A multi-modal logic for Galois connections

Alfredo Burrieza [1]

Departamento de Filosofía
Universidad de Málaga, Spain

Inmaculada Pérez de Guzmán [2]

Departamento de Matemática Aplicada
Universidad Málaga, Spain

Antonio Yuste-Ginel [3]

Departamento de Filosofía
Universidad de Málaga, Spain

Abstract

This paper provides an axiomatic characterization of Galois connections by means of a sound and complete system for a multi-modal language. Semantically, our frames consist of families of partially ordered sets (posets) which are (possibly) related by Galois connections. Syntactically, we use a language containing two types of modalities: One to move around inside each poset and the other to jump functionally from one poset to another. The completeness proof follows a step-by-step argument with some interesting particularities.

Keywords: Galois connections, modal logic, temporal×modal logic, multi-modal logic, step-by-step, completeness

1 Introduction

Galois connections are a very well-known concept within diverse areas of mathematics such as algebra, geometry and topology (see [8] for a monograph). Roughly speaking, a Galois connection is a couple of order-preserving or order-inverting maps between two ordered sets. This tool has also been extensively used in theoretical computer science, as it plays a fundamental role in the development of diverse theories with immediate applications (see [18]), Formal Concept Analysis (FCA) [11] being one of the most notable.

[1] burrieza@uma.es

[2] pguzman@uma.es

[3] antonioyusteginel@gmail.com

Concurrently, multi-modal logics are an excellent tool for studying properties of mathematical theories (with [12,16,17] as pioneering works in this field). Hence, it is unsurprising that, during the last few years, there has been a growing interest in the analysis of the links between (multi-)modal logics and Galois connections (e.g., [14,15,19,20,22]).

This paper is inserted in the previous tradition. However, and in contrast to the quoted approaches, our main objective is not to introduce operators which, due to its behaviour, form a Galois connection, nor to extend logics with new operators and to structure their semantics using Galois connections. Rather, we aim at characterising axiomatically what a Galois connection is. This objective is pursued through the development of a multi-modal logic, which we denote as $\mathcal{L}_{GP}$.[4] In other words, our main goal is to define a multi-modal logic provided with a Kripke semantics in which frames are partially ordered sets which are in turn related by means of Galois connections, and to give a sound and complete axiomatic system for such a logic.

As we will see, our representation naturally calls for the use of two types of modal operators: One of them is used to move forward and back inside each poset, while the other one connects these posets functionally. In this sense, our framework builds on the spirit and style of [4,3,5,6], where a very similar setting is used to axiomatise different properties of functions between linear orders. Moreover, and differently to what is usually done, (i) we use partial functions instead of total functions; and (ii) we study Galois connections between arbitrary families of posets, instead of restricting our attention to a finite number of them (usually one or two).

The rest of this paper is organised as follows. Section 2 presents the needed mathematical background. In particular, we recall the notion of Galois connections for partial functions. In Section 3, we introduce the language and semantics of our logic $\mathcal{L}_{GP}$. We present and comment on our axiomatisation in Section 4. The main technical contributions of the paper are contained in Section 5, where we provide a completeness proof for the mentioned axiom system following an elaborated step-by-step construction. In Section 6, we briefly discuss some closely related work. Finally, we close the paper in Section 7 by depicting open paths for future work. Some of the proofs are to be found in the Appendix.

2 Mathematical Preliminaries

In this section, we provide the necessary ingredients for a proper understanding of our semantics. Essentially, we introduce (alternative characterisations of) the notion of a Galois connection for partial functions. Although this notion makes sense for weaker order-theoretic structures, such as preorders (see e.g., [8, Chapter 1]), we restrict our attention to the most common case of partially ordered sets. So, let us first of all recall the definition.

Definition 2.1 Given a non-empty set A and a binary relation R on A, we

[4] As an acronym of **"$\mathcal{L}$"ogic of "G"alois connections between "P"osets**.

say that (A, R) is a **partially ordered set** (*poset*, for short) if R is reflexive, antisymmetric and transitive.

Notation: In what follows we shall use the following notation: If $(A, \leq_A)$ is a poset and $a \in A$:

$$(i)\ a{\uparrow} = \{a' \in A \mid a \leq_A a'\}; \qquad (ii)\ a{\downarrow} = \{a' \in A \mid a' \leq_A a\}.$$

We adopt the following definition of a Galois connection, which generalises the usual one from total functions to partial ones: [5]

Definition 2.2 Let $(A, \leq_A)$ and $(B, \leq_B)$ be two posets and $f\colon (A, \leq_A) \longrightarrow (B, \leq_B)$, $g : (B, \leq_B) \longrightarrow (A, \leq_A)$ be partial functions. We say that the pair (f, g) is a **Galois connection** (between $(A, \leq_A)$ and $(B, \leq_B)$) iff:

1.1 *$Im(f) \subseteq Dom(g)$ and $Im(g) \subseteq Dom(f)$; and*

1.2 For all $a \in Dom(f)$ and for all $b \in Dom(g)$, we have that:

$$a \leq_A g(b) \quad \textit{if and only if} \quad f(a) \leq_B b.$$

Given a Galois connection (f, g), we say that f is a **residuated function** (sometimes denoted by $f^{\rightarrow}$) and g is called its **residual function** (sometimes denoted by $f^{\leftarrow}$).

Our definition is well-behaved since, once item **1.1** is assumed, then a typical, alternative characterisation of Galois connections is equivalent to **1.2**. Let us introduce a couple of preliminary, needed notions.

Definition 2.3 Let $(A, \leq_A)$ and $(B, \leq_B)$ be posets and $f : A \to B$ a partial function. We say that f is **monotone** if, for all $a_1, a_2 \in Dom(f)$, we have that, if $a_1 \leq_A a_2$, then $f(a_1) \leq_B f(a_2)$.

Definition 2.4 Let $(A, \leq)$ be a poset and $f : A \to A$ a partial function. We say that f is **inflationary** if for all $a \in Dom(f)$ we have that $a \leq f(a)$. We say that f is **deflationary** if for all $a \in Dom(f)$, we have that $f(a) \leq a$

Proposition 2.5 *Let $(A, \leq_A)$ and $(B, \leq_B)$ be two posets and $f\colon (A, \leq_A) \longrightarrow (B, \leq_B)$ a partial function. Then, the following conditions are equivalent:*

1. *There exists a partial function $g\colon (B, \leq_B) \longrightarrow (A, \leq_A)$ s.t. (f, g) is a Galois connection.*
2. *$f : (A, \leq_A) \longrightarrow (B, \leq_B)$ is monotone and there exists monotone $g : (B, \leq_B) \longrightarrow (A, \leq_A)$ s.t.*
   - *2.1 $Im(f) \subseteq Dom(g)$ and $Im(g) \subseteq Dom(f)$.*
   - *2.2 $g \circ f$ is inflationary and $f \circ g$ is deflationary.*

Remark 2.6 [Lack of uniqueness of residual partial functions] In Galois connections for total functions, each residuated function uniquely determines its residual (see e.g., [10]). Interestingly, this property is lost when our definition for partial function is adopted. Figure 1 provides an example of two different Galois connections between posets $\mathbb{P}_0 = (A_0, \leq_0)$ and $\mathbb{P}_1 = (A_1, \leq_1)$ which,

[5] See [21] for further generalizations of the notion.

however, have the same resituated function. Each element of $\leq_0$ and $\leq_1$ is represented as a simple, directed arrow, except for reflexive arrows, which have been omitted for the sake of clarity. Residuated functions are depicted as double arrows, while residual functions are depicted as dashed, double arrows.

$$
\begin{tikzcd}
\mathbb{P}_0 & 0_0 \arrow[r] \arrow[d, Rightarrow] & 1_0 \arrow[dl, Rightarrow] \\
\mathbb{P}_1 & 0_1 \arrow[r] \arrow[ur, Rightarrow, dashed, bend right] & 1_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{P}_0 & 0_0 \arrow[r] \arrow[d, Rightarrow] & 1_0 \arrow[dl, Rightarrow] \\
\mathbb{P}_1 & 0_1 \arrow[r] \arrow[ur, Rightarrow, dashed, bend right] & 1_1 \arrow[u, Rightarrow, dashed]
\end{tikzcd}
$$

Fig. 1. Counterexample to uniqueness of residual partial functions.

Example 1 (Concept forming operators in FCA) In *Formal Concept Analysis* (FCA) [11], a *formal context* is a tuple $(\mathcal{O}, \mathcal{A}, \mathcal{R})$ where $\mathcal{O}$ is a set of objects, $\mathcal{A}$ is a set of attributes and $\mathcal{R} \subseteq \mathcal{O} \times \mathcal{A}$ is a relation among objects and attributes. The concept forming operator $\uparrow_R : 2^{\mathcal{O}} \longrightarrow 2^{\mathcal{A}}$ is defined as $\uparrow_R (O) = \{a \in \mathcal{A} \mid \forall o \in O, (o, a) \in \mathcal{R}\}$ for any $O \subseteq \mathcal{O}$ (and something analogous is done for $\downarrow_R : 2^{\mathcal{A}} \longrightarrow 2^{\mathcal{O}}$). These operators are used to set up the notion of a *formal concept*, i.e., a pair (O, A) s.t. $\uparrow_R (O) = A$ and $\downarrow (A) = O$. As it has been thoroughly exploited within FCA, we have that $(\uparrow_R, \downarrow_R)$ is a Galois connection between $(2^{\mathcal{O}}, \subseteq)$ and $(2^{\mathcal{A}}, \supseteq)$.

3 The logic $\mathcal{L}_{GP}$

In this section we introduce the multi-modal logic $\mathcal{L}_{GP}$. We start by introducing the language of this logic, and then move to define an adequate class of models for it.

3.1 Syntax

We assume a denumerable set of atoms $\mathcal{V}$ as fixed from now on. The language L_{GP} is the one generated by the following grammar:

$$A ::= \perp \mid p \mid \neg A \mid (A \wedge A) \mid FA \mid PA \mid \langle \overset{i\,j}{\rightarrow} \rangle A \mid \langle \overset{i\,j}{\leftarrow} \rangle A$$

where p ranges over $\mathcal{V}$, and both i and j range over $\mathbb{N}$.

FA reads "A is true at a state of the current poset which is $\leq$-accessible from the current state", while PA reads "A is true at a state of the current poset from which the current state is $\leq$-accessible". Moreover, $\langle \overset{i\,j}{\rightarrow} \rangle A$ reads "we are at the state s_i of poset $\mathbb{P}_i$ and A is true at the image of s_i in poset $\mathbb{P}_j$ (by a residuated function)", while the meaning of $\langle \overset{i\,j}{\leftarrow} \rangle A$ is "we are at the state s_j of poset $\mathbb{P}_j$ and A is true at the image of s_j in poset $\mathbb{P}_i$ (by a residual function)". The rest of Boolean connectives are defined and read as usual. The duals of $F, P, \langle \overset{i\,j}{\rightarrow} \rangle, \langle \overset{i\,j}{\leftarrow} \rangle$ are denoted $G, H, [\overset{i\,j}{\rightarrow}], [\overset{i\,j}{\leftarrow}]$ and defined $\neg F \neg, \neg P \neg, \neg \langle \overset{i\,j}{\rightarrow} \rangle \neg, \neg \langle \overset{i\,j}{\leftarrow} \rangle \neg$ respectively. The need of taking into account both

$\langle \overset{ij}{\rightarrow} \rangle$ and $\langle \overset{ij}{\leftarrow} \rangle$ as primitive operators will be clear when we define the semantics of L_{GP} in Section 3.2.

We also introduce the notion of the **Galois mirror image** of a formula. If A is a formula, its Galois mirror image is a formula A' obtained from A by replacing in A each occurrence of $P, F, H, G, \langle \overset{ij}{\rightarrow} \rangle, \langle \overset{ij}{\leftarrow} \rangle, [\overset{ij}{\rightarrow}]$ and $[\overset{ij}{\leftarrow}]$ by $F, P, G, H, \langle \overset{ij}{\leftarrow} \rangle, \langle \overset{ij}{\rightarrow} \rangle, [\overset{ij}{\leftarrow}]$ and $[\overset{ij}{\rightarrow}]$ respectively.

Example 2 The Galois mirror image of $[\overset{01}{\rightarrow}]\, H\, p \vee \langle \overset{21}{\leftarrow} \rangle\, q$ is $[\overset{01}{\leftarrow}]\, G\, p \vee \langle \overset{21}{\rightarrow} \rangle\, q$.

3.2 Semantics of L_{GP}

As announced, the skeleton of our models consists of a family of posets connected by residuated and residual functions. A formal definition follows below:

Definition 3.1 A **Galois frame** for L_{GP} is a tuple $\Sigma = (\Lambda, \mathcal{P}_{oset}, \mathcal{F})$ s.t.:

(i) $\varnothing \neq \Lambda \subseteq \mathbb{N}$, whose elements are called **labels**.

(ii) $\mathcal{P}_{oset} = \{(\mathbb{P}_i, \leq_i) \mid i \in \Lambda\}$ is a non-empty set of pairwise disjoint posets s.t. $\mathbb{P}_i \neq \varnothing$ for every label $i \in \Lambda$.
The elements of the disjoint union $\mathbb{S}_\Lambda = \bigsqcup_{i \in \Lambda} \mathbb{P}_i$, denoted by $s, s', etc.$, are called **states**. If we want to specify that a state s belongs to a poset $\mathbb{P}_i$ we denote it by s_i.

(iii) $\mathcal{F} \subseteq \{f : \mathbb{P}_i \longrightarrow \mathbb{P}_j \mid i, j \in \Lambda\}$ is a set of partial functions s.t.:
  (a) for each $f \in \mathcal{F}$, we have that $Dom(f) \neq \varnothing$.
  (b) for an arbitrary pair $i, j \in \Lambda$, it holds that:
   - if $i \neq j$, then there is at most one function $f \in \mathcal{F}$ s.t. $f\colon \mathbb{P}_i \longrightarrow \mathbb{P}_j$.
   - if $i = j$, then there are at most two functions $f, f' \in \mathcal{F}$ s.t. $f\colon \mathbb{P}_i \longrightarrow \mathbb{P}_j$ and $f'\colon \mathbb{P}_i \longrightarrow \mathbb{P}_j$.
  (c) for every $f \in \mathcal{F}$, f is either a residuated function or its residual. If f is a residuated function from $\mathbb{P}_i$ to $\mathbb{P}_j$, then we denote it as $f_{ij}^{\rightarrow}$ and its residual as $f_{ij}^{\leftarrow}$. Moreover, for every $i, j \in \Lambda$, we have that $f_{ij}^{\rightarrow} \in \mathcal{F}$ if and only if $f_{ij}^{\leftarrow} \in \mathcal{F}$. In the special case where $f, f' \in \mathcal{F}$, $f : \mathbb{P}_i \longrightarrow \mathbb{P}_j$ and $f' : \mathbb{P}_j \longrightarrow \mathbb{P}_i$ and both (f, f') and (f', f) are Galois connections (e.g., when f is an isomorphism and $f' = f^{-1}$), then we have to explicitly indicate which of the two functions is considered residuated and which one is considered residual.

Definition 3.2 A **Galois model** for L_{GP} is a tuple $\mathcal{M} = (\Sigma, h)$, where $\Sigma = (\Lambda, \mathcal{P}_{oset}, \mathcal{F})$ is a Galois frame and h is a function, called an **interpretation**, assigning to each atom $p \in \mathcal{V}$ a subset of $\mathbb{S}_\Lambda$. An interpretation h is recursively extended to a function (still denoted by h) defined for every formula of L_{GP}, by interpreting Boolean constants and connectives in a standard way and satisfying the following conditions:

- $h(FA) = \{s \in \mathbb{S}_\Lambda \mid s{\uparrow} \cap h(A) \neq \varnothing\}$
- $h(PA) = \{s \in \mathbb{S}_\Lambda \mid s{\downarrow} \cap h(A) \neq \varnothing\}$

- $h(\langle \overset{i\,j}{\rightarrow} \rangle A) = \{s_i \in \mathbb{P}_i \mid f_{ij}^{\rightarrow} \in \mathcal{F},\ s_i \in Dom(f_{ij}^{\rightarrow}) \textit{ and } f_{ij}^{\rightarrow}(s_i) \in h(A)\}$
- $h(\langle \overset{i\,j}{\leftarrow} \rangle A) = \{s_j \in \mathbb{P}_j \mid f_{ij}^{\leftarrow} \in \mathcal{F}, s_j \in Dom(f_{ij}^{\leftarrow}) \textit{ and } f_{ij}^{\leftarrow}(s_j) \in h(A)\}$

Hence, we can deduce the semantics of non-primitive connectives:

- $h(GA) = \{s \in \mathbb{S}_\Lambda \mid s{\uparrow} \subseteq h(A)\}$
- $h(HA) = \{s \in \mathbb{S}_\Lambda \mid s{\downarrow} \subseteq h(A)\}$
- $h([\overset{i\,j}{\rightarrow}]A) = \{s_i \in \mathbb{P}_i \mid \text{ If } f_{ij}^{\rightarrow} \in \mathcal{F} \text{ and } s_i \in Dom(f_{ij}^{\rightarrow}), \text{then}$
  $f_{ij}^{\rightarrow}(s_i) \in h(A)\} \cup \{s_k \in \mathbb{P}_k \mid k \neq i\}$
- $h([\overset{i\,j}{\leftarrow}]A) = \{s_j \in \mathbb{P}_j \mid \text{ If } f_{ij}^{\leftarrow} \in \mathcal{F} \text{ and } s_j \in Dom(f_{ij}^{\leftarrow}), \text{ then}$
  $f_{ij}^{\leftarrow}(s_j) \in h(A)\} \cup \{s_k \in \mathbb{P}_k \mid k \neq j\}$

The class of all Galois-models is denoted by $\mathcal{M}_{GP}$. The semantic notions of *satisfiability*, *validity* and related ones are defined as usual (see e.g., [1]).

Example 3 Figure 2 depicts a Galois frame with three posets, $(\mathbb{P}_0, \leq_0)$, $(\mathbb{P}_1, \leq_1)$ and $(\mathbb{P}_2, \leq_2)$. Each element of $\leq_i$ is represented as a simple, directed arrow, except for reflexive and transitive arrows, which have been omitted for the sake of clarity. Residuated functions are depicted as double arrows, while residual functions are depicted as dashed, double arrows.

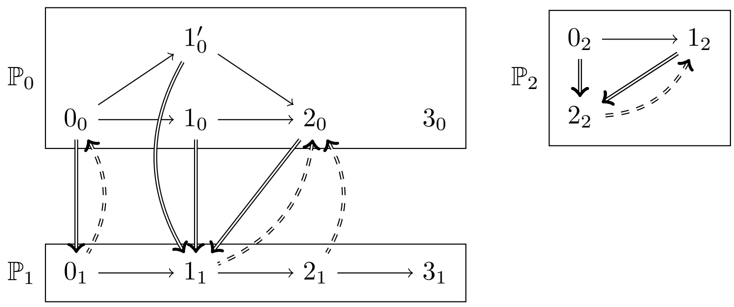

Fig. 2. Galois frame of Example 3.

Let us now define a model based on the previous Galois frame by adding the interpretation h defined as follows $h(p) = \{1_0, 1_1, 2_1, 3_1, 1_2\}$ and $h(q) = \mathbb{P}_0 \cup \mathbb{P}_1 \cup \mathbb{P}_2$ (for all $q \neq p$). Let us evaluate some formulae in this model:

Formula	**True at**	**Formula**	**True at**
$\langle \overset{0\,1}{\rightarrow} \rangle\, p$	$1_0, 1'_0, 2_0$	$\langle \overset{1\,2}{\leftarrow} \rangle\, p$	*none*
$[\overset{0\,1}{\rightarrow}]\, p$	*every state except* 0_0	$\langle \overset{2\,2}{\leftarrow} \rangle p$	2_2
$\langle \overset{0\,1}{\leftarrow} \rangle \neg p$	$0_1, 1_1, 2_1$	$PF\langle \overset{0\,1}{\leftarrow} \rangle \langle \overset{0\,1}{\rightarrow} \rangle\, p$	$0_1, 1_1, 2_1, 3_1$
$[\overset{1\,2}{\rightarrow}]\, p$	*every state*	$F\,\langle \overset{0\,1}{\rightarrow} \rangle H[\overset{0\,1}{\leftarrow}] \neg p$	$0_0, 1_0, 1'_0, 2_0$

4 An axiom system for L_{GP}

We now introduce an axiomatic calculus intended to produce as theorems all formulae that are valid in $\mathcal{M}_{GP}$. We denote this axiom system by S_{GP}. In what follows, we will use the symbol λ to denote a finite sequence (included the empty one) of the operators F and P.

Axiom schemata

1. A, where A is a truth-functional tautology

2. Axiom schemata for non-indexed connectives:

2.1 $G(A \to B) \to (GA \to GB)$

2.2 $A \to GPA$

2.3 $GA \to GGA$

2.4 $GA \to A$

2.5 The Galois mirror images of 2.1–2.4

3. Axiom schemata for indexed connectives: For each $i, j \in \mathbb{N}$,

3.1 $[\overset{i\,j}{\rightarrow}](A \to B) \to ([\overset{i\,j}{\rightarrow}]A \to [\overset{i\,j}{\rightarrow}]B)$

3.2 $\langle\overset{i\,j}{\rightarrow}\rangle A \to [\overset{i\,j}{\rightarrow}]A$

3.3 For all $i, j, k, l \in \mathbb{N}$:

$$\begin{cases} 3.3.1 \quad \langle\overset{i\,j}{\rightarrow}\rangle\lambda\langle\overset{k\,l}{\leftarrow}\rangle\top \to \bot, & \text{if } j \neq l \\ 3.3.2 \quad \langle\overset{i\,j}{\rightarrow}\rangle\lambda\langle\overset{k\,l}{\rightarrow}\rangle\top \to \bot & \text{if } j \neq k \\ 3.3.3 \quad \langle\overset{i\,j}{\leftarrow}\rangle\lambda\langle\overset{k\,l}{\leftarrow}\rangle\top \to \bot & \text{if } i \neq l \\ 3.3.4 \quad \langle\overset{i\,j}{\leftarrow}\rangle\lambda\langle\overset{k\,l}{\rightarrow}\rangle\top \to \bot & \text{if } i \neq k \end{cases}$$

$$\begin{cases} 3.3.5 \quad (\langle\overset{i\,j}{\rightarrow}\rangle\top \wedge \lambda\langle\overset{k\,l}{\leftarrow}\rangle\top) \to \bot & \text{if } i \neq l \\ 3.3.6 \quad (\langle\overset{i\,j}{\rightarrow}\rangle\top \wedge \lambda\langle\overset{k\,l}{\rightarrow}\rangle\top) \to \bot & \text{if } i \neq k \\ 3.3.7 \quad (\langle\overset{i\,j}{\leftarrow}\rangle\top \wedge \lambda\langle\overset{k\,l}{\leftarrow}\rangle\top) \to \bot & \text{if } j \neq l \end{cases}$$

3.4 For all $i, j \in \mathbb{N}$: $\langle\overset{i\,j}{\rightarrow}\rangle\top \to \langle\overset{i\,j}{\rightarrow}\rangle\langle\overset{i\,j}{\leftarrow}\rangle\top$

3.5 For all $i, j \in \mathbb{N}$: $\langle\overset{i\,j}{\rightarrow}\rangle F \langle\overset{i\,j}{\leftarrow}\rangle A \to FA$

3.6 The Galois mirror images of 3.1, 3.2, 3.4 and 3.5.

3.7 For all $i, j \in \mathbb{N}$: $(\langle\overset{i\,j}{\rightarrow}\rangle\top \wedge \lambda\langle\overset{j\,i}{\leftarrow}\rangle\top) \to \bot$ if $i \neq j$

4 Duality axioms: $\blacklozenge A \leftrightarrow \neg\blacksquare\neg A$ where $\blacklozenge \in \{F, P, \langle\overset{i\,j}{\rightarrow}\rangle, \langle\overset{i\,j}{\leftarrow}\rangle\}$ and $\blacksquare$ denotes the dual of $\blacklozenge$ (e.g., if $\blacklozenge = \langle\overset{i\,j}{\rightarrow}\rangle$, then $\blacksquare = [\overset{i\,j}{\rightarrow}]$)

Inference rules

$$(MP): \quad \frac{A,\ A \to B}{B}$$

$$(NG): \quad \frac{A}{GA} \qquad (NH): \quad \frac{A}{HA}$$

$$(N[\overset{ij}{\rightarrow}]): \quad \frac{A}{[\overset{ij}{\rightarrow}]A} \qquad (N[\overset{ij}{\leftarrow}]): \quad \frac{A}{[\overset{ij}{\leftarrow}]A} \qquad (\text{for all } i, j \in \mathbb{N})$$

Remark 4.1 Let us comment a bit on some of the schemata.

Schemata 2.1–2.5 conform the standard axiomatisation of posets in a temporal modal language (see e.g., [2]).

Schema 3.2 expresses the fact that each $f_{ij}^{\rightarrow}$ is a partial function. The same happens with $f_{ij}^{\leftarrow}$ and the Galois mirror image of 3.2.

Schemata 3.3.1–3.3.7 captures the fact that labels are unique for each poset, so that the same poset is not named twice.

Schema 3.4. and its Galois mirror image express the required conditions in item 1.1 of Definition 2.2. Specifically, they express $Im(f_{ij}^{\rightarrow}) \subseteq Dom(f_{ij}^{\leftarrow})$ and $Im(f_{ij}^{\leftarrow}) \subseteq Dom(f_{ij}^{\rightarrow})$, respectively.

Schema 3.5 and its Galois mirror image express the two implications contained in item 1.2 of Definition 2.2. More in detail, they capture, respectively, (a) if $f_{ij}^{\rightarrow}(s_i) \leq_j s_j$, then $s_i \leq_i f_{ij}^{\leftarrow}(s_j)$, and (b) if $s_i \leq_i f_{ij}^{\leftarrow}(s_j)$, then $f_{ij}^{\rightarrow}(s_i) \leq_j s_j$.

Finally, schema 3.7 tells us that, given two different indices i, j, we cannot have at the same time a resituated function and a residual from i to j.

5 Soundness and completeness of S_{GP}

This section contains the main technical results of the paper. The notions of *proof* in S_{GP} and *theorem* of S_{GP} (noted $\vdash \varphi$) are standard (see, e.g., [1, Chapter 1]). Let us first formally state the soundness of our axiom system:

Theorem 5.1 (Soundness) *S_{GP} is sound w.r.t. $\mathcal{M}_{GP}$, that is, every formula A of L_{GP} that is provable in S_{GP} is valid in $\mathcal{M}_{GP}$.*

More interestingly, our system is complete w.r.t. $\mathcal{M}_{GP}$.

Theorem 5.2 (Completeness) *S_{GP} is complete w.r.t. $\mathcal{M}_{GP}$, that is, every formula A of L_{GP} that is valid in $\mathcal{M}_{GP}$ is provable in S_{GP}.*

The rest of this section is devoted to the proof of this theorem, which is based on the step-by-step method (see e.g., [2] or [1, Chapter 4.6] for an introduction to this kind of constructions). In short, we will build, through a sequence of steps, a model satisfying each consistent formula A. At each step we will have a finite frame that does not necessarily satisfy all the properties of a Galois frame. However, this process approaches to a limit satisfying all the desired requirements. Moreover, at each step we have a frame which is "good enough". The step-by-step method is useful for dealing with frame properties that are not definable in the entertained modal language. In our case, the need of the method is ultimately triggered by the antisymmetry of each poset within a Galois frame, which is clearly not definable with the operators we take into account. Moreover, the idiosyncrasy of our frames makes the main argument a bit more elaborated since we have to take care that indexed modalities are well-behaved through the construction.

The proof is structured as follows. We first present a couple of theorems of S_{GP} that will be useful for the rest of the work (Proposition 5.3). After that, we will state and prove some properties of maximally consistent sets of formulae which are specific to our system. Finally, we will proceed to build our step-by-step model.

Proposition 5.3 *The following formulae are theorems of S_{GP}:*

T1. $(\langle \overset{ij}{\rightarrow} \rangle A \wedge \langle \overset{ij}{\rightarrow} \rangle B) \rightarrow \langle \overset{ij}{\rightarrow} \rangle (A \wedge B)$

T2. $(\langle \overset{ij}{\leftarrow} \rangle A \wedge \langle \overset{ij}{\leftarrow} \rangle B) \rightarrow \langle \overset{ij}{\leftarrow} \rangle (A \wedge B)$

Results about maximal consistent sets

The syntactical notions of *consistency* of a set of formulae Γ (denoted $\Gamma \nvdash \perp$) and *maximal consistency* of Γ in S_{GP} are defined in the usual way. Familiarity with basic properties of maximally consistent sets (*mc*-sets, for short) is assumed (see [1, Chapter 4]). We denote by $\mathcal{MC}$ the class of all *mc*-sets in S_{GP}.

Definition 5.4 Let $\Gamma_1, \Gamma_2 \in \mathcal{MC}$, and $i, j \in \mathbb{N}$. Then we define:

(a) $\Gamma_1 \preceq_{\mathbb{P}} \Gamma_2$ iff $\{A \mid GA \in \Gamma_1\} \subseteq \Gamma_2$.

(b) $\Gamma_1 \prec_{ij}^{\rightarrow} \Gamma_2$ iff $\varnothing \neq \{A \mid \langle \overset{ij}{\rightarrow} \rangle A \in \Gamma_1\} \subseteq \Gamma_2$.

(c) $\Gamma_1 \prec_{ij}^{\leftarrow} \Gamma_2$ iff $\varnothing \neq \{A \mid \langle \overset{ij}{\leftarrow} \rangle A \in \Gamma_1\} \subseteq \Gamma_2$.

As a consequence of this definition we have:

Proposition 5.5 *Let $\Gamma_1, \Gamma_2 \in \mathcal{MC}$, and $i, j \in \mathbb{N}$. Then:*

(i) $\Gamma_1 \preceq_{\mathbb{P}} \Gamma_2$ *iff* $\{FA \mid A \in \Gamma_2\} \subseteq \Gamma_1$ *iff* $\{A \mid HA \in \Gamma_2\} \subseteq \Gamma_1$ *iff* $\{PA \mid A \in \Gamma_1\} \subseteq \Gamma_2$.

(ii) $\Gamma_1 \prec_{ij}^{\rightarrow} \Gamma_2$ *iff* $\{A \mid [\overset{ij}{\rightarrow}]A \in \Gamma_1\} \subseteq \Gamma_2$ *iff* $\{\langle \overset{ij}{\rightarrow} \rangle A \mid A \in \Gamma_2\} \subseteq \Gamma_1$.

(iii) $\Gamma_1 \prec_{ij}^{\leftarrow} \Gamma_2$ *iff* $\{A \mid [\overset{ij}{\leftarrow}]A \in \Gamma_1\} \subseteq \Gamma_2$ *iff* $\{\langle \overset{ij}{\leftarrow} \rangle A \mid A \in \Gamma_2\} \subseteq \Gamma_1$.

We move to state some typical basic results. Namely, the *Lindenbaum's Lemma*, the so-called *Existence Lemma* for each of our diamond modalities, and the definability of reflexivity and transitivity in the basic temporal language.

Proposition 5.6 *The following properties are satisfied:*

1. *(Lindenbaum's Lemma) Any consistent set of formulae in S_{GP} can be extended to a mc-set in S_{GP}.*
2. *(Existence Lemmas) Let $\Gamma_1 \in \mathcal{MC}$ and $i, j \in \mathbb{N}$, then we have:*
   (a) If $FA \in \Gamma_1$, then there exists $\Gamma_2 \in \mathcal{MC}$ s.t. $\Gamma_1 \preceq_{\mathbb{P}} \Gamma_2$ and $A \in \Gamma_2$.
   (b) If $PA \in \Gamma_1$, then there exists $\Gamma_2 \in \mathcal{MC}$ s.t. $\Gamma_2 \preceq_{\mathbb{P}} \Gamma_1$ and $A \in \Gamma_2$.
   (c) If $\langle \overset{ij}{\rightarrow} \rangle A \in \Gamma_1$, then there exists $\Gamma_2 \in \mathcal{MC}$ s.t. $\Gamma_1 \prec_{ij}^{\rightarrow} \Gamma_2$ and $A \in \Gamma_2$.
   (d) If $\langle \overset{ij}{\leftarrow} \rangle A \in \Gamma_1$, then there exists $\Gamma_2 \in \mathcal{MC}$ s.t. $\Gamma_1 \prec_{ij}^{\leftarrow} \Gamma_2$ and $A \in \Gamma_2$.
3. *($\preceq_{\mathbb{P}}$ is a preorder) Let $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{MC}$. Then we have:*

(*a*) $\Gamma_1 \preceq_{\mathbb{P}} \Gamma_1$.

(*b*) *If* $\Gamma_1 \preceq_{\mathbb{P}} \Gamma_2$ *and* $\Gamma_2 \preceq_{\mathbb{P}} \Gamma_3$*, then* $\Gamma_1 \preceq_{\mathbb{P}} \Gamma_3$.

The following proposition states that the relations between mc-sets, $\prec_{ij}^{\rightarrow}$ and $\prec_{ij}^{\leftarrow}$, satisfy condition 1.1 of Definition 2.2.

Proposition 5.7 *Let* $\Gamma_1, \Gamma_2 \in \mathcal{MC}$ *and* $i, j \in \mathbb{N}$*. Then we have:*

(i) *If* $\Gamma_1 \prec_{ij}^{\rightarrow} \Gamma_2$*, then there exists* $\Gamma_3 \in \mathcal{MC}$ *s.t.* $\Gamma_2 \prec_{ij}^{\leftarrow} \Gamma_3$.

(ii) *If* $\Gamma_1 \prec_{ij}^{\leftarrow} \Gamma_2$*, then there exists* $\Gamma_3 \in \mathcal{MC}$ *s.t.* $\Gamma_2 \prec_{ij}^{\rightarrow} \Gamma_3$.

The following proposition states that the relations between mc-sets, $\prec_{ij}^{\rightarrow}$ and $\prec_{ij}^{\leftarrow}$, satisfy property 1.2 of Definition 2.2.

Proposition 5.8 *Let* $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \in \mathcal{MC}$ *and* $i, j \in \mathbb{N}$*. Then:*

(i) *If* $\Gamma_1 \prec_{ij}^{\rightarrow} \Gamma_2$, $\Gamma_2 \preceq_{\mathbb{P}} \Gamma_3$ *and* $\Gamma_3 \prec_{ij}^{\leftarrow} \Gamma_4$, *then* $\Gamma_1 \preceq_{\mathbb{P}} \Gamma_4$.

(ii) *If* $\Gamma_1 \prec_{ij}^{\leftarrow} \Gamma_2$, $\Gamma_3 \preceq_{\mathbb{P}} \Gamma_2$ *and* $\Gamma_3 \prec_{ij}^{\rightarrow} \Gamma_4$, *then* $\Gamma_4 \preceq_{\mathbb{P}} \Gamma_1$.

Step-by-step method

It is now time to make formally precise what does it mean that, at each step of our construction, although our frame might not be a Galois frame, it is "good enough":

Definition 5.9 Let $(A, \leq_A)$ and $(B, \leq_B)$ be two posets and $f : (A, \leq_A) \longrightarrow (B, \leq_B)$, $g : (B, \leq_B) \longrightarrow (A, \leq_A)$ a pair of partial functions. We say that the pair (f, g) is a **quasi-Galois connection** if for all $a \in Dom(f)$ and for all $b \in Dom(g)$, we have that $a \leq_A g(b)$ iff $f(a) \leq_B b$. If (f, g) is a quasi-Galois connection, we call f **quasi-residuated**, and denote it by $f^{q,\rightarrow}$, and the function g is called the **quasi-residual of** f, and denoted by $f^{q,\leftarrow}$.

Remark 5.10 Since the previous definition does not require that $Im(f) \subseteq Dom(g)$ and $Im(g) \subseteq Dom(f)$, then by Definition 2.2 we have that a quasi-Galois connection, (f, g), is not always a Galois connection. Moreover, according to the proof of Proposition 2.5, neither have we assured that f and g are monotone; nor, likewise, that $g \circ f$ is inflationary and $f \circ g$ deflationary.

Definition 5.11 A **quasi-Galois frame** for L_{GP} is a tuple $\Sigma = (\Lambda, \mathcal{P}_{oset}, \mathcal{F})$ where every component is just as in a Galois frame (Definition 3.1) except for $\mathcal{F}$, where the condition (iii)(c) of Definition 3.1 is replaced by:

(c') for every $f \in \mathcal{F}$, it is either a quasi-residuated function or its quasi-residual. If f is a quasi-residuated function from $\mathbb{P}_i$ to $\mathbb{P}_j$, then we denote it by $f_{ij}^{q,\rightarrow}$, and we use $f_{ij}^{q,\leftarrow}$ to denote its quasi-residual.

Note that, unlike what we did in (iii)(c) (Definition 3.1), we do not require $f^{q,\rightarrow} \in \mathcal{F}$ iff $f^{q,\leftarrow} \in \mathcal{F}$ for quasi-Galois frames.

In our construction, the quasi-Galois frame entertained at each step will be an *extension* of the previous one. Let us make this notion precise:

Definition 5.12 Let $\Sigma_1 = (\Lambda_1, \mathcal{P}_{oset1}, \mathcal{F}_1)$ and $\Sigma_2 = (\Lambda_2, \mathcal{P}_{oset2}, \mathcal{F}_2)$ be a pair of quasi-Galois frames. We say that Σ_2 is an **extension** of Σ_1 if the following conditions hold:

(i) $\Lambda_1 \subseteq \Lambda_2$.

(ii) for any $(\mathbb{P}_i, \leq_i) \in \mathcal{P}_{oset1}$, the poset $(\mathbb{P}'_i, \leq'_i) \in \mathcal{P}_{oset2}$ satisfies:
- $\mathbb{P}_i \subseteq \mathbb{P}'_i$;
- $\leq_i = \leq'_i \cap (\mathbb{P}_i \times \mathbb{P}_i)$.

We also define a special type of quasi-Galois frame that will be useful in the construction.

Definition 5.13 Given $k \in \mathbb{N}$, a **simple quasi-Galois frame at** k is a quasi-Galois frame $\Sigma^k = (\Lambda, \mathcal{P}_{oset}, \mathcal{F})$ where $\mathcal{F} = \varnothing$, $\Lambda = \{k\}$, and $\mathcal{P}_{oset} = \{(\mathcal{P}_k, \leq_k)\}$. Given $j \in \mathbb{N}$, we say that a simple quasi-Galois frame is a j**-renamed frame of** Σ^k, denoted $\Sigma^{j/k}$, if it is obtained by replacing in Σ^k every occurrence of k by j.

As usual in the *step-by-step* completeness method, we introduce a function (called *trace*) that associates elements of $\mathcal{MC}$ to states of a quasi-Galois frame.

Definition 5.14 Let $\Sigma = (\Lambda, \mathcal{P}_{oset}, \mathcal{F})$ be a quasi-Galois frame for L_{GP}. A **trace** of Σ is a function $\Phi_\Sigma : \mathbb{S}_\Lambda \to \mathcal{MC}$. Moreover, if Σ^k is a simple quasi-Galois frame and Φ_{Σ^k} is a trace of it, the **renamed trace** $\Phi_{\Sigma^{j/k}}$ is obtained by replacing all occurrences of k in the domain of Φ_{Σ^k} by j.

We now introduce the desired properties of a trace:

Definition 5.15 Let Φ_Σ be a trace of a quasi-Galois frame $\Sigma = (\Lambda, \mathcal{P}_{oset}, \mathcal{F})$. Then Φ_Σ is called:

- **nominally-coherent**, if for all $s \in \mathbb{S}_\Lambda$ and $i, j \in \mathbb{N}$ we have that:
  (nc_1) : if $\lambda\langle \overset{i\,j}{\rightarrow} \rangle A \in \Phi_\Sigma(s)$, then $s = s_i$.
  (nc_2) : if $\lambda\langle \overset{i\,j}{\leftarrow} \rangle A \in \Phi_\Sigma(s)$, then $s = s_j$.
- **poset-coherent**, if for all $s, s' \in \mathbb{S}_\Lambda$ we have that:
  if $s' \in s{\uparrow}$, then $\Phi_\Sigma(s) \preceq_{\mathbb{P}} \Phi_\Sigma(s')$.
- **functionally-coherent**, if for all $s_i, s_j \in \mathbb{S}_\Lambda$ with $i, j \in \Lambda$ we have that:
  (fc_1) : *if* $s_j = f^{\rightarrow}_{ij}(s_i)$, *then* $\Phi_\Sigma(s_i) \prec^{\rightarrow}_{ij} \Phi_\Sigma(s_j)$.
  (fc_2) : *if* $s_i = f^{\leftarrow}_{ij}(s_j)$, *then* $\Phi_\Sigma(s_j) \prec^{\leftarrow}_{ij} \Phi_\Sigma(s_i)$.
- **$\uparrow$-projectable** if for all $A \in L_{GP}$ and $s \in \mathbb{S}_\Lambda$ we have that:
  if $FA \in \Phi_\Sigma(s)$, there exists $s' \in s{\uparrow}$ s.t. $A \in \Phi_\Sigma(s')$.
- **$\downarrow$-projectable** if for all $A \in L_{GP}$ and $s \in \mathbb{S}_\Lambda$ we have that:
  if $PA \in \Phi_\Sigma(s)$, there exists $s' \in s{\downarrow}$ s.t. $A \in \Phi_\Sigma(s')$.
- **$\langle\rightarrow\rangle$-projectable** if for all $A \in L_{GP}$, $i, j \in \mathbb{N}$ and $s_k \in \mathbb{S}_\Lambda$ we have that:
  if $\langle \overset{i\,j}{\rightarrow} \rangle A \in \Phi_\Sigma(s_k)$, then there exists $s_j = f^{\rightarrow}_{kj}(s_k)$ s.t. $A \in \Phi_\Sigma(s_j)$.
- **$\langle\leftarrow\rangle$-projectable** if for all $A \in L_{GP}$, $i, j \in \mathbb{N}$ and $s_k \in \mathbb{S}_\Lambda$ we have that:
  if $\langle \overset{i\,j}{\leftarrow} \rangle A \in \Phi_\Sigma(s_k)$, then there exists $s_i = f^{\leftarrow}_{ik}(s_k)$ s.t. $A \in \Phi_\Sigma(s_i)$.
- **quasi-coherent** if it is poset-coherent, functionally-coherent (but not necessarily nominally-coherent).

- **coherent** if it is nominally-coherent, poset-coherent and functionally-coherent.
- **full** if it is coherent, $\uparrow$-projectable, $\downarrow$-projectable, $\langle\rightarrow\rangle$-projectable, and $\langle\leftarrow\rangle$-projectable.

Remark 5.16 It is worth noting the following consideration in the definition of the nominally-coherent condition: If $\lambda\langle\overset{i\,j}{\rightarrow}\rangle A \in \Phi_\Sigma(s)$, then any formula in $\Phi_\Sigma(s)$ other than $\lambda\langle\overset{i\,j}{\rightarrow}\rangle A$ of the form $\lambda'\langle\overset{k\,l}{\rightarrow}\rangle B$ implies $i = k$ and of the form $\lambda'\langle\overset{k\,l}{\leftarrow}\rangle B$ implies $i = l$ (by axioms 3.3.5 and 3.3.6 and the fact that $\Phi_\Sigma(s)$ is an mc-set). Similar considerations can be done if $\lambda\langle\overset{i\,j}{\leftarrow}\rangle A \in \Phi_\Sigma(s)$.

Remark 5.17 We will refer to the conditionals introduced in the previous definition (e.g., in the definition of $\uparrow$-projectable trace) just as we refer to a trace Φ_Σ that satisfies them. In general, we will also use the expression "conditional for Φ_Σ" to mean that it is a $\uparrow$-projectable ($\downarrow$-projectable, $\langle\rightarrow\rangle$-projectable or $\langle\leftarrow\rangle$-projectable) conditional for Φ_Σ. Moreover, given a conditional (α) for Φ_Σ, if we replace the index Σ' by Σ, where Σ' is an extension of Σ, then (α) is a conditional for Φ'_Σ, but we can say that we refer to the same conditional in both cases.

Definition 5.18 Let Φ_Σ be a trace of a quasi-Galois frame.

- Consider a $\uparrow$-projectable conditional:

  "If $FA \in \Phi_\Sigma(s)$, then there exists $s' \in s\uparrow$ s.t. $A \in \Phi_\Sigma(s')$"

  We say that it is **active** for Φ_Σ if $FA \in \Phi_\Sigma(s)$, but there is no $s' \in s\uparrow$ s.t. $A \in \Phi_\Sigma(s')$. On the other hand, we say that it is **exhausted** for Φ_Σ if there exists a state s' s.t. $s' \in s\uparrow$ and $A \in \Phi_\Sigma(s')$.

  The notions of activeness and exhaustedness are defined in a similar way for $\downarrow$-projectable conditionals.
- Consider a $\langle\rightarrow\rangle$-projectable conditional:

  "If $\langle\overset{i\,j}{\rightarrow}\rangle A \in \Phi_\Sigma(s_k)$, then there exists $s_j = f^{\rightarrow}_{kj}(s_k)$ s.t. $A \in \Phi_\Sigma(s_j)$."

  We say that it is **active** for Φ_Σ if $\langle\overset{i\,j}{\rightarrow}\rangle A \in \Phi_\Sigma(s_k)$, but there is no $s_j = f^{\rightarrow}_{kj}(s_k)$ s.t. $A \in \Phi_\Sigma(s_j)$. On the other hand, the conditional is **exhausted** for Φ_Σ if there exists $s_j = f^{\rightarrow}_{kj}(s_k)$ s.t. $A \in \Phi_\Sigma(s_j)$.

  The notions of activeness and exhaustedness are defined in a similar way for $\langle\leftarrow\rangle$-projectable conditionals.

Lemma 5.19 (Truth) *Let Φ_Σ be a full trace of a Galois frame Σ. Let h be an interpretation assigning to each propositional variable p the set $h(p) = \{s \in \mathbb{S}_\Sigma \mid p \in \Phi_\Sigma(s)\}$. Then, for any formula A, we have $h(A) = \{s \in \mathbb{S}_\Sigma \mid A \in \Phi_\Sigma(s)\}$.*

As we will see, at every stage of the step-by-step construction, the trace of the constructed quasi-Galois frame is quasi-coherent or coherent. However, this is not necessarily so for the projectable properties, which are only satisfied at the end of the process. Let us then state the lemmas that allow exhausting active conditionals at each step.

Lemma 5.20 (Basic exhausting lemma) *Let Φ_{Σ^k} be a quasi-coherent trace of a simple quasi-Galois frame Σ^k and let (α) be an active conditional for Φ_{Σ^k}, then:*

(i) *If (α) is a $\langle\rightarrow\rangle$-($\langle\leftarrow\rangle$-)conditional, then there is $i \in \mathbb{N}$ s.t. $\Phi_{\Sigma^{i/k}}$ is coherent.*

(ii) *If (α) is a $\uparrow$-($\downarrow$-)conditional, then there is an extension of Σ^k, Σ^k_1, and a quasi-coherent trace $\Phi_{\Sigma^k_1}$ of Σ^k_1 s.t. $\Phi_{\Sigma^k_1} \subseteq \Phi_{\Sigma^k}$ and (α) is exhausted for $\Phi_{\Sigma^k_1}$. Moreover, if Φ_{Σ^k} was coherent, then $\Phi_{\Sigma^k_1}$ is coherent.*

Lemma 5.21 (General exhausting lemma) *Let Φ_{Σ_1} be a coherent trace of a finite quasi-Galois frame Σ_1 and let (α) be an active conditional for Φ_{Σ_1}. Then there is a finite quasi-Galois frame Σ_2, which is an extension of Σ_1, and a coherent trace Φ_{Σ_2} of Σ_2 s.t. $\Phi_{\Sigma_1} \subseteq \Phi_{\Sigma_2}$ and (α) is exhausted for Φ_{Σ_2}.*

We are finally able to give a **proof of Theorem 5.2**:

Proof. [Sketch] By a standard argument, it suffices to show that given a consistent formula A, this formula is satisfiable. In order to do so, we will build a model $\mathcal{M} = (\Sigma, h)$ where $\Sigma = (\Lambda, \mathcal{P}_{oset}, \mathcal{F})$ is a Galois frame. Given that A is consistent, there exists a *mc*-set Γ_0 containing A (*Lindenbaum's Lemma*). We start our construction with the finite simple quasi-Galois frame $\Sigma_0 = (\{0\}, \{\{\{s_0\}, \{(s_0, s_0)\}\}\}, \varnothing)$. The corresponding trace Φ_{Σ_0} is defined as $\Phi_{\Sigma_0}(s_0) = \Gamma_0$. It is straightforward to show that Φ_{Σ_0} is quasi-coherent. Now, we aim at constructing:

- a denumerable sequence, $\Sigma_0, \Sigma_1, \ldots, \Sigma_n \ldots$, of finite quasi-Galois frames whose union will be a Galois frame, and
- a denumerable sequence of the corresponding quasi-coherent or coherent traces, Φ_{Σ_0}, Φ_{Σ_1}, … , Φ_{Σ_n}, ….

In order to do so, let $A_0, A_1, \ldots$ be an enumeration of all the existential formulae of the language [6] in which every formula occurs infinitely many times. Assume that $\Sigma_n = (\Lambda_n, \mathcal{P}_{oset_n}, \mathcal{F}_n)$ and Φ_{Σ_n} (with $n \geq 0$) are given and take the existential formula A_n of the above enumeration. Consider the finite set $\mathbb{S}_{\Sigma_n}$ and $S \subseteq \mathbb{S}_{\Sigma_n}$ the set of all states s.t. for every $s \in S$, "$A_n \in \Phi_{\Sigma_n}(s)$" is the antecedent of an active conditional. Let us now define Σ_{n+1} inductively:

(A) If $S = \varnothing$, then we establish $\Sigma_{n+1} = \Sigma_n$ and $\Phi_{\Sigma_{n+1}} = \Phi_{\Sigma_n}$, and continue the process considering the existential formula A_{n+1}. Clearly, if Φ_{Σ_n} is (quasi-)coherent, then so it is $\Phi_{\Sigma_{n+1}}$.

(B) If $S \neq \varnothing$, $\Lambda_n = \{0\}$ and A_n is of the form $\langle\overset{i\,j}{\rightarrow}\rangle B$ or $\langle\overset{j\,i}{\leftarrow}\rangle B$ with $i \neq 0$, then we apply Lemma 5.20.(i) by setting $\Sigma_{n+1} = \Sigma^{i/0}$, and we reconsider A_n, so that we reconfigure the enumeration of existential formulae by setting $A_{n+1} \mapsto A_n$, $A_{n+2} \mapsto A_{n+1}$,... It is easy to see that Σ_n is quasi-coherent but

[6] These are formulae with the prefixes F, P, $\langle\overset{i\,j}{\rightarrow}\rangle$ or $\langle\overset{i\,j}{\leftarrow}\rangle$ (for $i, j \in \mathbb{N}$).

not coherent, and Σ_{n+1} is coherent.

(C) If $S \neq \varnothing$, $\Lambda_n = \{0\}$ and A_n is of the form FB or PB, then, by several applications of Lemma 5.20.(ii), we can obtain a sequence, Σ_{n_1}, ..., Σ_{n_m}, of finite, simple quasi-Galois frames s.t. each of them is an extension of the previous one, and a corresponding sequence of (quasi-)coherent traces $\Phi_{\Sigma_{n_1}}, \ldots, \Phi_{\Sigma_{n_m}}$ s.t. $\Phi_{\Sigma_n} = \Phi_{\Sigma_{n_1}} \subseteq \ldots \subseteq \Phi_{\Sigma_{n_m}}$, so that each active conditional $(\alpha)_i$ (for $1 \leq i < m$) with antecedent $A_n \in \Phi_{\Sigma_{n_i}}(s)$ is exhausted for $\Phi_{\Sigma_{n_{i+1}}}$. Moreover, we set $\Sigma_{n+1} = \Sigma_{n_m}$ and $\Phi_{\Sigma_{n+1}} = \Phi_{\Sigma_{n_m}}$.

(D) If $S \neq \varnothing$ and either (i) $\Lambda_n \neq \{0\}$ or (ii) $\Lambda_n = \{0\}$ and A_n is of the form $\langle \overset{0\,j}{\rightarrow} \rangle B$ or $\langle \overset{j\,0}{\leftarrow} \rangle B$, then we can guarantee that Φ_{Σ_n} is coherent, so that we can do the same as in the previous case, but applying Lemma 5.21. Moreover, the coherence of $\Phi_{\Sigma_{n+1}}$ is also ensured.

Finally, we define $\Sigma = (\Lambda, \mathcal{P}_{oset}, \mathcal{F})$ with $\Lambda = \bigcup_{n\in\omega} \Lambda_n$, $\mathcal{P}_{oset} = \bigcup_{n\in\omega} \mathcal{P}_{oset_n}$, where $\bigcup_{n\in\omega} \mathcal{P}_{oset_n}$ means the pointwise union of all posets with the same index, and $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$. Let us show that Σ is a Galois frame. If $\mathcal{F} = \varnothing$, then this is trivial. Otherwise:

- Conditions (i), (ii) and (iii)(a) of Definition 3.1 are straightforwardly guaranteed by construction. As for condition (iii)(b), details are left to the reader, but note that for each state $s \in \Lambda$ and each $i, j \in \mathbb{N}$ with $i \neq j$ we found at most one active $\langle\rightarrow\rangle$-($\langle\leftarrow\rangle$-)conditional during the construction process –axiom 3.7. is needed to show this– (and at most two when $i = j$).
- Regarding condition (iii)(c), we have to show that Σ satisfies both requirements of Definition 2.2 (1.1 and 1.2) and that a residuated function is defined in $\mathcal{F}$ if and only if its residual is also in $\mathcal{F}$ (in symbols, $f_{ij}^{\rightarrow} \in \mathcal{F}$ iff $f_{ij}^{\leftarrow} \in \mathcal{F}$). In effect, assume that, at a given step of the construction, we have a quasi-residuated function $f_{ij}^{q,\rightarrow}$ with $f_{ij}^{q,\rightarrow}(s_i) = s_j$, but $f_{ij}^{q,\leftarrow}(s_j)$ does not exist, so that $Im(f_{ij}^{q,\rightarrow}) \not\subseteq Dom(f_{ij}^{q,\leftarrow})$. Then at a later step we will create a new point $f_{ij}^{q,\leftarrow}(s_j)$ where $f_{ij}^{q,\leftarrow}(s_j)$ will be associated with a new mc-set. It occurs similarly if we consider a quasi-residual function. Both cases are justified by Proposition 5.7 and the construction. This ensures property 1.1. Moreover, this argument also guarantees that $f_{ij}^{\rightarrow} \in \mathcal{F}$ iff $f_{ij}^{\leftarrow} \in \mathcal{F}$. Finally, Σ satisfies property 1.2 of Definition 2.2, because all members of the sequence of quasi-Galois frames satisfy that property.

Let us now show that Φ_Σ is coherent. If Φ_{Σ_0} is coherent, then it is clear that coherence is preserved through the construction. If Φ_{Σ_0} is not coherent, then it is not nominally coherent which means that there exists some formula in $\Phi_{\Sigma_0}(s_0)$ of the form $\lambda\langle \overset{i\,j}{\rightarrow} \rangle A$ or $\langle \overset{j\,i}{\leftarrow} \rangle A$ (with $i \neq 0$). Hence, (B) is reached at some point of the construction, so that we obtain a coherent trace (by Lemma 5.20.(i)), and this is preserved through the rest of the construction. Note that (B) is reached at most once in the whole process. Furthermore, by lemmas 5.20.(ii) and 5.21, each active conditional for a given Φ_{Σ_n} is exhausted sooner or later for a given Φ_{Σ_m}. Thus Φ_Σ is full.

Finally, we can define a Galois model (Σ, h) where $h(p) = \{s \in \mathbb{S}_\Sigma \mid p \in \Phi_\Sigma(s)\}$ for every propositional variable p. So, by the Truth Lemma (5.19), A is satisfiable. $\square$

6 Related work

The current paper can be understood as an integration of two different lines of research: the modal study of Galois connections (e.g., [13,15,19,22]) and the investigation of temporal×modal functional frames (e.g., [3,4,5,6]). Let us comment briefly on both, focusing on how they compare to our approach.

In the first place, there are several works in the literature looking at some interactions between modal operators and Galois connections (among others, [7,19,20,14,22,15,13]). In general, these works are based on the identification of a certain Galois connection within a standard Kripke frame (i.e., a pair (W, R) where $W \neq \emptyset$ and R is a relation on W), usually followed by a deep algebraic analysis (e.g., in [22,19,13]). Such identifications have been used with diverse particular purposes. For instance, [15] presents the *Information Logic of Galois Connections* (ILGC) as a means for reasoning about approximate information (represented mathematically as *rough sets*). ILGC is essentially the basic tense logic K_t taking F and H as primitive operators. The axiomatisation of ILGC is based on the fact that, when interpreted on Kripke frames, F and H form a Galois connection. More precisely, given a Kripke frame (W, R), we can look at F and H as semantic operators $2^W \longrightarrow 2^W$ by setting $F(X) = \{w \in W \mid \exists u.(w, u) \in R \quad \text{and} \quad u \in X\}$ and $H(X) = \{w \in W \mid \forall u, (u, w) \in R \quad \text{implies} \quad u \in X\}$; and it is clear that (F, H) is a Galois connection in $(2^W, \subseteq)$. In [13], the author does something similar, but using two-sorted Kripke frames (frames where the domain and range of the relation R are disjoint and form a partition of W). As a third example, [19] uses these sorted frames, which are nothing but formal contexts, in order to provide a logical tool for Formal Concept Analysis (see Example 1).

The second line of research is the one initiated in [4], where functional temporal×modal frames are introduced, and later developed in a series of papers ([3,5,6] among others). Our language is essentially the one used in [6] for the axiomatisation of surjective functions. The changes in the semantics of both works can be identified at first sight, the most important being the addition of condition (iii) in our definition of frame (Definition 3.1), and in the semantic clause for $\langle \overset{ij}{\longleftarrow} \rangle$.

By integrating both branches of research, we generalize existing frameworks for the modal study of Galois connections in at least three well-differentiated aspects. First, we consider Galois connections for partial functions, instead of the more usual and restricted definition that looks only at total functions. Second, our frames take into account connections among an arbitrary number of posets, instead of considering a single connection among two (possibly different) posets. Third, these frames can accommodate the representation of *any* poset $(\mathbb{P}_i, \leq_i)$, while in the quoted works these are always the complete lattice generated by a power set of worlds and the set inclusion relation among them $(2^W, \subseteq)$. Moreover, the pre-orders $\leq_i$ are first-class citizen in our semantics, described in turn with the temporal modalities F and P, while the subset inclusion among sets of worlds is left implicit in the logical treatment of the quoted papers.

7 Future work

Let us just mention three open lines for future work. First, computational aspects of reasoning tasks associated to S_{GP}, e.g., decidability and complexity of the provability problem, have been left out of this work. We believe, however, that the logic is indeed decidable. This would open the door to using our results as a first step towards an automated modal prover for the study of Galois connections. Second, as mentioned elsewhere in this paper, the notion of a Galois connection makes also sense among pre-ordered sets (sets endowed with a reflexive and transitive relation), hence a natural extension of our work consists in relaxing our set of assumptions on $\leq_i$ and studying the resulting logic. Finally, the link between our approach and Gaggle theory [9], a natural generalisation of Galois connections that has been shown very fruitful for logic, remains to be studied too.

Acknowledgements

We are very thankful for the AiML2022 reviewers' work, whose comments helped us to spot some mistakes in the initial version of this paper, as well as to substantially improve its final outlook. The research activity of Antonio Yuste-Ginel was partially funded by a grant from "Plan Propio de Investigación de la Universidad de Málaga". The authors also gratefully acknowledge funding from the projects PID2020-117871GB-I00 (Spanish Ministry of Science and Innovation) and PY20_01140 (Regional Government of Andalusia).

Appendix

In this section, we shall use the following conventions: by PC (Propositional Calculus), we denote proofs in the classical logic; and by ML we denote proofs in the basic multi-modal logic.

[**Proposition 2.5**]

Proof. For $1 \Rightarrow 2$: Let g be the function whose existence is affirmed in item 1.

- Let us see that $g \circ f$ is inflationary. In effect, given that for all $a \in Dom(f)$ we have that $f(a) \leq_B f(a)$ and, by hypothesis, we have that $Im(f) \subseteq Dom(g)$, item 1 ensures that $a \leq_A g(f(a))$.
- Let us see that $f \circ g$ is deflationary. In effect, given that for all $b \in Dom(g)$ we have that $g(b) \leq_A g(b)$ and, by hypothesis, we have that $Im(g) \subseteq Dom(f)$, item 1 guarantees that $f(g(b)) \leq_B b$.
- Let us see that f is monotone. Let $a_1, a_2 \in Dom(f)$ be s.t. $a_1 \leq_A a_2$,

since $g \circ f$ is inflationary we obtain $a_2 \leq_A g(f(a_2))$ and, as a consequence, $a_1 \leq_A g(f(a_2))$. Now, the hypothesis ensures that $f(a_1) \leq_B f(a_2)$.

- Let us see that g is monotone. Let $b_1, b_2 \in Dom(g)$ be s.t. $b_1 \leq_B b_2$, since $f \circ g$ is deflationary, we have that $f(g(b_1)) \leq_B b_1$ and, as a consequence, $f(g(b_1)) \leq_B b_2$. Now, the hypothesis ensures that $g(b_1) \leq_A g(b_2)$.

As for $2 \Rightarrow 1$: Let us see that for all $b \in Dom(g)$ we have that $f^{-1}(b\downarrow) = g(b)\downarrow \cap Dom(f)$ where g is the function whose existence is affirmed in item 2.

Assume $a \in f^{-1}(b\downarrow)$. Then $f(a) \leq_B b$. Since g is monotone, $(g \circ f)(a)$ is defined (given that $Im(f) \subseteq Dom(g)$) and $b \in Dom(g)$, we have that $g(f(a)) \leq_A g(b)$. Now, since $g \circ f$ is inflationary, we obtain $a \leq_A g(f(a)) \leq_A g(b)$, that is, $a \in g(b)\downarrow$.

Reciprocally, assume that $a \in g(b)\downarrow \cap Dom(f)$, that is, $a \leq_A g(b)$. Since f is monotone, $a \in Dom(f)$ and $Im(g) \subseteq Dom(f)$, we have that $f(a) \leq_B f(g(b))$ and given that $f \circ g$ is deflationary we obtain $f(a) \leq_B f(g(b)) \leq_B b$. Therefore, $f(a) \in b\downarrow$ and $a \in f^{-1}(b\downarrow)$.

Let us see that $a \leq_A g(b)$ if and only if $f(a) \leq_B b$.

- Assume $a \leq_A g(b)$, that is, $a \in g(b)\downarrow \cap Dom(f) = f^{-1}(b\downarrow)$ and, as a consequence, $f(a) \leq_B b$.
- Assume $f(a) \leq_B b$, that is, $f(a) \in b\downarrow$. Now, given that the function $f^{-1} : (2^B, \subseteq) \longrightarrow (2^A, \subseteq)$ defined by $f^{-1}(Y) = \{a \in A \mid f(a) \in Y\}$ is monotone we obtain $a \in f^{-1}(f(a)) \subseteq f^{-1}(b\downarrow) = g(b)\downarrow \cap Dom(f)$ and, as a consequence, $a \leq_A g(b)$.

$\square$

[**Theorem 5.1**]

Proof. Soundness follows S_{GP} from a standard inductive argument on the length of derivations. Let us just show, as an illustration, the validity of schema 3.3.1: $\langle \overset{i\,j}{\rightarrow} \rangle \lambda \langle \overset{k\,l}{\leftarrow} \rangle \top \rightarrow \bot$, where $j \neq l$.
Let $(\Lambda, \mathcal{P}_{oset}, \mathcal{F}, h)$ be a Galois model, $s \in \mathbb{S}_\Lambda$ and $j \neq l$. Suppose, for the sake of contradiction, that $s \in h(\langle \overset{i\,j}{\rightarrow} \rangle \lambda \langle \overset{k\,l}{\leftarrow} \rangle \top)$, which implies that there is an $f_{ij}^{\rightarrow} \in \mathcal{F}$ s.t. $f_{ij}^{\rightarrow}(s) \in h(\lambda \langle \overset{k\,l}{\leftarrow} \rangle \top)$. Since $f_{ij}^{\rightarrow}(s) \in h(\lambda \langle \overset{k\,l}{\leftarrow} \rangle \top)$, there is a $s_j \in \mathbb{P}_j$ s.t. $s_j \in h(\langle \overset{k\,l}{\leftarrow} \rangle \top)$ (this can be shown by induction on the length of λ). But then, by the semantic clause for $\langle \overset{k\,l}{\leftarrow} \rangle$, there is an $f_{kl}^{\leftarrow} \in \mathcal{F}$ s.t. $s_j \in Dom(f_{kl}^{\leftarrow})$, which in turn implies that $s_j \in \mathbb{P}_l$ (with $j \neq l$), and this is absurd. $\square$

[**Proposition 5.3**]

Proof. We only prove **T1** (**T2** is its Galois mirror image):

1.	$\langle\overset{ij}{\rightarrow}\rangle A \rightarrow [\overset{ij}{\rightarrow}]A$	Axiom 3.2
2.	$\langle\overset{ij}{\rightarrow}\rangle B \rightarrow [\overset{ij}{\rightarrow}]B$	Axiom 3.2
3.	$(\langle\overset{ij}{\rightarrow}\rangle A \wedge \langle\overset{ij}{\rightarrow}\rangle B) \rightarrow ([\overset{ij}{\rightarrow}]A \wedge [\overset{ij}{\rightarrow}]B)$	from 1, 2 by PC
4.	$([\overset{ij}{\rightarrow}]A \wedge [\overset{ij}{\rightarrow}]B) \rightarrow [\overset{ij}{\rightarrow}](A \wedge B)$	ML
5.	$(\langle\overset{ij}{\rightarrow}\rangle A \wedge \langle\overset{ij}{\rightarrow}\rangle B) \rightarrow [\overset{ij}{\rightarrow}](A \wedge B)$	from 3, 4 by PC
6.	$(\langle\overset{ij}{\rightarrow}\rangle A \wedge \langle\overset{ij}{\rightarrow}\rangle B \wedge [\overset{ij}{\rightarrow}](A \wedge B)) \rightarrow \langle\overset{ij}{\rightarrow}\rangle(A \wedge B)$	ML
7.	$(\langle\overset{ij}{\rightarrow}\rangle A \wedge \langle\overset{ij}{\rightarrow}\rangle B) \rightarrow \langle\overset{ij}{\rightarrow}\rangle(A \wedge B)$	from 5, 6 by PC

□

[**Proposition 5.5**]

Proof. The proof of (i) is standard in modal logic. With respect to (ii) we first show that $\Gamma_1 \prec^{\rightarrow}_{ij} \Gamma_2$ iff $\{A \mid [\overset{ij}{\rightarrow}]A \in \Gamma_1\} \subseteq \Gamma_2$.

The left-to-right direction is proved as follows: Assume $\Gamma_1 \prec^{\rightarrow}_{ij} \Gamma_2$ and also $[\overset{ij}{\rightarrow}]A \in \Gamma_1$. We have to show that $A \in \Gamma_2$. Now suppose the contrary, i.e., $A \notin \Gamma_2$. Hence $\langle\overset{ij}{\rightarrow}\rangle A \notin \Gamma_1$, because $\Gamma_1 \prec^{\rightarrow}_{ij} \Gamma_2$ iff $\varnothing \neq \{A \mid \langle\overset{ij}{\rightarrow}\rangle A \in \Gamma_1\} \subseteq \Gamma_2$ by Definition 5.4(b), and so $[\overset{ij}{\rightarrow}]\neg A \in \Gamma_1$ by ML, hence $[\overset{ij}{\rightarrow}]\bot \in \Gamma_1$ by ML again (since $[\overset{ij}{\rightarrow}]A \in \Gamma_1$). Since the set $\{A \mid \langle\overset{ij}{\rightarrow}\rangle A \in \Gamma_1\}$ is non-empty, it should be clear that $\langle\overset{ij}{\rightarrow}\rangle\top \in \Gamma_1$ which, by ML, leads to a contradiction.

For the right-to-left direction, suppose $\{A \mid [\overset{ij}{\rightarrow}]A \in \Gamma_1\} \subseteq \Gamma_2$ (†). We first show that $\{A \mid \langle\overset{ij}{\rightarrow}\rangle A \in \Gamma_1\} \neq \varnothing$. Assume the contrary, then we have that $\langle\overset{ij}{\rightarrow}\rangle\top \notin \Gamma_1$, hence $[\overset{ij}{\rightarrow}]\bot \in \Gamma_1$ by ML, so $\bot \in \Gamma_2$ by (†), and Γ_2 would be inconsistent, which is impossible. Now we shall show that $\{A \mid \langle\overset{ij}{\rightarrow}\rangle A \in \Gamma_1\} \subseteq \Gamma_2$. Consider $\langle\overset{ij}{\rightarrow}\rangle A \in \Gamma_1$, given axiom 3.2 we obtain $[\overset{ij}{\rightarrow}]A \in \Gamma_1$, so by (†) we get $A \in \Gamma_2$. This completes the proof of that direction. Moreover, the proof of $\{A \mid [\overset{ij}{\rightarrow}]A \in \Gamma_1\} \subseteq \Gamma_2$ iff $\{\langle\overset{ij}{\rightarrow}\rangle A \mid A \in \Gamma_2\} \subseteq \Gamma_1$ is standard in modal logic.

The proof of item (iii) is similar to that of (ii). □

[**Proposition 5.7**]

Proof. For (i), assume $\Gamma_1 \prec^{\rightarrow}_{ij} \Gamma_2$. All we have to prove is that the set $\{A \mid [\overset{ij}{\leftarrow}]A \in \Gamma_2\}$ is consistent. If not, then there are formulae $A_1, \ldots, A_n$ such that $[\overset{ij}{\leftarrow}]A_1, \ldots [\overset{ij}{\leftarrow}]A_n \in \Gamma_2$ and $\vdash \neg(A_1 \wedge \ldots \wedge A_n)$. Then, by ML, $\vdash ([\overset{ij}{\leftarrow}]A_1 \wedge \ldots \wedge [\overset{ij}{\leftarrow}]A_{n-1}) \rightarrow [\overset{ij}{\leftarrow}]\neg A_n$. Therefore $[\overset{ij}{\leftarrow}]\neg A_n \in \Gamma_2$, hence $[\overset{ij}{\leftarrow}]\bot \in \Gamma_2$ using ML again (since $[\overset{ij}{\leftarrow}]A_n \in \Gamma_2$). Now, given $\Gamma_1 \prec^{\rightarrow}_{ij} \Gamma_2$, we obtain $\langle\overset{ij}{\rightarrow}\rangle[\overset{ij}{\leftarrow}]\bot \in \Gamma_1$ (using $[\overset{ij}{\leftarrow}]\bot \in \Gamma_2$ and Proposition 5.5(ii)). Moreover, by the assumption and Definition 5.4(b), we also obtain $\langle\overset{ij}{\rightarrow}\rangle\top \in \Gamma_1$ (because $\top \in \Gamma_2$). So, by axiom 3.4, we get $\langle\overset{ij}{\rightarrow}\rangle\langle\overset{ij}{\leftarrow}\rangle\top \in \Gamma_1$. Thus, by **T1.** of Proposition 5.3 and the fact that $\langle\overset{ij}{\rightarrow}\rangle[\overset{ij}{\leftarrow}]\bot \in \Gamma_1$, we get $\langle\overset{ij}{\rightarrow}\rangle(\langle\overset{ij}{\leftarrow}\rangle\top \wedge [\overset{ij}{\leftarrow}]\bot) \in \Gamma_1$ and so $\langle\overset{ij}{\rightarrow}\rangle\langle\overset{ij}{\leftarrow}\rangle\bot \in \Gamma_1$, by

ML, which leads to a contradiction using items 2(c) and 2(d) of Proposition 5.6, since there would then exist an mc-set Γ such that $\perp \in \Gamma$, which is impossible. The proof of item (ii) is similar. $\square$

[**Proposition 5.8**]

Proof. We shall prove item (i). Assume $\Gamma_1 \prec_{ij}^{\rightarrow} \Gamma_2$, $\Gamma_2 \preceq_{\mathbb{P}} \Gamma_3$ and $\Gamma_3 \prec_{ij}^{\leftarrow} \Gamma_4$ and $GA \in \Gamma_1$. We will show that $A \in \Gamma_4$. Now, the axiom 3.5 establish that $\langle \overset{i\,j}{\rightarrow} \rangle F \langle \overset{i\,j}{\leftarrow} \rangle X \rightarrow FX$, so, from $GA \in \Gamma_1$ we obtain that $[\overset{i\,j}{\rightarrow}] G [\overset{i\,j}{\leftarrow}] A \in \Gamma_1$ (by ML). Since $\Gamma_1 \prec_{ij}^{\rightarrow} \Gamma_2$, from Proposition 5.5(ii), we get $G[\overset{i\,j}{\leftarrow}]A \in \Gamma_2$, and from $\Gamma_2 \preceq_{\mathbb{P}} \Gamma_3$, by Definition 5.4(a), we obtain $[\overset{i\,j}{\leftarrow}]A \in \Gamma_3$. Finally, since $\Gamma_3 \prec_{ij}^{\leftarrow} \Gamma_4$, by Proposition 5.5(iii), we get $A \in \Gamma_4$ as required. Thus $\Gamma_1 \preceq_{\mathbb{P}} \Gamma_4$.

The proof of item (ii) is similar. $\square$

[**Lemma 5.20**]

Proof. For item (i), let $\lambda \langle \overset{i\,j}{\rightarrow} \rangle B \in \Phi_{\Sigma^k}(s)$ (resp. $\lambda \langle \overset{j\,i}{\leftarrow} \rangle \in \Phi_{\Sigma^k}(s)$) the antecedent of (α), then the renaming we are looking for is just $\Sigma^{i/k}$. Item (ii) is proved using the standard construction for temporal logic [2]. $\square$

[**Lemma 5.21**]

Proof. Let Φ_{Σ_1} be a coherent trace of a finite quasi-Galois frame $\Sigma_1 = (\Lambda_1, \mathcal{P}_{oset_1}, \mathcal{F}_1)$, and let (α) be an active conditional for Φ_{Σ_1}. We want to construct an extension of Σ_1, call it Σ_2, together with a coherent trace Φ_{Σ_2} s.t. $\Phi_{\Sigma_1} \subseteq \Phi_{\Sigma_2}$ and (α) is exhausted for Φ_{Σ_2}.

In order to do so, if (α) is either a $\uparrow$-projectable or a $\downarrow$-projectable conditional, then such a construction is carried out following the standard way in temporal logic (see e.g., [2] or [1, Chapter 4.6]). So, let us consider only the case in which (α) is a $\langle\rightarrow\rangle$-projectable conditional.

Hence, assume $i, j \in \mathbb{N}$ and let (α) be the following active $\langle\rightarrow\rangle$-projectable conditional for Φ_Σ:

"If $\langle \overset{i\,j}{\rightarrow} \rangle A \in \Phi_{\Sigma_1}(s_i)$, then there exists $s_j = f_{ij}^{\rightarrow}(s_i)$ such that $A \in \Phi_{\Sigma_1}(s_j)$" [7]

Thus, we have that $\langle \overset{i\,j}{\rightarrow} \rangle A \in \Phi_{\Sigma_1}(s_i)$, but there is no $s_j = f_{ij}^{\rightarrow}(s_i)$ s.t. $A \in \Phi_{\Sigma_1}(s_j)$. Moreover, by item 2(c) of Proposition 5.6, there exists an mc-set, Γ, s.t. $\Phi_{\Sigma_1}(s_i) \prec_{ij}^{\rightarrow} \Gamma$ and $A \in \Gamma$. Furthermore, as $\langle \overset{i\,j}{\rightarrow} \rangle A \in \Phi_{\Sigma_1}(s_i)$, we have $i \in \Lambda_1$, so we continue by cases:

Case 1: $j \notin \Lambda_1$.

Then, we need a new poset labelled with j, namely $\mathbb{P}_j$, which requires extending Λ_1 and which contains one element, called s_j, associated with Γ. We also need to introduce a new function, $f_{ij}^{q,\rightarrow}$, extending $\mathcal{F}_1$ so that s_j is the image of s_i in $\mathbb{P}_j$. That is, we define $\Sigma_2 = (\Lambda_2, \mathcal{P}_{oset_2}, \mathcal{F}_2)$, an extension of Σ_1, and Φ_{Σ_2}, an extension of Φ_{Σ_1}, as follows:

[7] The match between the indices in the formula $\langle \overset{i\,j}{\rightarrow} \rangle A$ and the state s_i are guaranteed by the assumptions that (α) is active and Φ_{Σ_1} is coherent.

- $\Lambda_2 = \Lambda_1 \cup \{j\}$;
- $\mathcal{P}_{oset_2} = \mathcal{P}_{oset_1} \cup \{(\mathbb{P}_j, \leq_j)\}$, where $\mathbb{P}_j = \{s_j\}$ and $\leq_j = \{(s_j, s_j)\}$;
- $\mathcal{F}_2 = \mathcal{F}_1 \cup \{f_{ij}^{q,\rightarrow}\}$, where $f_{ij}^{q,\rightarrow} = \{(s_i, s_j)\}$;
- $\Phi_{\Sigma_2} = \Phi_{\Sigma_1} \cup \{(s_j, \Gamma)\}$.

It should be clear that Σ_2, as defined, is a quasi-Galois-frame, given that Σ_1 is a quasi-Galois frame. Let us show that Φ_{Σ_2} preserves coherency.

- Φ_{Σ_2} is poset-coherent by Proposition 5.6(3).
- Moreover, taking into account the definition of Φ_{Σ_2}, it is easy to see that it is functionally-coherent.
- As for the preservation of nominal coherence, the only new element in the frame Σ_2 is s_j, therefore we will focus our attention exclusively on it. The fact that $\langle\overset{i\,j}{\rightarrow}\rangle A \in \Phi_{\Sigma_1}(s_i)$ together with axioms 3.3.1, 3.3.2, 3.3.5 and 3.3.6 prevent that a formula of the form $\lambda\langle\overset{k\,l}{\rightarrow}\rangle\top$ or of the form $\lambda\langle\overset{l\,k}{\leftarrow}\rangle\top$ (being $k \neq j$) appears in $\Phi_{\Sigma_1}(s_j)(=\Gamma)$, hence Φ_{Σ_2} is nominally coherent too.

  The fact that $\langle\overset{i\,j}{\rightarrow}\rangle A \in \Phi_{\Sigma_1}(s_i)$ and the comments of Remark 5.16 ensure us that Φ_{Σ_2} is nominally coherent too.

Case 2: $j \in \Lambda_1$.

We distinguish two relevant subcases:

Case 2.1: $f_{ij}^{q,\leftarrow} \notin \mathcal{F}_1$.

We define $\Sigma_2 = (\Lambda_2, \mathcal{P}_{oset_2}, \mathcal{F}_2)$ analogously to what we did in **Case 1**. However, we do not need to create a new poset $\mathbb{P}_j$, because it already exists, but just to introduce a new point s_j that will be the $f_{ij}^{q,\rightarrow}$-image of s_i. So, we define:

- $\Lambda_2 = \Lambda_1$;
- $\mathcal{P}_{oset_2} = (\mathcal{P}_{oset_1} \setminus \{(\mathbb{P}_j, \leq_j)\}) \cup \{(\mathbb{P}'_j, \leq'_j)\}$ where $\mathbb{P}'_j = \mathbb{P}_j \cup \{s_j\}$ and $\leq'_j = \leq_j \cup \{(s_j, s_j)\}$;
- $\mathcal{F}_2 = \begin{cases} (\mathcal{F}_1 \setminus \{f_{ij}^{q,\rightarrow}\}) \cup \{f_{ij}^{\prime q,\rightarrow}\}, \text{ where } f_{ij}^{\prime q,\rightarrow} = f_{ij}^{q,\rightarrow} \cup \{(s_i, s_j)\} \\ \qquad\qquad\qquad\qquad\qquad \text{if } f_{ij}^{q,\rightarrow} \in \mathcal{F}_1; \\ \mathcal{F}_1 \cup \{f_{ij}^{q,\rightarrow'}\}, \text{ where } f_{ij}^{q,\rightarrow'} = \{(s_i, s_j)\} \\ \qquad\qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}$
- $\Phi_{\Sigma_2} = \Phi_{\Sigma_1} \cup \{(s_j, \Gamma)\}$.

It is easy to check that Σ_2 is a quasi-Galois frame. Note that we don't have to check that $(f_{ij}^{\prime q,\rightarrow}, f_{ij}^{\prime q,\leftarrow'})$ form a quasi-Galois connection, because $f_{ij}^{\prime q,\leftarrow'}$ does not exist by hypothesis. Moreover, it is also easy to check that Φ_{Σ_2} is coherent.

Case 2.2: $f_{ij}^{q,\leftarrow} \in \mathcal{F}_1$.

Let us define the set:

$$X = \{s \in Dom(f_{ij}^{q,\leftarrow}) \mid f_{ij}^{q,\leftarrow}(s) \in s_i \uparrow\}.$$

Now, we define $\Sigma_2 = (\Lambda_2, \mathcal{P}_{oset_2}, \mathcal{F}_2)$ and Φ_{Σ_2}, where Λ_2, $\mathcal{F}_2$, and Φ_{Σ_2} are just as in **Case 2.1**, and $\mathcal{P}_{oset_2} = (\mathcal{P}_{oset_1} - \{(P_j, \leq_j)\}) \cup \{(P'_j, \leq'_j)\}$, where $\mathbb{P}'_j = \mathbb{P}_j \cup \{s_j\}$ and $\leq'_j$ is the transitive closure of the relation $\leq_j \cup \{(s_j, s_j)\} \cup \{(s_j, s) \mid s \in X\}$.

The frame so defined is a quasi-Galois frame. It suffices to show that $(f_{ij}^{\prime q,\rightarrow}, f_{ij}^{\prime q,\leftarrow})$ form a quasi-Galois connection. So let $x_i \in Dom(f_{ij}^{\prime q,\rightarrow}), x_j \in Dom(f_{ij}^{\prime q,\leftarrow})$, we need to show that

$$x_i \leq'_i f_{ij}^{\prime q,\leftarrow}(x_j) \text{ iff } f_{ij}^{\prime q,\rightarrow}(x_i) \leq'_j x_j.$$

For the left-to-right direction, suppose $x_i \leq'_i f_{ij}^{\prime q,\leftarrow}(x_j)$. We analyse two cases. First, if $x_i \neq s_i$, then we have $x_i \leq_i f_{ij}^{q,\leftarrow}(x_j)$ (because $f_{ij}^{\prime q,\leftarrow} = f_{ij}^{q,\leftarrow}$), and then $f_{ij}^{q,\rightarrow}(x_i) \leq_j x_j$ (because $(f_{ij}^{q,\rightarrow}, f_{ij}^{q,\leftarrow})$ is a quasi-Galois connection, by hypothesis), which implies $f_{ij}^{\prime q,\rightarrow}(x_i) \leq'_j x_j$ (because $f_{ij}^{q,\rightarrow} \subseteq f_{ij}^{\prime q,\rightarrow}$ and $\leq_j \subseteq \leq'_j$ by construction). If $x_i = s_i$, then $f_{ij}^{\prime q,\rightarrow}(x_i) \leq'_j x_j$ follows immediately by definition of $\leq'_j$ (because $x_j \in X$). The right-to-left direction is analogous.

As for the treatment of $\langle\leftarrow\rangle$-projectable conditionals, it is similar to the previous case. However, it is worth noticing that when we arrive to the case that is analogous to **Case 2.2** above (i.e., $f_{ij}^{q,\rightarrow} \in \mathcal{F}_1$, with $\langle \overset{ij}{\leftarrow} \rangle A \in \Phi_\Sigma(s_j)$ and $s_i = f_{ij}^{\prime q,\leftarrow}(s_j)$) we have to consider the set $Y = \{s \in Dom(f_{ij}^{q,\rightarrow}) \mid f_{ij}^{q,\rightarrow}(s) \in s_j \downarrow\}$ instead of X. Moreover, $\leq'_i$ is the transitive closure of the relation $\leq_i \cup \{(s_i, s_i)\} \cup \{(s, s_i) \mid s \in Y\}$. □

References

[1] Blackburn, P., M. De Rijke and Y. Venema, "Modal Logic," Cambridge University Press, 2002.

[2] Burgess, J. P., *Basic tense logic*, in: D. M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic* (2002), pp. 1–42.

[3] Burrieza, A. and I. P. de Guzmán, *A functional approach for temporalXmodal logics*, Acta informatica **39** (2003), pp. 71–96.

[4] Burrieza, A., I. P. de Guzman and E. Muñoz, *Indexed flows in temporalXmodal logic with functional semantics*, in: M. Fisher and A. Artale, editors, *Proceedings Ninth International Symposium on Temporal Representation and Reasoning*, IEEE, 2002, pp. 146–153.

[5] Burrieza, A., I. P. de Guzmán and E. Muñoz-Velasco, *Analyzing completeness of axiomatic functional systems for temporal×modal logics*, Mathematical Logic Quarterly **56** (2010), pp. 89–102.

[6] Burrieza, A., I. Fortes and I. P. de Guzmán, *Completeness of a functional system for surjective functions*, Mathematical Logic Quarterly **63** (2017), pp. 574–597.

[7] Demri, S. P. and E. Orlowska, "Incomplete information: Structure, inference, complexity," Springer, 2013.

[8] Denecke, K., M. Erné and S. L. Wismath, "Galois connections and applications," Springer, 2004.

[9] Dunn, J. M., *Gaggle theory: An abstraction of galois connections and residuation, with applications to negation, implication, and various logical operators*, in: J. van Eijck, editor, *Logics in AI* (1991), pp. 31–51.

[10] Erné, M., J. Koslowski, A. Melton and G. E. Strecker, *A primer on galois connections*, Annals of the New York Academy of Sciences **704** (1993), pp. 103–125.

[11] Ganter, B. and R. Wille, "Formal concept analysis: mathematical foundations," Springer, 2012.

[12] Gödel, K., *Eine interpretation des intuitionistischen aussagenkalküls*, in: E. Dierker and K. Sigmund, editors, *Ergebnisse eines mathematischen Kolloquiums 4 (1933)*, pp. 39–40, reprinted and translated in: S. Feferman et al.(eds.), Kurt Gödel. Collected Works. Vol. 1, 1986.

[13] Hartonas, C., *Lattice logic as a fragment of (2-sorted) residuated modal logic*, Journal of Applied Non-Classical Logics **29** (2019), pp. 152–170.

[14] Horn, A., *Dynamic epistemic algebra with post-conditions to reason about robot navigation*, in: L. D. Beklemishev and R. J. G. B. de Queiroz, editors, *Proceedings of WoLLIC 2011*, Lecture Notes in Computer Science **6642** (2011), pp. 161–175.

[15] Järvinen, J., M. Kondo and J. Kortelainen, *Logics from galois connections*, Int. J. Approx. Reason. **49** (2008), pp. 595–606.
URL `https://doi.org/10.1016/j.ijar.2008.06.003`

[16] McKinsey, J. C. C., *A solution of the decision problem for the Lewis systems S2 and s4, with an application to topology*, J. Symb. Log. **6** (1941), pp. 117–134.

[17] McKinsey, J. C. C. and A. Tarski, *The algebra of topology*, Annals of mathematics (1944), pp. 141–191.

[18] Melton, A., D. A. Schmidt and G. E. Strecker, *Galois connections and computer science applications*, in: D. Pitt, S. Abramsky, A. Poigné and D. Rydeheard, editors, *Category Theory and Computer Programming* (1986), pp. 299–312.

[19] Orlowska, E. and I. Rewitzky, *Context algebras, context frames, and their discrete duality*, Trans. Rough Sets **9** (2008), pp. 212–229.

[20] Orlowska, E. and I. Rewitzky, *Algebras for Galois-style connections and their discrete duality*, Fuzzy Sets Syst. **161** (2010), pp. 1325–1342.

[21] Pöschel, R., *Galois connections for operations and relations*, in: K. Denecke, M. Erné and S. L. Wismath, editors, *Galois Connections and Applications*, Springer, 2004 pp. 231–258.

[22] von Karger, B., *Temporal algebra*, Math. Struct. Comput. Sci. **8** (1998), pp. 277–320.