrow \Diamond B}\,\mathsf{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Rightarrow A \qquad A \Rightarrow B}{\Rightarrow B}\,\mathsf{cut}}{\Gamma, \Gamma' \Rightarrow \Diamond B}\,\mathsf{P}^{\mathsf{i}}_{\Diamond}$$

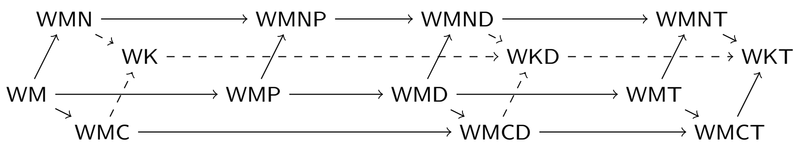

Fig. 6. Dyagram of constructive modal logics.

□

3.1 Axiom systems

For each constructive calculus S.WL, we now define an equivalent axiomatic system. The logics WL are defined in the language $\mathcal{L}$ extending (any axiomatisation of) IPL with the following modal axioms and rules from Fig. 1:

WM := $dual_\wedge + mon_\Box + mon_\Diamond$	WMND := WMN + D
WMN := WM + $N_\Box$	WMCD := WMC + D + $P_\Diamond$
WMC := WM + $C_\Box$ + $K_\Diamond$	WKD := WK + D
WK := WMC + $N_\Box$	WMT := WM + $T_\Box$ + $T_\Diamond$
WMP := WM + $P_\Diamond$	WMNT := WMN + $T_\Box$ + $T_\Diamond$
WMNP := WMN + $P_\Diamond$	WMCT := WMC + $T_\Box$ + $T_\Diamond$
WMD := WM + D + $P_\Diamond$	WKT := WK + $T_\Box$ + $T_\Diamond$

In the following, we will refer to these systems as *W-logics*. Moreover, we denote W* any W-logic, and we denote WC*, resp. WD*, resp. WT* any W-logic with axioms $C_\Box$ and $K_\Diamond$, resp. with axiom D, resp. with axioms $T_\Box$ and $T_\Diamond$. As usual, we say that A is a theorem of W*, written W$^* \vdash A$, if there is a finite sequence of formulas ending with A in which every formula is an axiom of W*, or it is obtained from previous formulas by the application of a rule of W*. Moreover, we say that A is deducible in W* from a set of formulas Σ, written $\Sigma \vdash_{\mathsf{W}^*} A$, if there is a finite set $\{B_1, ..., B_n\} \subseteq \Sigma$ such that $\vdash_{\mathsf{W}^*} B_1 \wedge ... \wedge B_n \supset A$. Furthermore, given two axiomatic systems L$_1$ and L$_2$, we say that L$_1$ is included in L$_2$ if L$_1 \vdash A$ entails L$_2 \vdash A$ for all $A \in \mathcal{L}$, and that L$_1$ and L$_2$ are equivalent if they derive exactly the same theorems.

Concerning W-logics specifically, note that $C_\Diamond$ is not an axiom of WMC because it is not derivable in S.WMC, $C_\Diamond$ must be replaced by $K_\Diamond$ which is instead derivable in the calculus. Note also that $P_\Diamond$ must be included in the axiomatisation of WMD and WMCD as it is not derivable from D in the intuitionistic systems. The relations among the W-logics are displayed in Fig. 6.

We prove some basic results about W-logics.

Proposition 3.3 *(i)* WM $\vdash Rdual_\wedge$. *(ii)* WMN $\vdash nec$. *(iii)* WMN $\vdash N_\Diamond$. *(iv)* WMC $\vdash K_\Box$. *(v)* WMP $\vdash P_\Box$. *(vi)* WMND $\vdash P_\Diamond$.

Proof. (i) From $\neg(A \wedge B)$, by IPL we obtain $B \supset \neg A$, then by $mon_\Diamond$, $\Diamond B \supset \Diamond\neg A$, thus $\neg\Diamond\neg A \supset \neg\Diamond B$. Moreover from $dual_\wedge$ we have $\Box A \supset \neg\Diamond\neg A$, thus $\Box A \supset \neg\Diamond B$, therefore $\neg(\Box A \wedge \Diamond B)$. (ii) From A, by IPL we have $\top \supset A$, then by $mon_\Box$, $\Box\top \supset \Box A$, then by $N_\Box$, $\Box A$. (iii) By $Rdual_\wedge$, $\Box\top \supset \neg\Diamond\bot$, then by $N_\Box$, $\neg\Diamond\bot$. (iv) By $C_\Box$, $\Box(A \supset B) \wedge \Box A \supset \Box((A \supset B) \wedge A)$, and by $mon_\Box$,

$\Box((A \supset B) \wedge A) \supset \Box B$, thus by IPL, $\Box(A \supset B) \supset (\Box A \supset \Box B)$. (v) By $dual_\wedge$, $\Diamond\top \supset \neg\Box\bot$, then by $P_\Diamond$, $\neg\Box\bot$. (vi) By D, $\Box\top \supset \Diamond\top$, then by $N_\Box$, $\Diamond\top$. □

We recall that Wijesekera's original axiomatisation of WK [38] was given by $\mathsf{IPL} + nec + K_\Box + K_\Diamond + N_\Diamond$. It is easy to verify that $dual_\wedge$, $mon_\Box$, $mon_\Diamond$, $C_\Box$, and $N_\Box$ are all derivable in Wijesekera's original system. Then from Proposition 3.3 it follows in particular that the axiomatisation of WK considered here is equivalent to Wijesekera's one.

We now prove that the systems W^* are equivalent to the corresponding calculi.

Theorem 3.4 *$\mathsf{S.W}^* \vdash \Gamma \Rightarrow \Delta$ if and only if $\mathsf{W}^* \vdash \bigwedge\Gamma \supset \bigvee\Delta$.*

Proof. From right to left, it is easy to see that all modal axioms are derivable in the corresponding calculi (cf. derivations in Fig. 5), observing that initial sequents init can be generalised as usual to arbitrary formulas A. For $mon_\Box$, from $\Rightarrow A \supset B$ and $A \supset B, A \Rightarrow B$, by cut (which has been proved admissible) we obtain $A \Rightarrow B$, then by $\mathsf{M}^{\mathsf{i}}_\Box$, $\Box A \Rightarrow \Box B$, and by $\supset^{\mathsf{i}}_{\mathsf{R}}$, $\Rightarrow \Box A \supset \Box B$. $mon_\Diamond$ is derived similarly. The derivations of the intuitionistic axioms are standard, moreover modus ponens is simulated by cut in the usual way.

For the other direction, we can consider standard derivations of the propositional rules. Here we show that for every modal sequent rule of $\mathsf{S.W}^*$ with premiss $\Gamma \Rightarrow \Delta$ and conclusion $\Gamma' \Rightarrow \Delta'$, the Hilbert-style rule $\bigwedge\Gamma \supset \bigvee\Delta$ / $\bigwedge\Gamma' \supset \bigvee\Delta'$ is derivable in the corresponding system W^*. We only consider some relevant examples, the other derivations are similar.

($\mathsf{C}^{\mathsf{i}}_\Box$) From $\bigwedge\Gamma \wedge A \supset B$, by $mon_\Box$ we get $\Box(\bigwedge\Gamma \wedge A) \supset \Box B$, moreover by $C_\Box$, $\bigwedge\Box\Gamma \wedge \Box A \supset \Box(\bigwedge\Gamma \wedge A)$, then $\bigwedge\Gamma' \wedge \bigwedge\Box\Gamma \wedge \Box A \supset \Box B$.

($\mathsf{C}^{\mathsf{i}}_\Diamond$) From $\bigwedge\Gamma \wedge A \supset B$, we get $\bigwedge\Gamma \supset (A \supset B)$, then by $mon_\Box$, $\Box\bigwedge\Gamma \supset \Box(A \supset B)$. By $C_\Box$ we have $\bigwedge\Box\Gamma \supset \Box(A \supset B)$, then by $K_\Diamond$, $\bigwedge\Box\Gamma \supset (\Diamond A \supset \Diamond B)$, thus $\bigwedge\Gamma' \wedge \bigwedge\Box\Gamma \wedge \Diamond A \supset \Diamond B$.

($\wedge$-$\mathsf{dual}^{\mathsf{i}}_{\mathsf{C}}$) From $\bigwedge\Gamma \wedge A \wedge B \supset \bot$, we get $\bigwedge\Gamma \wedge A \supset \neg B$, then by $mon_\Box$, $\Box(\bigwedge\Gamma \wedge A) \supset \Box\neg B$, and by $C_\Box$, $\bigwedge\Box\Gamma \wedge \Box A \supset \Box\neg B$. Moreover by $dual_\wedge$, $\Box\neg B \supset \neg\Diamond B$, thus $\bigwedge\Gamma' \wedge \bigwedge\Box\Gamma \wedge \Box A \wedge \Diamond B \supset C$ for any C.

(CD^{i}) If $\Gamma \neq \emptyset$, then from $\bigwedge\Gamma \supset B$, by $mon_\Box$ we get $\Box\bigwedge\Gamma \supset \Box B$, then by $C_\Box$, $\bigwedge\Box\Gamma \supset \Box B$, and by D, $\bigwedge\Box\Gamma \supset \Diamond B$, thus $\bigwedge\Gamma' \wedge \bigwedge\Box\Gamma \supset \Diamond B$. If $\Gamma = \emptyset$, then from B we get $\top \supset B$, then by $mon_\Diamond$, $\Diamond\top \supset \Diamond B$, and by $P_\Diamond$, $\Diamond B$, thus $\bigwedge\Gamma' \supset \Diamond B$.

($\mathsf{CD}^{\mathsf{i}}_\Box$) From $\bigwedge\Gamma \supset \bot$ we get $\bigwedge\Gamma'' \supset \neg B$ for some $B \in \Gamma$ and $\Gamma'' = \Gamma \setminus B$. Then by $mon_\Diamond$, $\Diamond\bigwedge\Gamma'' \supset \Diamond\neg B$, thus by D, $\Box\bigwedge\Gamma'' \supset \Diamond\neg B$, by $C_\Box$, $\bigwedge\Box\Gamma'' \supset \Diamond\neg B$, and by $dual_\wedge$, $\bigwedge\Box\Gamma'' \supset \neg\Box B$, thus $\bigwedge\Gamma' \wedge \bigwedge\Box\Gamma \supset C$ for any C.

□

3.2 Some properties of W-logics

Cut-free sequent calculi are a very powerful tool for the analysis of logical systems. In this subsection, we present some fundamental properties of W-logics that easily follow from those of their sequent calculi.

We start considering the following result, which establishes how classical and Wijesekera-style modal logics are related from the point of view of the axiomatic systems.

Theorem 3.5 *Let* L *be any classical modal logic from Sec. 2, and* WL *be the corresponding W-logic. Then* L *is equivalent to* $\mathsf{WL} + A \vee \neg A + \Box A \vee \Diamond\neg A$.

Proof. From left to right, it is easy to verify that all axioms of WL, as well as $A \vee \neg A$ and $\Box A \vee \Diamond\neg A$, are derivable in L. For the opposite direction, adding $A \vee \neg A$ one derives as usual all theorems of CPL, while adding $\Box A \vee \Diamond\neg A$ one derives *dual*. □

Note that $\mathsf{WL} + A \vee \neg A + \Box A \vee \Diamond\neg A$ is a proper extension of WL, as it is stated by the following proposition.

Proposition 3.6 *For every W-logic* W^*, $\mathsf{W}^* \nvdash p \vee \neg p$ *and* $\mathsf{W}^* \nvdash \Box p \vee \Diamond\neg p$.

Proof. If $\mathsf{W}^* \vdash p \vee \neg p$, then by Theorem 3.4, $\mathsf{S.W}^* \vdash \Rightarrow p \vee \neg p$. The only rule of $\mathsf{S.W}^*$ with a consequence of the form $\Rightarrow p \vee \neg p$ is $\vee^{\mathsf{i}}_{\mathsf{R}}$, thus $\vee^{\mathsf{i}}_{\mathsf{R}}$ must be the last rule applied in the derivation, with premiss either $\Rightarrow p$ or $\Rightarrow \neg p$. However, by inspecting the rules of $\mathsf{S.W}^*$ it is easy to verify that $\mathsf{S.W}^* \nvdash \Rightarrow p$ and $\mathsf{S.W}^* \nvdash \Rightarrow \neg p$, therefore $\mathsf{W}^* \nvdash p \vee \neg p$. $\mathsf{W}^* \nvdash \Box p \vee \Diamond\neg p$ is proved similarly, observing that $\mathsf{S.W}^* \nvdash \Rightarrow \Box p$ and $\mathsf{S.W}^* \nvdash \Rightarrow \Diamond\neg p$. □

In a similar way we can prove that W-logics satisfy the disjunction property.

Proposition 3.7 (Disjunction property) *For all W-logics* W^* *and all formulas* A, B *of* $\mathcal{L}$, *if* $\mathsf{W}^* \vdash A \vee B$, *then* $\mathsf{W}^* \vdash A$ *or* $\mathsf{W}^* \vdash B$.

Proof. If $\mathsf{W}^* \vdash A \vee B$, then by Theorem 3.4, $\mathsf{S.W}^* \vdash \Rightarrow A \vee B$. The only rule of $\mathsf{S.W}^*$ with a consequence of the form $\Rightarrow A \vee B$ is $\vee^{\mathsf{i}}_{\mathsf{R}}$, thus $\vee^{\mathsf{i}}_{\mathsf{R}}$ must be the last rule applied in the derivation, with premiss either $\Rightarrow A$ or $\Rightarrow B$. Then $\mathsf{S.W}^* \vdash \Rightarrow A$ or $\mathsf{S.W}^* \vdash \Rightarrow B$, therefore $\mathsf{W}^* \vdash A$ or $\mathsf{W}^* \vdash B$. □

Furthermore, we can prove that derivability in W-logics is decidable. To this aim, observe that for every rule R of $\mathsf{S.W}^*$, the premisses of R have a smaller complexity than its conclusion, with the only exceptions of $\supset_{\mathsf{L}}$ and $\mathsf{T}^{\mathsf{i}}_{\Box}$ which copy the principal formula into one premiss. It follows that bottom-up proof search in $\mathsf{S.W}^*$ is not strictly terminating, however, similarly to [37, Ch. 4], termination can be gained by controlling the applications of $\supset_{\mathsf{L}}$ and $\mathsf{T}^{\mathsf{i}}_{\Box}$ with a simple loop-checking in order to avoid redundant applications of these rules, preserving at the same time the completeness of the calculi. Adopting this restriction, it turns out that every proof tree for a root sequent $\Gamma \Rightarrow \Delta$ is finite, moreover there are only finitely many distinct proof trees for it. Then, given the equivalence between $\mathsf{S.W}^*$ and W^*, it follows that derivability of A in W^* is decidable for any A: the decision procedure trivially consists in checking all possible derivations of $\Rightarrow A$ in $\mathsf{S.W}^*$.

Theorem 3.8 (Decidability) *Given a W-logic* W^* *and a formula* A *of* $\mathcal{L}$, *it is decidable whether* A *is derivable in* W^*.

Finally, we can prove that all W-logics enjoy Craig interpolation. For every formula A of $\mathcal{L}$ and every multiset $\Gamma = B_1, ..., B_n$, we define $var(A) = \{\bot\} \cup \{p \in Atm \mid p \text{ occurs in } A\}$, and $var(\Gamma) = var(B_1) \cup ... \cup var(B_n)$. Then Craig interpolation amounts to the following property.

Definition 3.9 A logic W^* enjoys *Craig interpolation* if for all $A, B \in \mathcal{L}$, if $\mathsf{W}^* \vdash A \supset B$, then there is $C \in \mathcal{L}$ such that $\mathsf{W}^* \vdash A \supset C$, $\mathsf{W}^* \vdash C \supset B$, and $var(C) \subseteq var(A) \cap var(B)$.

The proof of Craig interpolation is based on the following lemma.

Lemma 3.10 *For every calculus* $\mathsf{S.W}^*$, *if* $\mathsf{S.W}^* \vdash \Gamma_1, \Gamma_2 \Rightarrow \Delta$, *then there is* $C \in \mathcal{L}$ *such that* $\mathsf{S.W}^* \vdash \Gamma_1 \Rightarrow C$, $\mathsf{S.W}^* \vdash C, \Gamma_2 \Rightarrow \Delta$, *and* $var(C) \subseteq var(\Gamma_1) \cap var(\Gamma_2, \Delta)$.

Proof. By induction on the height h of the derivation of $\Gamma_1, \Gamma_2 \Rightarrow \Delta$, taking into account the last rule applied in the derivation. If $h = 0$ or the last rule applied is propositional we refer to [30]. Here we consider just one significant case involving a modal rule, for the other rules the proof is analogous.

Let $\mathsf{C}^{\mathsf{i}}_{\Diamond}$ be the last rule applied in the derivation. Then $\Gamma_1, \Gamma_2 \Rightarrow \Delta$ has the form $\Gamma_1', \Box\Gamma, \Diamond A, \Gamma_2' \Rightarrow \Diamond B$ and it is obtained from the premiss $\Gamma, A \Rightarrow B$. There are four possible partitions of $\Gamma_1', \Box\Gamma, \Diamond A, \Gamma_2'$ into Γ_1, Γ_2.

(i) $\Gamma_1 = \Gamma_1'$ and $\Gamma_2 = \Box\Gamma, \Diamond A, \Gamma_2'$. Then $\top$ is an interpolant: $\Gamma_1' \Rightarrow \top$ is derivable, and from $\Gamma, A \Rightarrow B$, by $\mathsf{C}^{\mathsf{i}}_{\Diamond}$ we obtain $\top, \Box\Gamma, \Diamond A, \Gamma_2' \Rightarrow \Diamond B$.

(ii) $\Gamma_1 = \Gamma_1', \Box\Gamma, \Diamond A$ and $\Gamma_2 = \Gamma_2'$. By i.h., there is C such that $\Gamma, A \Rightarrow C$ and $C \Rightarrow B$ are derivable, and $var(C) \subseteq var(\Gamma, A) \cap var(B)$. Then by $\mathsf{C}^{\mathsf{i}}_{\Diamond}$ we obtain $\Gamma_1', \Box\Gamma, \Diamond A \Rightarrow \Diamond C$ and $\Diamond C, \Gamma_2' \Rightarrow \Diamond B$. Since $var(\Diamond C) = var(C)$, $\Diamond C$ is an interpolant.

The following two partitions are possible if $\Gamma = D_1, ..., D_n$ and $n \geq 2$.

(iii) $\Gamma_1 = \Gamma_1', \Box D_1, ..., \Box D_k$ and $\Gamma_2 = \Box D_{k+1}, ..., \Box D_n, \Diamond A, \Gamma_2'$ (for $1 \leq k < n$). By i.h., there is C such that $D_1, ..., D_k \Rightarrow C$ and $C, D_{k+1}, ..., D_n, A \Rightarrow B$ are derivable, and $var(C) \subseteq var(D_1, ..., D_k) \cap var(D_{k+1}, ..., D_n, A, B)$. Then by $\mathsf{C}^{\mathsf{i}}_{\Box}$, $\Gamma_1', \Box D_1, ..., \Box D_k \Rightarrow \Box C$ is derivable, and by $\mathsf{C}^{\mathsf{i}}_{\Diamond}$, $\Box C, \Box D_{k+1}, ..., \Box D_n, \Diamond A, \Gamma_2' \Rightarrow \Diamond B$ is derivable. Then $\Box C$ is an interpolant.

(iv) $\Gamma_1 = \Gamma_1', \Box D_1, ..., \Box D_k, \Diamond A$ and $\Gamma_2 = \Box D_{k+1}, ..., \Box D_n, \Gamma_2'$ (for $1 \leq k < n$). By i.h., there is C such that $D_1, ..., D_k, A \Rightarrow C$ and $C, D_{k+1}, ..., D_n \Rightarrow B$ are derivable, and $var(C) \subseteq var(D_1, ..., D_k, A) \cap var(D_{k+1}, ..., D_n, B)$. Then by $\mathsf{C}^{\mathsf{i}}_{\Diamond}$, $\Gamma_1', \Box D_1, ..., \Box D_k, \Diamond A \Rightarrow \Diamond C$ and $\Diamond C, \Box D_{k+1}, ..., \Box D_n, \Gamma_2' \Rightarrow \Diamond B$ are derivable. Then $\Diamond C$ is an interpolant.

□

Theorem 3.11 *Every W-logic* W^* *enjoys Craig interpolation.*

Proof. Suppose that $\mathsf{W}^* \vdash A \supset B$. Then $\mathsf{S.W}^* \vdash A \Rightarrow B$. By Lemma 3.10, there is $C \in \mathcal{L}$ such that $var(C) \subseteq var(A) \cap var(B)$, $\mathsf{S.W}^* \vdash A \Rightarrow C$, and $\mathsf{S.W}^* \vdash C \Rightarrow B$, thus $\mathsf{W}^* \vdash A \supset C$ and $\mathsf{W}^* \vdash C \supset B$. □

4 Semantics

We now define constructive neighbourhood models (CNMs) that characterise the constructive modal logics defined in Sec. 3. CNMs are defined analogously to Wijesekera's relational models [38]: we enrich intuitionistic Kripke models with a neighbourhood function (rather than a binary relation as in Wijesekera's models), moreover we generalise the classical satisfaction clauses for modal formulas to all $\leq$-successors, so that hereditariness is built into the clauses: in order that w satisfies $\Box A$, we require that for all successors of w there is a neighbourhood α such that $\alpha \Vdash^{\forall} A$, and similarly for $\Diamond A$. We show that, for every classical logic characterised by neighbourhood models satisfying some conditions from Def. 2.2, the corresponding W-logic is characterised by the CNMs satisfying exactly the same conditions. CNMs are defined as follows.

Definition 4.1 A *constructive neighbourhood model* (CNM) is a tuple $\mathcal{M} = \langle \mathcal{W}, \leq, \mathcal{N}, \mathcal{V} \rangle$, where $\mathcal{W}$ is a non-empty set of worlds, $\leq$ is a preorder on $\mathcal{W}$, $\mathcal{N} : \mathcal{P}(\mathcal{W}) \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}))$ is a neighbourhood function, and $\mathcal{V} : Atm \longrightarrow \mathcal{P}(\mathcal{W})$ is a hereditary valuation function (i.e., if $w \in \mathcal{V}(p)$ and $w \leq v$, then $v \in \mathcal{V}(p)$). The forcing relation $\mathcal{M}, w \Vdash A$ is defined as in Def. 2.2 for $A = p, \bot, B \wedge C, B \vee C$, otherwise it is as follows:

$\mathcal{M}, w \Vdash B \supset C$	iff	for all $v \geq w$, $\mathcal{M}, v \Vdash B$ implies $\mathcal{M}, v \Vdash C$.
$\mathcal{M}, w \Vdash \Box B$	iff	for all $v \geq w$, there is $\alpha \in \mathcal{N}(v)$ such that $\alpha \Vdash^{\forall} B$.
$\mathcal{M}, w \Vdash \Diamond B$	iff	for all $v \geq w$, for all $\alpha \in \mathcal{N}(v)$, $\alpha \Vdash^{\exists} B$.

We consider the following properties on CNMs:

(C) If $\alpha, \beta \in \mathcal{N}(w)$, then $\alpha \cap \beta \in \mathcal{N}(w)$.
(N) $\mathcal{N}(w) \neq \emptyset$.
(D) If $\alpha, \beta \in \mathcal{N}(w)$, then $\alpha \cap \beta \neq \emptyset$.
(P) $\emptyset \notin \mathcal{N}(w)$.
(T) If $\alpha \in \mathcal{N}(w)$, then $w \in \alpha$.

We say that $\mathcal{M}$ is a model for a logic W^*, or is a W^*-model, if for every modal axiom X of W^* (among $C_\Box$, $N_\Box$, $T_\Box$, D, $P_\Diamond$), $\mathcal{M}$ satisfies the corresponding condition (X).

CNMs represent the simplest way of combining intuitionistic Kripke models and neighbourhood models. From the definition of $\mathcal{V}$ and of the satisfaction clauses, it immediately follows that CNMs enjoy the hereditary property.

Proposition 4.2 (Hereditary property) *For all $A \in \mathcal{L}$ and all CNMs $\mathcal{M}$, if $\mathcal{M}, w \Vdash A$ and $w \leq v$, then $\mathcal{M}, v \Vdash A$.*

Proof. By induction on the construction of A. We only consider the inductive cases $A = \Box B, \Diamond B$ as the other cases are standard. $(A = \Box B)$ If $w \Vdash \Box B$, then for all $u \geq w$, there is $\alpha \in \mathcal{N}(u)$ such that $\alpha \Vdash^{\forall} B$, then for all $u \geq v$, there is $\alpha \in \mathcal{N}(u)$ such that $\alpha \Vdash^{\forall} B$, thus $v \Vdash \Box B$. $(A = \Diamond B)$ If $w \Vdash \Diamond B$, then for all $u \geq w$, for all $\alpha \in \mathcal{N}(u)$, $\alpha \Vdash^{\exists} B$, then for all $u \geq v$, for all $\alpha \in \mathcal{N}(u)$, $\alpha \Vdash^{\exists} B$, thus $v \Vdash \Diamond B$. $\Box$

We show that, for any classical modal logic characterised by neighbourhood models satisfying some conditions among (C), (N), (T), (D), (P), the corresponding W-logic is characterised by the CNMs satisfying the same conditions. We first show that W-logics are sound with respect to their classes of CNMs, then prove their completeness by a canonical model construction.

Theorem 4.3 (Soundness) *For every W-logic* W^**, if* $\mathsf{W}^* \vdash A$*, then* $\mathcal{M} \models A$ *for all* W^**-models* $\mathcal{M}$*.*

Proof. As usual, we need to show that the axioms of W^* are valid in all W^*-models, and that the rules of W^* preserve the validity in W^*-models.

($\mathsf{M}_\Box$) Assume that $\mathcal{M} \models A \supset B$ and $w \Vdash \Box A$. Then for all $v \geq w$, there is $\alpha \in \mathcal{N}(v)$ such that $\alpha \Vdash^\forall A$. If follows that $\alpha \Vdash^\forall B$. Therefore $w \Vdash \Box B$.

($\mathsf{M}_\Diamond$) Assume that $\mathcal{M} \models A \supset B$ and $w \Vdash \Diamond A$. Then for all $v \geq w$, for all $\alpha \in \mathcal{N}(v)$, $\alpha \Vdash^\exists A$. If follows that $\alpha \Vdash^\exists B$. Therefore $w \Vdash \Diamond B$.

($dual_\wedge$) Assume that $w \Vdash \Box A \wedge \Diamond\neg A$. Then for all $v \geq w$, there is $\alpha \in \mathcal{N}(v)$ such that $\alpha \Vdash^\forall A$, and for all $\beta \in \mathcal{N}(v)$, $\beta \Vdash^\exists \neg A$. Thus there is $\gamma \in \mathcal{N}(w)$ such that $\gamma \Vdash^\exists A \wedge \neg A$, which is impossible. Therefore $\mathcal{M} \models \neg(\Box A \wedge \Diamond\neg A)$.

($C_\Box$) Assume that $\mathcal{M}$ satisfies condition (C) and $w \Vdash \Box A \wedge \Box B$. Then for all $v \geq w$, there is $\alpha \in \mathcal{N}(v)$ such that $\alpha \Vdash^\forall A$, and there is $\beta \in \mathcal{N}(v)$ such that $\beta \Vdash^\forall B$. By (C), $\alpha \cap \beta \in \mathcal{N}(v)$, moreover $\alpha \cap \beta \Vdash^\forall A \wedge B$. Thus $w \Vdash \Box(A \wedge B)$.

($K_\Diamond$) Assume by contradiction that $\mathcal{M}$ satisfies (C), $w \Vdash \Box(A \supset B)$, $w \Vdash \Diamond A$ and $w \nVdash \Diamond B$. By $w \nVdash \Diamond B$, there are $v \geq w$ and $\alpha \in \mathcal{N}(v)$ such that $\alpha \nVdash^\exists B$. Then by $w \Vdash \Box(A \supset B)$, there is $\beta \in \mathcal{N}(v)$ such that $\beta \Vdash^\forall A \supset B$. Thus by (C), $\alpha \cap \beta \in \mathcal{N}(v)$. It follows $\alpha \cap \beta \nVdash^\exists B$ and $\alpha \cap \beta \Vdash^\forall A \supset B$. However by $w \Vdash \Diamond A$, $\alpha \cap \beta \Vdash^\exists A$, thus $\alpha \cap \beta \Vdash^\exists B$, which gives a contradiction.

($N_\Box$) If $\mathcal{M}$ satisfies the condition (N), then for all w there is $\alpha \in \mathcal{N}(w)$. Moreover, $\alpha \Vdash^\forall \top$, therefore $\mathcal{M} \models \Box\top$.

($P_\Diamond$) If $\mathcal{M}$ satisfies the condition (P), then for all w and all $\alpha \in \mathcal{N}(v)$, $\alpha \neq \emptyset$, thus $\alpha \Vdash^\exists \top$. Therefore $\mathcal{M} \models \Diamond\top$.

(D) Assume by contradiction that $\mathcal{M}$ satisfies the condition (D), $w \Vdash \Box A$, and $w \nVdash \Diamond A$. Then there are $v \geq w$ and $\alpha \in \mathcal{N}(v)$ such that $\alpha \nVdash^\exists A$. Moreover, there is $\beta \in \mathcal{N}(v)$ such that $\beta \Vdash^\forall A$. By (D), there is $u \in \alpha \cap \beta$. Thus $u \nVdash A$ and $u \Vdash A$. Therefore $w \Vdash \Diamond A$.

($T_\Box$, $T_\Diamond$) Suppose that $\mathcal{M}$ satisfies the condition (T). Then if $w \Vdash \Box A$, there is $\alpha \in \mathcal{N}(w)$ such that $\alpha \Vdash^\forall A$. By (T), $w \in \alpha$, thus $w \Vdash A$. Moreover, if $w \Vdash A$, then by Prop. 4.2, $v \Vdash A$ for all $v \geq w$. By (T), $v \in \alpha$ for all $\alpha \in \mathcal{N}(v)$, thus $\alpha \Vdash^\exists A$. Therefore $w \Vdash \Diamond A$.

$\Box$

4.1 Completeness

We now prove that W-logics are complete with respect to the corresponding classes of CNMs. As usual, for every logic W^*, we call W^*-*prime* any set Σ of formulas of $\mathcal{L}$ such that $\Sigma \nvdash_{\mathsf{W}^*} \bot$ (consistency), if $\Sigma \vdash_{\mathsf{W}^*} A$, then $A \in \Sigma$

(closure under derivation), and if $A \vee B \in \Sigma$, then $A \in \Sigma$ or $B \in \Sigma$ (disjunction property). Moreover, for every set of formulas Σ, we denote $\Box^{-}\Sigma$ the set $\{A \mid \Box A \in \Sigma\}$. One can prove in a standard way the following lemma.

Lemma 4.4 (Lindenbaum) *If* $\Sigma \nvdash_{\mathsf{W}^*} A$*, then there is a* W^**-prime set* Π *such that* $\Sigma \subseteq \Pi$ *and* $A \notin \Pi$.

We also consider the following notion of segment (we adopt the terminology of [38]), and prove the subsequent lemma that will be needed in the following.

Definition 4.5 For every logic W^*, a W^*-*segment* is a pair $(\Sigma, \mathscr{C})$, where Σ is a W^*-prime set, and $\mathscr{C}$ is a class of sets of W^*-prime sets such that:

- if $\Box A \in \Sigma$, then there is $\mathscr{U} \in \mathscr{C}$ such that for all $\Pi \in \mathscr{U}$, $A \in \Pi$; and
- if $\Diamond A \in \Sigma$, then for all $\mathscr{U} \in \mathscr{C}$, there is $\Pi \in \mathscr{U}$ such that $A \in \Pi$.

WC^*-, WD^*- and WT^*-segments must satisfy also the following conditions: (WC^*) If $\mathscr{U}, \mathscr{U}' \in \mathscr{C}$, then $\mathscr{U} \cap \mathscr{U}' \in \mathscr{C}$. ($\mathsf{WD}^*$) If $\mathscr{U}, \mathscr{U}' \in \mathscr{C}$, then $\mathscr{U} \cap \mathscr{U}' \neq \emptyset$. ($\mathsf{WT}^*$) For all $\mathscr{U} \in \mathscr{C}$, $\Sigma \in \mathscr{U}$.

Lemma 4.6 *For every* W^**-prime set* Σ*, there exists a* W^**-segment* $(\Sigma, \mathscr{C})$.

Proof. Given a W^*-prime set Σ, we construct a W^*-segment $(\Sigma, \mathscr{C})$ as follows. If there is no $\Box A \in \Sigma$, we put $\mathscr{C} = \emptyset$. If there is no $\Diamond A \in \Sigma$, we put $\mathscr{C} = \{\emptyset\}$. Otherwise we distinguish two cases.

(i) W^* does not contain $C_\Box$, $K_\Diamond$. Let $\Box A, \Diamond B \in \Sigma$. Then $A, B \nvdash_{\mathsf{W}^*} \bot$ (otherwise by $Rdual_\wedge$, $\Box A, \Diamond B \vdash_{\mathsf{W}^*} \bot$, against the consistency of Σ). Then by Lemma 4.4, there is Σ'_{AB} W^*-prime such that $A, B \in \Sigma'_{AB}$. For all $\Box A \in \Sigma$, we define $\mathscr{U}_A = \{\Sigma'_{AB} \mid \Diamond B \in \Sigma\}$ if W^* does not contain $T_\Box$, and $\mathscr{U}_A = \{\Sigma'_{AB} \mid \Diamond B \in \Sigma\} \cup \{\Sigma\}$ if it contains $T_\Box$. Moreover we define $\mathscr{C} = \{\mathscr{U}_A \mid \Box A \in \Sigma\}$. Then $(\Sigma, \mathscr{C})$ is a W^*-segment: if $\Box A \in \Sigma$, then $\mathscr{U}_A \in \mathscr{C}$ and for all $\Sigma' \in \mathscr{U}_A$, $A \in \Sigma'$. If $\Diamond B \in \Sigma$, then for all $\mathscr{U}_A \in \mathscr{C}$, there is $\Sigma'_{AB} \in \mathscr{U}_A$ such that $B \in \Sigma'_{AB}$. Moreover for WD^*, if $\mathscr{U}_A, \mathscr{U}_B \in \mathscr{C}$, $\mathscr{U}_A \neq \mathscr{U}_B$, then $\Box A, \Box B \in \Sigma$, then by axiom D, $\Diamond A, \Diamond B \in \Sigma$, thus $\Sigma'_{AB} \in \mathscr{U}_A \cap \mathscr{U}_B$.

(ii) W^* contains $C_\Box$, $K_\Diamond$. Let $\Diamond B \in \Sigma$. Then $\Box^{-}\Sigma \cup \{B\} \nvdash_{\mathsf{W}^*} \bot$ (otherwise by $Rdual_\wedge$ and $C_\Box$, $\Sigma \vdash_{\mathsf{W}^*} \bot$). Then by Lemma 4.4, there is Σ'_B W^*-prime such that $\Box^{-}\Sigma \subseteq \Sigma'_B$ and $B \in \Sigma'_B$. We define $\mathscr{U} = \{\Sigma'_B \mid \Diamond B \in \Sigma\}$ if W^* does not contain $T_\Box$, and $\mathscr{U} = \{\Sigma'_B \mid \Diamond B \in \Sigma\} \cup \{\Sigma\}$ if it contains $T_\Box$. Moreover we define $\mathscr{C} = \{\mathscr{U}\}$. It is easy to verify that $(\Sigma, \mathscr{C})$ is a WC^*-segment.

□

We consider the following definition of canonical model.

Definition 4.7 For every logic W^*, the *canonical model* for W^* is the tuple $\mathcal{M} = \langle \mathcal{W}, \leq, \mathcal{N}, \mathcal{V} \rangle$, where:

- $\mathcal{W}$ is the class of all W^*-segments;
- $(\Sigma, \mathscr{C}) \leq (\Sigma', \mathscr{C}')$ if and only if $\Sigma \subseteq \Sigma'$;

- for every set $\mathscr{U}$ of W^*-prime sets, $\alpha_{\mathscr{U}} = \{(\Sigma, \mathscr{C}) \mid \Sigma \in \mathscr{U}\}$;
- $\alpha_{\mathscr{U}} \in \mathcal{N}((\Sigma, \mathscr{C}))$ if and only if $\mathscr{U} \in \mathscr{C}$;
- $(\Sigma, \mathscr{C}) \in \mathcal{V}(p)$ if and only if $p \in \Sigma$.

We prove the following two lemmas which entail completeness of W^*-logics.

Lemma 4.8 *The canonical model for* W^* *is a CNM for* W^*.

Proof. We show that the canonical model for W^* satisfies the conditions of CNMs for W^*. (C), (D), (T) and hereditariness of $\mathcal{V}$ are immediate by Defs. 4.5 and 4.7. (N) For all WN^*-prime sets Σ, $\Box\top \in \Sigma$, then for all WN^*-segments $(\Sigma, \mathscr{C})$, $\mathscr{C} \neq \emptyset$, thus $\mathcal{N}((\Sigma, \mathscr{C})) \neq \emptyset$. (P) For all WP^*-prime sets Σ, $\Diamond\top \in \Sigma$, then for all WP^*-segments $(\Sigma, \mathscr{C})$, for all $\mathscr{U} \in \mathscr{C}$, $\mathscr{U} \neq \emptyset$, thus $\emptyset \notin \mathcal{N}((\Sigma, \mathscr{C}))$.□

Lemma 4.9 *Let* W^* *be a W-logic, and* $\mathcal{M} = \langle \mathcal{W}, \leq, \mathcal{N}, \mathcal{V}\rangle$ *be the canonical model for* W^*. *Then for all* $(\Sigma, \mathscr{C}) \in \mathcal{W}$, $(\Sigma, \mathscr{C}) \Vdash A$ *if and only if* $A \in \Sigma$.

Proof. By induction on the construction of A. If $A = p$ or $A = \bot$, the proof is immediate by definition of $\mathcal{V}$ or by consistency of Σ, moreover for $A = B \wedge C, B \vee C$ the proof is immediate by applying the inductive hypothesis. We consider the remaining cases.

• $A = B \supset C$: If $B \supset C \in \Sigma$, then suppose $(\Sigma, \mathscr{C}) \leq (\Sigma', \mathscr{C}')$ and $(\Sigma', \mathscr{C}') \Vdash B$. Then $\Sigma \subseteq \Sigma'$, thus $B \supset C \in \Sigma'$. Moreover by i.h., $B \in \Sigma'$, then $C \in \Sigma'$, thus by i.h., $(\Sigma', \mathscr{C}') \Vdash C$. Therefore $(\Sigma, \mathscr{C}) \Vdash B \supset C$. If instead $B \supset C \notin \Sigma$, then $\Sigma \not\vdash B \supset C$, thus $\Sigma \cup \{B\} \not\vdash C$. By Lemma 4.4, there is Σ' W^*-prime such that $\Sigma \cup \{B\} \subseteq \Sigma'$ and $C \notin \Sigma'$. Then by Lemma 4.6 and Def. 4.7, there is a W^*-segment $(\Sigma', \mathscr{C}') \in \mathcal{W}$. By definition, $(\Sigma, \mathscr{C}) \leq (\Sigma', \mathscr{C}')$, and by i.h., $(\Sigma', \mathscr{C}') \Vdash B$ and $(\Sigma', \mathscr{C}') \nVdash C$. Therefore $(\Sigma, \mathscr{C}) \nVdash B \supset C$.

• $A = \Box B$: If $\Box B \in \Sigma$, then for all $(\Sigma', \mathscr{C}') \geq (\Sigma, \mathscr{C})$, $\Box B \in \Sigma'$. By definition of segment, there is $\mathscr{U}' \in \mathscr{C}'$ such that for all $\Sigma'' \in \mathscr{U}'$, $B \in \Sigma''$. Then $\alpha_{\mathscr{U}'} \in \mathcal{N}((\Sigma', \mathscr{C}'))$, moreover by i.h., $(\Sigma'', \mathscr{C}'') \Vdash B$ for all $(\Sigma'', \mathscr{C}'') \in \alpha_{\mathscr{U}'}$. Therefore $(\Sigma, \mathscr{C}) \Vdash \Box B$. Now suppose that $\Box B \notin \Sigma$. If there is no $\Box C \in \Sigma$, then $(\Sigma, \emptyset)$ is a W^*-segment, moreover $(\Sigma, \mathscr{C}) \leq (\Sigma, \emptyset)$ and $\mathcal{N}((\Sigma, \emptyset)) = \emptyset$, thus $(\Sigma, \mathscr{C}) \nVdash \Box B$. If instead there is $\Box C \in \Sigma$, we distinguish two cases:

(i) W^* does not contain $C_\Box$, $K_\Diamond$. Then for all $\Box D \in \Sigma$, $D \not\vdash B$ (otherwise by $mon_\Box$, $\Box D \vdash \Box B$, whence $\Box B \in \Sigma$). Then there is Σ'_D W^*-prime such that $D \in \Sigma'_D$ and $B \notin \Sigma'_D$. Moreover, for all $\Diamond C \in \Sigma$, $C, D \not\vdash \bot$ (otherwise by $Rdual_\wedge$, $\Diamond C, \Box D \vdash \bot$, whence $\bot \in \Sigma$). Then there is Σ'_{CD} W^*-prime such that $C, D \in \Sigma'_{CD}$. For all $\Box D \in \Sigma$, we define $\mathscr{U}'_D = \{\Sigma'_D\} \cup \{\Sigma'_{CD} \mid \Diamond C \in \Sigma\}$ if W^* does not contain $T_\Box$, and $\mathscr{U}'_D = \{\Sigma'_D\} \cup \{\Sigma'_{CD} \mid \Diamond C \in \Sigma\} \cup \{\Sigma\}$ if it contains $T_\Box$. Moreover, we define $\mathscr{C}' = \{\mathscr{U}'_D \mid \Box D \in \Sigma\}$. It is easy to verify that $(\Sigma, \mathscr{C}')$ is a W^*-segment. Moreover, for all $\mathscr{U}'_D \in \mathscr{C}'$, $\Sigma'_D \in \mathscr{U}'_D$ and $B \notin \Sigma'_D$, thus by i.h., $(\Sigma'_D, \mathscr{C}'') \nVdash B$ for any $(\Sigma'_D, \mathscr{C}'') \in \mathcal{W}$. It follows that for all $\alpha_{\mathscr{U}} \in \mathcal{N}((\Sigma, \mathscr{C}'))$, $\alpha_{\mathscr{U}} \nVdash^{\forall} B$. Thus $(\Sigma, \mathscr{C}') \nVdash \Box B$, and since $(\Sigma, \mathscr{C}) \leq (\Sigma, \mathscr{C}')$, $(\Sigma, \mathscr{C}) \nVdash \Box B$.

(ii) W^* contains $C_\Box$, $K_\Diamond$. Then $\Box^-\Sigma \not\vdash B$ (otherwise by $mon_\Box$ and $C_\Box$, $\Sigma \vdash \Box B$), then there is Σ' WC^*-prime such that $\Box^-\Sigma \subseteq \Sigma'$ and $B \notin \Sigma'$.

Moreover, for all $\Diamond C \in \Sigma$, $\Box^- \Sigma \cup \{C\} \nvdash \bot$, then there is Σ'_C WC^*-prime such that $\Box^- \Sigma \subseteq \Sigma'_C$ and $C \in \Sigma'_C$. We define $\mathscr{U}' = \{\Sigma'\} \cup \{\Sigma'_C \mid \Diamond C \in \Sigma\}$ if WC^* does not contain $T_\Box$, and $\mathscr{U}' = \{\Sigma'\} \cup \{\Sigma'_C \mid \Diamond C \in \Sigma\} \cup \{\Sigma\}$ if it contains $T_\Box$. Moreover, we define $\mathscr{C}' = \{\mathscr{U}'\}$. It is easy to verify that $(\Sigma, \mathscr{C}')$ is a WC^*-segment. Moreover, since $B \notin \Sigma'$, by i.h., $(\Sigma', \mathscr{C}'') \nVdash B$ for any $(\Sigma', \mathscr{C}'') \in \mathcal{W}$, it follows that for all $\alpha_{\mathscr{U}} \in \mathcal{N}((\Sigma, \mathscr{C}'))$, $\alpha_{\mathscr{U}} \nVdash^\forall B$. Thus $(\Sigma, \mathscr{C}') \nVdash \Box B$, and since $(\Sigma, \mathscr{C}) \leq (\Sigma, \mathscr{C}')$, $(\Sigma, \mathscr{C}) \nVdash \Box B$.

• $A = \Diamond B$: If $\Diamond B \in \Sigma$, then for all $(\Sigma', \mathscr{C}') \geq (\Sigma, \mathscr{C})$, $\Diamond B \in \Sigma'$. By definition of segment, for all $\mathscr{U}' \in \mathscr{C}'$, there is $\Sigma'' \in \mathscr{U}'$ such that $B \in \Sigma''$. Then for all $\alpha_{\mathscr{U}'} \in \mathcal{N}((\Sigma', \mathscr{C}'))$, there is $(\Sigma'', \mathscr{C}'') \in \alpha_{\mathscr{U}'}$ such that $B \in \Sigma''$, thus by i.h., $(\Sigma'', \mathscr{C}'') \Vdash B$. It follows that $(\Sigma, \mathscr{C}) \Vdash \Diamond B$. Now suppose that $\Diamond B \notin \Sigma$. If there is no $\Diamond C \in \Sigma$, then $(\Sigma, \{\emptyset\})$ is a W^*-segment, moreover $(\Sigma, \mathscr{C}) \leq (\Sigma, \{\emptyset\})$ and $\mathcal{N}((\Sigma, \{\emptyset\})) = \{\emptyset\}$, thus $(\Sigma, \mathscr{C}) \nVdash \Diamond B$. If instead there is $\Diamond C \in \Sigma$, we distinguish two cases:

(i) W^* does not contain $C_\Box$, $K_\Diamond$. Then for all $\Diamond C \in \Sigma$, $C \nvdash B$ (otherwise by $mon_\Diamond$, $\Diamond C \vdash \Diamond B$, whence $\Diamond B \in \Sigma$). Then there is Σ'_C W^*-prime such that $C \in \Sigma'_C$ and $B \notin \Sigma'_C$. Moreover, for all $\Box D \in \Sigma$, $C, D \nvdash \bot$ (otherwise by $Rdual_\wedge$, $\Diamond C, \Box D \vdash \bot$, whence $\bot \in \Sigma$). Then there is Σ'_{CD} W^*-prime such that $C, D \in \Sigma'_{CD}$. If in addition W^* does not contain $T_\Box$, we define $\mathscr{U}' = \{\Sigma'_C \mid \Diamond C \in \Sigma\}$, and for all $\Box D \in \Sigma$, $\mathscr{U}'_D = \{\Sigma'_{CD} \mid \Diamond C \in \Sigma\}$; otherwise we define $\mathscr{U}' = \{\Sigma'_C \mid \Diamond C \in \Sigma\} \cup \{\Sigma\}$, and for all $\Box D \in \Sigma$, $\mathscr{U}'_D = \{\Sigma'_{CD} \mid \Diamond C \in \Sigma\} \cup \{\Sigma\}$. Moreover, we define $\mathscr{C}' = \{\mathscr{U}'\} \cup \{\mathscr{U}'_D \mid \Box D \in \Sigma\}$. It is easy to verify that $(\Sigma, \mathscr{C}')$ is a W^*-segment: for instance for WD^*, if $\mathscr{U}'_D, \mathscr{U}'_E \in \mathscr{C}'$, then $\mathscr{U}'_D \cap \mathscr{U}'_E \neq \emptyset$ (cf. proof of Lemma 4.6), moreover by $P_\Diamond$, $\Diamond\top \in \Sigma$, thus for every $\Box D \in \Sigma$, $\Sigma'_{\top D} \in \mathscr{U}'_D \cap \mathscr{U}'$, then $\mathscr{U}'_D \cap \mathscr{U}' \neq \emptyset$. In addition, by definition we have $\alpha_{\mathscr{U}'} \in \mathcal{N}((\Sigma, \mathscr{C}'))$, and for all $\Sigma' \in \mathscr{U}'$, $B \notin \Sigma'$ (in particular, by $T_\Diamond$, $B \notin \Sigma$). Thus by i.h., for all $\Sigma' \in \mathscr{U}'$ and all $(\Sigma', \mathscr{C}'') \in \mathcal{W}$, $(\Sigma', \mathscr{C}'') \nVdash B$, then $\alpha_{\mathscr{U}'} \nVdash^\exists B$. Therefore $(\Sigma, \mathscr{C}') \nVdash \Diamond B$, and since $(\Sigma, \mathscr{C}) \leq (\Sigma, \mathscr{C}')$, $(\Sigma, \mathscr{C}) \nVdash \Diamond B$.

(ii) W^* contains $C_\Box$, $K_\Diamond$. Then for all $\Diamond C \in \Sigma$, $\Box^- \Sigma \cup \{C\} \nvdash B$ (otherwise by $mon_\Box$, $C_\Box$ and $K_\Diamond$, $\Sigma \vdash \Diamond B$). Then there is Σ'_C WC^*-prime such that $\Box^- \Sigma \cup \{C\} \subseteq \Sigma'_C$ and $B \notin \Sigma'_C$. We define $\mathscr{U}' = \{\Sigma'_C \mid \Diamond C \in \Sigma\}$ if WC^* does not contain $T_\Box$, and $\mathscr{U}' = \{\Sigma'_C \mid \Diamond C \in \Sigma\} \cup \{\Sigma\}$ if it contains $T_\Box$. Moreover we define $\mathscr{C}' = \{\mathscr{U}'\}$. It is easy to verify that $(\Sigma, \mathscr{C}')$ is a WC^*-segment. Moreover, for all $\Sigma' \in \mathscr{U}'$, $B \notin \Sigma'$, then by i.h., for all $(\Sigma', \mathscr{C}'') \in \mathcal{W}$, $(\Sigma', \mathscr{C}'') \nVdash B$, thus $\alpha_{\mathscr{U}'} \nVdash^\exists B$. It follows that $(\Sigma, \mathscr{C}') \nVdash \Diamond B$, and since $(\Sigma, \mathscr{C}) \leq (\Sigma, \mathscr{C}')$, $(\Sigma, \mathscr{C}) \nVdash \Diamond B$.

□

Theorem 4.10 (Completeness) *For every W-logic* W^**, if* $\mathcal{M} \models A$ *for all* W^**-models* $\mathcal{M}$*, then* $\mathsf{W}^* \vdash A$.

Proof. Suppose that $\mathsf{W}^* \nvdash A$. Then by Lemma 4.4, there is a W^*-prime set Σ such that $A \notin \Sigma$, thus by Lemma 4.6, there exists a W^*-segment $(\Sigma, \mathscr{C})$. By Def. 4.7, $(\Sigma, \mathscr{C})$ belongs to the canonical model $\mathcal{M}$ for W^*. Then by Lemma 4.9,

$(\Sigma, \mathscr{C}) \not\Vdash A$, and by Lemma 4.8, $\mathcal{M}$ is a W^*-model. Therefore A is not valid in all W^*-models. □

5 Discussion and future work

In this paper, we have defined a family of 14 constructive modal logics both proof-theoretically and semantically motivated, corresponding each to a different classical modal logic. On the one hand, the logics correspond to the single-succedent restriction of standard sequent calculi for classical modal logics. On the other hand, the same logics are obtained by considering over intuitionistic Kripke models a natural generalisation of the classical satisfaction clauses for modal formulas in the neighbourhood semantics. The main result of this paper is that, despite being mutually independent, for the considered logics the two approaches return exactly the same systems.

In addition, we have provided some preliminary analysis of W-logics. First, we have shown how W-logics are related to the corresponding classical modal logics from the point of view of the axiomatic systems: each classical modal logic considered in this paper can be obtained by extending the corresponding W-logic with both excluded middle $A \vee \neg A$ and disjunctive duality $\Box A \vee \Diamond \neg A$. Moreover, basing on their sequent calculi we have proved some fundamental properties of W-logics, such as the disjunction property, decidability and Craig interpolation.

Simpson [35, Ch. 3] listed some requirements that one expects to be satisfied by any intuitionistic modal logic: they must be conservative over IPL; they must contain all axioms of IPL (over the whole language) and be closed under modus ponens; they must satisfy the disjunction property; the modalities must be independent; the addition of the axiom $A \vee \neg A$ must yield a standard classical modal logic. Basing on the results presented in this paper, it is easy to verify that all W-logics satisfy the first four requirements, by contrast they do not satisfy the last one. [5] However, it comes natural to ask whether there could be some modal principle, additional to excluded middle, that distinguishes between constructive and classical modalities. As a matter of fact, it is easy to identify such a principle for W-logics: as we observed above this principle is precisely $\Box A \vee \Diamond \neg A$.

This relation between classical and W-logics is not entirely trivial. For instance, the same relation does not hold between CK and K, in particular CK must be extended also with $\neg(\Box A \wedge \Diamond \neg A)$ (or equivalently with $\neg \Diamond \bot$) in order to obtain classical K. Moreover, we believe that failure of $\Box A \vee \Diamond \neg A$ is justifiable from a constructive perspective, as it can be seen as a modalised form of excluded middle.

Concerning the semantics of W-logics, the choice of considering neighbourhood models is motivated by the possibility to uniformly cover all considered logics, which include both normal and non-normal systems. However, WK,

[5] This requirement has been sometimes criticised as being too strong, see [22] for an argument against this requirement based on negative translations.

WKD and WKT have an equivalent characterisation in terms of constructive bi-relational models [38], and we conjecture that an analogous characterisation can be given for WMC and its extensions in terms of relational models with non-normal worlds (cf. e.g. [34]). As a byproduct of this work, we have provided a new semantics for WK alternative to its original relational semantics [38] and to the neighbourhood semantics in [18,9].

The possibility to define constructive counterparts of both normal and non-normal classical logics can be seen as providing additional justification for the present approach. To make a comparison, it is not obvious how to extend the family of intuitionistic modal logics (IK and extensions) to non-normal systems, given that their definition ultimately relies on the standard translation of modal formulas into first-order sentences, which in turn is based on the relational semantics. Interestingly, the constructive counterparts of non-normal logics that we have obtained are not entirely new. In particular, WM and WMN coincide with the logics IM and $\mathsf{IMN}_\Box$ introduced in [9], where they are given an alternative semantics with distinct neighbourhood functions for $\Box$ and $\Diamond$. By contrast, WMC is not equivalent to IMC in [9], since WMC contains $K_\Diamond$ which is not a theorem of IMC.

The results presented in this paper can be extended in several directions. In future work we plan to study the complexity of W-logics, possibly extending some optimal calculi for IPL (G3-style calculi are not adequate to establish good complexity bounds for constructive logics). Moreover, we would like to study whether Iemhoff's proof-theoretical method for proving uniform interpolation [16] can be adapted to W-logics. We would also like to define calculi for W-logics that allow for a direct extraction of countermodels from failed proofs, along the lines of [17,10,11].

Furthermore, one can extend the present analysis to further classical modal logics in order to enrich the family of W-logics, but also to inspect the limits of our approach. An obvious limit concerns the logics for which no standard cut-free Gentzen calculi exist, such as S5. For these logics one can study whether a similar analysis could be based on alternative kinds of calculi, like hyper- or nested sequent calculi. At the same time, it is known that incorporating hereditariness into the satisfaction clauses is not sufficient to provide a semantics for some constructive systems, this is the case for instance of the logics with axiom 4 [2]. Concerning instead weaker systems, it seems that for non-normal logic E [8] this approach returns a very weak form of duality analogous to the one of IE_1 in [9], but this requires further study.

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