Parametrized modal logic I: an introduction

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Abstract

In this paper, within the context of a two-typed parametrized modal language, we define a parametrized modal logic as a couple whose components are sets of formulas containing, in their respective types, all propositional tautologies and the distribution axiom and closed, in their respective types, under modus ponens, uniform substitution and generalization. We axiomatically introduce different two-typed parametrized modal logics and we prove their completeness with respect to appropriate classes of two-typed frames by means of an adaptation of the canonical model construction.

Keywords: Parametrized modal logic. Complete axiomatization. Canonical model construction. Bounded morphism Lemma. Epistemic reasoning.

1 Introduction

In an application domain such as reasoning about knowledge where states and agents have been identified as the primitive entities of interest, one usually considers relational structures of the form $(S, \equiv)$ where S is a nonempty set of states and $\equiv$ is a function associating an equivalence relation $\equiv_a$ on S to every element a of a fixed set A of agents [8,9,17]. In that setting, for all $a \in A$, two states s and t are equivalent modulo $\equiv_a$ exactly when a cannot distinguish between s and t. When one wants to reason about distributed knowledge, it is of interest to assume that $\equiv$ is also a function associating an equivalence relation $\equiv_B$ on S to every $B \in \wp(A)$ in such a way that for all $B \in \wp(A)$, $\equiv_B = \bigcap\{\equiv_a\colon\ a \in B\}$.

The modal language interpreted over relational structures of the form $(S, \equiv)$ traditionally consists of one type of formulas: state-formulas — to be interpreted by sets of states. State-formulas are constructed over the Boolean connectives and the modal connectives $[B]$ — B ranging over $\wp(A)$. The state-formula $[B]\varphi$ is true in a state s of some model if the state-formula φ is true in every state of that model that can be distinguished from s by no B-agents.

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Although agents are omnipresent in the standard syntax of modal languages used for talking about relational structures of the form $(S, \equiv)$, states are the only entities that these relational structures take as first-class citizens. However, there is no need to force oneself to find examples where, in addition to equivalence relations between states parametrized by sets of agents, one would like to have at hand binary relations between agents parametrized by sets of states.

Indeed, in many situations, one would like to use relational structures of the form $(S, A, \equiv, \triangleright)$ where on top of the above-considered elements S and $\equiv$, one can find a nonempty set A of agents and a function $\triangleright$ associating a binary relation $\triangleright_s$ on A to every element s of S. Which situations? Situations where relationships between agents such as the following ones have to be taken into account: "agent a trusts agent b in state s", "agent a is a friend of agent b in state s", etc [3]. In these situations, for all $s \in S$, two agents a and b are related by $\triangleright_s$ exactly when a trusts b in state s, a is a friend of b in state s, etc. Moreover, on top of the assumption that $\equiv$ is also a function associating an equivalence relation $\equiv_B$ on S to every $B \in \wp(A)$ in such a way that for all $B \in \wp(A)$, $\equiv_B = \bigcap\{\equiv_a : a \in B\}$, one will naturally assume that $\triangleright$ is also a function associating a binary relation $\triangleright_T$ on A to every $T \in \wp(S)$ in such a way that for all $T \in \wp(S)$, $\triangleright_T = \bigcap\{\triangleright_s : s \in T\}$.

The modal language interpreted over relational structures of the form $(S, A, \equiv, \triangleright)$ will naturally consist of two types of formulas: state-formulas — to be interpreted by sets of states — and agent-formulas — to be interpreted by sets of agents. State-formulas will be constructed over the Boolean connectives and the modal connectives $[\alpha]$ — α ranging over the set of all agent-formulas — whereas agent-formulas will be constructed over the Boolean connectives and the modal connectives $[\varphi]$ — φ ranging over the set of all state-formulas. The state-formula $[\alpha]\varphi$ will be true in a state s of some model if the state-formula φ is true in every state of that model that can be distinguished from state s by no α-agents whereas the agent-formula $[\varphi]\alpha$ will be true in an agent a of some model if the agent-formula α is true in every agent of that model that is trusted by agent a at all φ-states.

In this paper, within the context of a two-typed parametrized modal language, we define a parametrized modal logic as a couple whose components are sets of formulas containing, in their respective types, all propositional tautologies and the distribution axiom and closed, in their respective types, under modus ponens, uniform substitution and generalization. We axiomatically introduce different two-typed parametrized modal logics and we prove

[3] See [4,14,25] and [15,24,28] for examples of situations in which one would like to use relationships such as "trusts", "is a friend of", etc.

their completeness with respect to appropriate classes of two-typed frames by means of an adaptation of the canonical model construction [4].

2 Syntax

From now on, when we write "$\underline{1}$" we will mean "2" and when we write "$\underline{2}$" we will mean "1". For all $i \in \{1,2\}$, let $\mathcal{P}_i$ be a countably infinite set (with typical members denoted p_i, q_i, etc). For all $i \in \{1,2\}$, members of $\mathcal{P}_i$ will be called *atomic i-formulas.* An *atomic formula* is either an atomic 1-formula, or an atomic 2-formula. We will always assume that $\mathcal{P}_1$ and $\mathcal{P}_2$ are disjoint. A *tip* is a couple (Σ_1, Σ_2) where for all $i \in \{1,2\}$, Σ_i is a set of finite words over the alphabet $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \{\bot_1, \bot_2, \neg_1, \neg_2, \vee_1, \vee_2, \Box_1, \Box_2, (,)\}$ (with typical members denoted φ, ψ, etc). Let $\sqsubseteq$ be the partial order between tips defined by

- $(\Sigma_1, \Sigma_2) \sqsubseteq (\Delta_1, \Delta_2)$ if and only if for all $i \in \{1,2\}$, $\Sigma_i \subseteq \Delta_i$.

Let $(\mathcal{L}_1, \mathcal{L}_2)$ be the least tip such that for all $i \in \{1,2\}$, $\mathcal{P}_i \subseteq \mathcal{L}_i$ and

- $\bot_i \in \mathcal{L}_i$,
- for all $\varphi \in \mathcal{L}_i$, $\neg_i\varphi \in \mathcal{L}_i$,
- for all $\varphi, \psi \in \mathcal{L}_i$, $(\varphi \vee_i \psi) \in \mathcal{L}_i$,
- for all $\varphi \in \mathcal{L}_{\underline{i}}$ and for all $\psi \in \mathcal{L}_i$, $(\varphi\Box_i\psi) \in \mathcal{L}_i$.

Obviously, $\mathcal{L}_1$ and $\mathcal{L}_2$ are disjoint. For all $i \in \{1,2\}$, members of $\mathcal{L}_i$ will be called *i-formulas.* A *formula* is either a 1-formula, or a 2-formula. Let $\mathcal{L}_{\mathbf{PML}}$ —the *language of parametrized modal logic* — be the set of all formulas. For all $i \in \{1,2\}$, the Boolean connectives $\top_i$, $\wedge_i$, $\rightarrow_i$ and $\leftrightarrow_i$ are defined as usual. For all $i \in \{1,2\}$, the modal connective $\lozenge_i$ is defined by $(\varphi\lozenge_i\psi) ::= \neg_i(\varphi\Box_i\neg_i\psi)$, where φ ranges over $\mathcal{L}_{\underline{i}}$ and ψ ranges over $\mathcal{L}_i$. For all $i \in \{1,2\}$, for all $\varphi \in \mathcal{L}_{\underline{i}}$ and for all $\psi \in \mathcal{L}_i$, we will write "$[\varphi]_i\psi$" instead of "$(\varphi\Box_i\psi)$". For all $i \in \{1,2\}$, for all $\varphi \in \mathcal{L}_{\underline{i}}$ and for all $\psi \in \mathcal{L}_i$, we will write "$\langle\varphi\rangle_i\psi$" instead of "$(\varphi\lozenge_i\psi)$". For all $i \in \{1,2\}$, for all $\varphi \in \mathcal{L}_{\underline{i}}$ and for all sets Σ_i of i-formulas, let $[\varphi]\Sigma_i = \{\psi : [\varphi]_i\psi \in \Sigma_i\}$. A tip (Σ_1, Σ_2) is *readable* if $(\Sigma_1, \Sigma_2) \sqsubseteq (\mathcal{L}_1, \mathcal{L}_2)$. When writing formulas, most of the times, we will not make explicit the types of the connectives constituting them: if we know the type of a formula then we can inductively determine the types of its constituents. Therefore, for all $i \in \{1,2\}$, the result of uniformly replacing the atomic formulas of a given Boolean formula by arbitrary i-formulas can be seen as a i-formula. As a result, for all $i \in \{1,2\}$, we will talk about "the i-formula $(p \vee \neg q)$" instead of talking about "the formula $(p_i \vee_i \neg_i q_i)$", we will talk about "the i-formula $(p \vee (\neg q\Box\bot))$" instead of talking about "the formula $(p_i \vee_i (\neg_{\underline{i}} q_{\underline{i}} \Box_i \bot_i))$", etc. We adopt the standard rules for omission of the parentheses. A *substitution* is a couple (σ_1, σ_2) of functions $\sigma_1 : \mathcal{L}_1 \longrightarrow \mathcal{L}_1$ and $\sigma_2 : \mathcal{L}_2 \longrightarrow \mathcal{L}_2$ such that for all $i \in \{1,2\}$,

[4] See Propositions 5.5, 5.6, 6.9, 6.10, 6.20, and 6.21 for the main completeness results of the paper. A sketch of an alternative proof of Proposition 6.11 is included in the Appendix. The proofs of immediate consequences of the definitions are not given.

- $\sigma_i(\bot) = \bot$,
- $\sigma_i(\neg\varphi) = \neg\sigma_i(\varphi)$,
- $\sigma_i(\varphi \vee \psi) = \sigma_i(\varphi) \vee \sigma_i(\psi)$,
- $\sigma_i([\varphi]\psi) = [\sigma_{\underline{i}}(\varphi)]\sigma_i(\psi)$.

As a result, $\sigma_i(\langle\varphi\rangle\psi) = \langle\sigma_{\underline{i}}(\varphi)\rangle\sigma_i(\psi)$.

3 Relational semantics

A *frame* is a 4-tuple (W_1, W_2, R_1, R_2) where for all $i \in \{1, 2\}$, W_i is a nonempty set and $R_i : \wp(W_{\underline{i}}) \longrightarrow \wp(W_i \times W_i)$. A *frame of indiscernibility* is a frame (W_1, W_2, R_1, R_2) such that for all $i \in \{1, 2\}$ and for all $A \in \wp(W_{\underline{i}})$, $R_i(A)$ is an equivalence relation on W_i. A frame (W_1, W_2, R_1, R_2) is *conjunctive* if for all $i \in \{1, 2\}$ and for all $A \in \wp(W_{\underline{i}})$, $R_i(A) = \bigcap\{R_i(\{s_{\underline{i}}\}) : s_{\underline{i}} \in A\}$.

Example 3.1 *Let* **Ob** *be a nonempty set of* objects, **At** *be a nonempty set of* attributes, **Val** *be a nonempty set of* values *and* $m : \mathbf{Ob} \times \mathbf{At} \longrightarrow \wp(\mathbf{Val})$. *Systems such as the* 4*-tuple* $(\mathbf{Ob}, \mathbf{At}, \mathbf{Val}, m)$ *have been introduced and developed by Orłowska and Pawlak within the context of analysis of data and representation of nondeterministic information [18,20,21,23]. In* $(\mathbf{Ob}, \mathbf{At}, \mathbf{Val}, m)$*, the objects* o *and* o' *are* equivalent for the attribute a *if* $m(o, a) = m(o', a)$ *whereas the attributes* a *and* a' *are* equivalent for the object o *if* $m(o, a) = m(o, a')$. *Obviously, the frame* (W_1', W_2', R_1', R_2') *where*

- $W_1' = \mathbf{Ob}$,
- $W_2' = \mathbf{At}$,
- *for all* $At \in \wp(\mathbf{At})$, $R_1'(At)$ *is the binary relation on* **Ob** *defined by*
  - $oR_1'(At)o'$ *if and only if for all* $a \in At$, $m(o, a) = m(o', a)$,
- *for all* $Ob \in \wp(\mathbf{Ob})$, $R_2'(Ob)$ *is the binary relation on* **At** *defined by*
  - $aR_2(Ob)a'$ *if and only if for all* $o \in Ob$, $m(o, a) = m(o, a')$.

is conjunctive. In this frame, two objects are related by a set of attributes if and only if these objects are equivalent for all attributes in that set whereas two attributes are related by a set of objects if and only if these attributes are equivalent for all objects in that set[5] .

A frame (W_1, W_2, R_1, R_2) is *unitary* if for all $i \in \{1, 2\}$, $R_i(\emptyset) = W_i \times W_i$, i.e. $R_i(\emptyset)$ is the *universal relation on* W_i. Obviously, every conjunctive frame is unitary. A unitary frame (W_1, W_2, R_1, R_2) is *preconjunctive* if for all $i \in \{1, 2\}$ and for all $A, B \in \wp(W_{\underline{i}})$, $R_i(A \cup B) = R_i(A) \cap R_i(B)$. A unitary frame (W_1, W_2, R_1, R_2) is *paraconjunctive* if for all $i \in \{1, 2\}$ and for all $A, B \in \wp(W_{\underline{i}})$, if $A \subseteq B$ then $R_i(A) \supseteq R_i(B)$.

Proposition 3.2 *Every conjunctive frame is preconjunctive.*

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[5] For all sets At of attributes, the above-defined binary relation $R_1'(At)$ between objects is exactly the *strong indiscernibility relation* considered by Demri, Orłowska and Vakarelov [7,32,33]: two objects are in the relation $R_1'(At)$ if and only if they have the same values for all attributes in At.

Proposition 3.3 *Every preconjunctive frame is paraconjunctive.*

Example 3.4 *Let* (W_1, W_2, R_1, R_2) *be the frame where* $W_1 = \mathbb{N}$, $W_2 = \mathbb{N}$, *for all* $A \in \wp(\mathbb{N})$, *if* A *is finite then* $R_1(A) = \mathbb{N} \times \mathbb{N}$ *else* $R_1(A) = \emptyset$ *and for all* $A \in \wp(\mathbb{N})$, *if* A *is finite then* $R_2(A) = \mathbb{N} \times \mathbb{N}$ *else* $R_2(A) = \emptyset$. *Obviously, this frame is preconjunctive. However, it is not conjunctive, seeing that for all* $i \in \{1, 2\}$, $R_i(\mathbb{N}) = \emptyset$ *and* $\bigcap\{R_i(\{s_{\underline{i}}\}) : s_{\underline{i}} \in \mathbb{N}\} = \mathbb{N} \times \mathbb{N}$.

Example 3.5 *Let* (W_1, W_2, R_1, R_2) *be the frame where* $W_1 = \mathbb{N}$, $W_2 = \mathbb{N}$, *for all* $A \in \wp(\mathbb{N})$, *if* $\mathtt{Card}(A) < 2$ *then* $R_1(A) = \mathbb{N} \times \mathbb{N}$ *else* $R_1(A) = \emptyset$ *and for all* $A \in \wp(\mathbb{N})$, *if* $\mathtt{Card}(A) < 2$ *then* $R_2(A) = \mathbb{N} \times \mathbb{N}$ *else* $R_2(A) = \emptyset$. *Obviously, this frame is paraconjunctive. However, it is not preconjunctive, seeing that for all* $i \in \{1, 2\}$, $R_i(\{0, 1\}) = \emptyset$ *and* $R_i(\{0\}) \cap R_i(\{1\}) = \mathbb{N} \times \mathbb{N}$.

A *valuation on a frame* (W_1, W_2, R_1, R_2) is a couple (V_1, V_2) of functions $V_1 : \mathcal{L}_1 \longrightarrow \wp(W_1)$ and $V_2 : \mathcal{L}_2 \longrightarrow \wp(W_2)$ such that for all $i \in \{1, 2\}$,

- $V_i(\bot) = \emptyset$,
- $V_i(\neg\varphi) = W_i \setminus V_i(\varphi)$,
- $V_i(\varphi \vee \psi) = V_i(\varphi) \cup V_i(\psi)$,
- $V_i([\varphi]\psi) = \{s_i \in W_i : \forall t_i \in W_i \ (s_i R_i(V_{\underline{i}}(\varphi))t_i \ \Rightarrow \ t_i \in V_i(\psi))\}$.

As a result, $V_i(\langle\varphi\rangle\psi) = \{s_i \in W_i : \exists t_i \in W_i \ (s_i R_i(V_{\underline{i}}(\varphi))t_i \ \& \ t_i \in V_i(\psi))\}$. A *model* is a 6-tuple consisting of a frame and a valuation on that frame. A model is *conjunctive (resp., unitary, preconjunctive, paraconjunctive)* if it is based on a conjunctive (resp., unitary, preconjunctive, paraconjunctive) frame.

Lemma 3.6 *For all unitary models* $(W_1, W_2, R_1, R_2, V_1, V_2)$, *for all* $i \in \{1, 2\}$ *and for all i-formulas* φ, *the i-formula* $[\bot]\varphi$ *is such that*

- *if* $V_i(\varphi) = W_i$ *then* $V_i([\bot]\varphi) = W_i$,
- *otherwise,* $V_i([\bot]\varphi) = \emptyset$.

For all $i \in \{1, 2\}$, a i-formula φ is *true in a model* $(W_1, W_2, R_1, R_2, V_1, V_2)$ (in symbols $(W_1, W_2, R_1, R_2, V_1, V_2) \models \varphi$) if $V_i(\varphi) = W_i$.

Lemma 3.7 *For all* $i \in \{1, 2\}$ *and for all i-formulas* φ, *the i-formulas* $[\bot]\varphi \to \varphi$ *and* $\langle\bot\rangle\varphi \to [\bot]\langle\bot\rangle\varphi$ *are true in any unitary model.*

Lemma 3.8 *For all* $i \in \{1, 2\}$, *for all* $\underline{i}$*-formulas* φ, ψ *and for all i-formulas* χ, *the i-formula* $\langle\varphi \vee \psi\rangle\chi \to \langle\varphi\rangle\chi \wedge \langle\psi\rangle\chi$ *is true in any paraconjunctive model.*

A formula φ is *valid on a frame* (W_1, W_2, R_1, R_2) (in symbols $(W_1, W_2, R_1, R_2) \models \varphi$) if for all (W_1, W_2, R_1, R_2)-valuations (V_1, V_2), $(W_1, W_2, R_1, R_2, V_1, V_2) \models \varphi$.

Example 3.9 *In a frame* (W_1, W_2, R_1, R_2), *one may easily prove that for all* $i \in \{1, 2\}$,

- *the i-formula* $[\bot]p \to p$ *is valid if and only if for all* $s_i \in W_i$, $s_i R_i(\emptyset)s_i$,
- *the i-formula* $[\bot]p \to [\bot][\bot]p$ *is valid if and only if for all* $s_i, t_i, u_i \in W_i$, *if* $s_i R_i(\emptyset)t_i$ *and* $t_i R_i(\emptyset)u_i$ *then* $s_i R_i(\emptyset)u_i$,

- *the i-formula* $\langle\bot\rangle p \to [\bot]\langle\bot\rangle p$ *is valid if and only if for all* $s_i, t_i, u_i \in W_i$*, if* $s_i R_i(\emptyset) t_i$ *and* $s_i R_i(\emptyset) u_i$ *then* $t_i R_i(\emptyset) u_i$*.*

Example 3.10 *In a conjunctive frame* (W_1, W_2, R_1, R_2)*, one may easily prove that for all* $i \in \{1, 2\}$*,*

- *the i-formula* $\langle p\rangle q \to [p]q$ *is valid if and only if* $\texttt{Card}(W_i) \leq 1$*,*
- *the i-formula* $[p]q \to q$ *if and only if for all* $s_i \in W_i$ *and for all* $t_{\underline{i}} \in W_{\underline{i}}$*,* $s_i R_i(\{t_{\underline{i}}\}) s_i$*,*
- *the i-formula* $[p]q \to [p][p]q$ *if and only if for all* $s_i, t_i, u_i \in W_i$ *and for all* $v_{\underline{i}} \in W_{\underline{i}}$*, if* $s_i R_i(\{v_{\underline{i}}\}) t_i$ *and* $t_i R_i(\{v_{\underline{i}}\}) u_i$ *then* $s_i R_i(\{v_{\underline{i}}\}) u_i$*,*
- *the i-formula* $\langle p\rangle q \to [p]\langle p\rangle q$ *if and only if for all* $s_i, t_i, u_i \in W_i$ *and for all* $v_{\underline{i}} \in W_{\underline{i}}$*, if* $s_i R_i(\{v_{\underline{i}}\}) t_i$ *and* $s_i R_i(\{v_{\underline{i}}\}) u_i$ *then* $t_i R_i(\{v_{\underline{i}}\}) u_i$*.*

A formula φ is *valid on a class* $\mathcal{C}$ *of frames* (in symbols $\mathcal{C} \models \varphi$) if for all frames (W_1, W_2, R_1, R_2) in $\mathcal{C}$, $(W_1, W_2, R_1, R_2) \models \varphi$.

Lemma 3.11 *For all frames* (W_1, W_2, R_1, R_2)*,* (W_1, W_2, R_1, R_2) *is a frame of indiscernibility if and only if for all* $i \in \{1, 2\}$*, for all* $\underline{i}$*-formulas* φ *and for all i-formulas* ψ*,* $(W_1, W_2, R_1, R_2) \models [\varphi]\psi \to \psi$ *and* $(W_1, W_2, R_1, R_2) \models \langle\varphi\rangle\psi \to [\varphi]\langle\varphi\rangle\psi$*.*

A *bounded morphism from a model* $(W_1, W_2, R_1, R_2, V_1, V_2)$ *to a model* $(W'_1, W'_2, R'_1, R'_2, V'_1, V'_2)$ is a couple (f_1, f_2) of functions $f_1 : W_1 \longrightarrow W'_1$ and $f_2 : W_2 \longrightarrow W'_2$ such that for all $i \in \{1, 2\}$,

Atomic condition: for all $p \in \mathcal{P}_i$, $f_i^{-1}[V'_i(p)] = V_i(p)$,

Forward condition: for all $s_i, t_i \in W_i$ and for all $\underline{i}$-formulas φ, if $s_i R_i(V_{\underline{i}}(\varphi)) t_i$ then $f_i(s_i) R'_i(V'_{\underline{i}}(\varphi)) f_i(t_i)$,

Backward condition: for all $s_i \in W_i$, for all $t'_i \in W'_i$ and for all $\underline{i}$-formulas φ, if $f_i(s_i) R'_i(V'_{\underline{i}}(\varphi)) t'_i$ then there exists $t_i \in W_i$ such that $f_i(t_i) = t'_i$ and $s_i R_i(V_{\underline{i}}(\varphi)) t_i$.

Proposition 3.12 *For all models* $(W_1, W_2, R_1, R_2, V_1, V_2)$ *and* $(W'_1, W'_2, R'_1, R'_2, V'_1, V'_2)$*, for all bounded morphisms* (f_1, f_2) *from* $(W_1, W_2, R_1, R_2, V_1, V_2)$ *to* $(W'_1, W'_2, R'_1, R'_2, V'_1, V'_2)$*, for all* $i \in \{1, 2\}$ *and for all i-formulas* φ*,* $f_i^{-1}[V'_i(\varphi)] = V_i(\varphi)$*.*

Proof. Similar to the proof of Bounded Morphism Lemma [5, Proposition 2.14]. □

Proposition 3.13 is an immediate consequence of Proposition 3.12.

Proposition 3.13 *Let* (W_1, W_2, R_1, R_2) *and* (W'_1, W'_2, R'_1, R'_2) *be frames and* (f_1, f_2) *be a couple of surjective functions* $f_1 : W_1 \longrightarrow W'_1$ *and* $f_2 : W_2 \longrightarrow W'_2$*. If for all* (W'_1, W'_2, R'_1, R'_2)*-valuations* (V'_1, V'_2)*, there exists a* (W_1, W_2, R_1, R_2)*-valuation* (V_1, V_2) *such that* (f_1, f_2) *is a bounded morphism from* $(W_1, W_2, R_1, R_2, V_1, V_2)$ *to* $(W'_1, W'_2, R'_1, R'_2, V'_1, V'_2)$ *then for all formulas* φ*, if* $(W_1, W_2, R_1, R_2) \models \varphi$ *then* $(W'_1, W'_2, R'_1, R'_2) \models \varphi$*.*

4 Parametrized modal logics

A *parametrized modal logic (PML)* is a readable tip $(\mathbf{L}_1, \mathbf{L}_2)$ such that for all $i \in \{1, 2\}$, $\mathbf{L}_i$ satisfies the following conditions:

($\mathbf{Taut}_i$) $\mathbf{L}_i$ contains all i-formulas obtained from propositional tautologies after having uniformly replaced their atomic formulas by arbitrary i-formulas,

($\mathbf{Dist}_i$) $\mathbf{L}_i$ contains all i-formulas of the form $[\varphi](\psi \to \chi) \to ([\varphi]\psi \to [\varphi]\chi)$,

($\mathbf{MP}_i$) if $\varphi \in \mathbf{L}_i$ and $\varphi \to \psi \in \mathbf{L}_i$ then $\mathbf{L}_i$ contains the i-formula ψ,

($\mathbf{Gen}_i$) if $\varphi \in \mathbf{L}_i$ then $\mathbf{L}_i$ contains all i-formulas of the form $[\psi]\varphi$,

($\mathbf{RE}_i$) if $\varphi \leftrightarrow \psi \in \mathbf{L}_{\underline{i}}$ then $\mathbf{L}_i$ contains all i-formulas of the form $[\varphi]\chi \leftrightarrow [\psi]\chi$,

($\mathbf{UOI}_i$) if $\bot \in \mathbf{L}_{\underline{i}}$ then $\mathbf{L}_i$ contains the i-formula $\bot$,

($\mathbf{US}_i$) if $\varphi \in \mathbf{L}_i$ and (σ_1, σ_2) is a substitution then $\mathbf{L}_i$ contains the i-formula $\sigma_i(\varphi)$.

There is a greatest PML, namely $(\mathcal{L}_1, \mathcal{L}_2)$. There is also a least PML (denoted $(\mathbf{K}_1, \mathbf{K}_2)$), seeing that for all collections $(\mathbf{L}_1^m, \mathbf{L}_2^m)_{m \in I}$ of PMLs, $(\bigcap\{\mathbf{L}_1^m : m \in I\}, \bigcap\{\mathbf{L}_2^m : m \in I\})$ is a PML. Let $(\mathbf{Equiv}_1^g, \mathbf{Equiv}_2^g)$ be the least PML such that for all $i \in \{1, 2\}$, $\mathbf{Equiv}_i^g$ contains all i-formulas of the form $[\varphi]\psi \to \psi$ and $\langle\varphi\rangle\psi \to [\varphi]\langle\varphi\rangle\psi$. A PML $(\mathbf{L}_1, \mathbf{L}_2)$ is *paraconjunctive* if for all $i \in \{1, 2\}$, $\mathbf{L}_i$ satisfies the following conditions:

($\mathbf{UNI}_i$) $\mathbf{L}_i$ contains all i-formulas of the form $[\bot]\varphi \to \varphi$ and $\langle\bot\rangle\varphi \to [\bot]\langle\bot\rangle\varphi$,

($\mathbf{RI}_i^-$) if $\bigwedge_{k=1\ldots m}[\bot](\langle\psi'_k\rangle\chi'_k \to \langle\varphi'_k\rangle\chi'_k) \to \bigvee_{l=1\ldots n}[\bot](\varphi_l \to \psi_l) \in \mathbf{L}_{\underline{i}}$ then $\mathbf{L}_i$ contains all i-formulas of the form $\bigwedge_{k=\ldots m}[\bot](\varphi'_k \to \psi'_k) \to \bigvee_{l=1\ldots n}[\bot](\langle\psi_l\rangle\chi_l \to \langle\varphi_l\rangle\chi_l)$.

Let $(\mathbf{ParCon}_1, \mathbf{ParCon}_2)$ be the least paraconjunctive PML. Let $(\mathbf{Equiv}_1^p, \mathbf{Equiv}_2^p)$ be the least paraconjunctive PML such that for all $i \in \{1, 2\}$, $\mathbf{Equiv}_i^p$ contains all i-formulas of the form $[\varphi]\psi \to \psi$ and $\langle\varphi\rangle\psi \to [\varphi]\langle\varphi\rangle\psi$. For all PMLs $(\mathbf{L}_1, \mathbf{L}_2)$ and for all readable tips (Σ_1, Σ_2), let $(\mathbf{L}_1, \mathbf{L}_2) + (\Sigma_1, \Sigma_2)$ be the least PML containing $(\mathbf{L}_1 \cup \Sigma_1, \mathbf{L}_2 \cup \Sigma_2)$.

Problem 4.1 *We do not know if there exists a finite readable tip* (Σ_1, Σ_2) *such that* $(\mathbf{ParCon}_1, \mathbf{ParCon}_2) = (\mathbf{K}_1, \mathbf{K}_2) + (\Sigma_1, \Sigma_2)$.

Problem 4.2 *We do not know if there exists a finite readable tip* (Σ_1, Σ_2) *such that* $(\mathbf{Equiv}_1^p, \mathbf{Equiv}_2^p) = (\mathbf{K}_1, \mathbf{K}_2) + (\Sigma_1, \Sigma_2)$.

A PML $(\mathbf{L}_1, \mathbf{L}_2)$ is *consistent* if for all $i \in \{1, 2\}$, $\mathbf{L}_i \neq \mathcal{L}_i$. Obviously, thanks to the conditions ($\mathbf{UOI}_1$) and ($\mathbf{UOI}_2$), $(\mathcal{L}_1, \mathcal{L}_2)$ is the one and only inconsistent PML. For all PMLs $(\mathbf{L}_1, \mathbf{L}_2)$ and for all $i \in \{1, 2\}$, we will say that a set s_i of i-formulas is $(\mathbf{L}_1, \mathbf{L}_2)$-*consistent,* if for all $n \in \mathbb{N}$ and for all $\varphi_1, \ldots, \varphi_n \in s_i$, $\neg(\varphi_1 \wedge \ldots \wedge \varphi_n) \notin \mathbf{L}_i$.

Lemma 4.3 *For all PMLs* $(\mathbf{L}_1, \mathbf{L}_2)$*, for all* $i \in \{1, 2\}$ *and for all* $(\mathbf{L}_1, \mathbf{L}_2)$*-consistent sets* s_i *of* i*-formulas, there exists a maximal* $(\mathbf{L}_1, \mathbf{L}_2)$*-consistent set* t_i *of* i*-formulas such that* $s_i \subseteq t_i$.

Proof. Similar to the proof of Lindenbaum's Lemma [6, Lemma 5.1]. □

Lemma 4.4 *For all PMLs* $(\mathbf{L}_1, \mathbf{L}_2)$, *for all* $i \in \{1, 2\}$, *for all maximal* $(\mathbf{L}_1, \mathbf{L}_2)$-*consistent sets* s_i *of* i-*formulas, for all* $\underline{i}$-*formulas* φ *and for all* i-*formulas* ψ, *if* $[\varphi]\psi \notin s_i$ *then* $[\varphi]s_i \cup \{\neg\psi\}$ *is a* $(\mathbf{L}_1, \mathbf{L}_2)$-*consistent set of* i-*formulas.*

Proof. Similar to the proof of Existence Lemma [13, Proposition 2.8.4]. □

Notice that for all consistent PMLs $(\mathbf{L}_1, \mathbf{L}_2)$ and for all $i \in \{1, 2\}$, $\mathbf{L}_i$ is a $(\mathbf{L}_1, \mathbf{L}_2)$-consistent set of i-formulas. A PML $(\mathbf{L}_1, \mathbf{L}_2)$ is *sound with respect to a class* $\mathcal{C}$ *of frames* if for all $i \in \{1, 2\}$ and for all i-formulas φ, if $\varphi \in \mathbf{L}_i$ then $\mathcal{C} \models \varphi$. A PML $(\mathbf{L}_1, \mathbf{L}_2)$ is *complete with respect to a class* $\mathcal{C}$ *of frames* if for all $i \in \{1, 2\}$ and for all i-formulas φ, if $\mathcal{C} \models \varphi$ then $\varphi \in \mathbf{L}_i$. The proofs of the soundness statements expressed in Proposition 4.5 are as expected.

Proposition 4.5 *In Table 1, the PMLs listed in the left column are sound with respect to the corresponding classes of frames listed in the right column.*

PMLs	Classes of frames
$(\mathbf{K}_1, \mathbf{K}_2)$	All frames
$(\mathbf{Equiv}^g_1, \mathbf{Equiv}^g_2)$	All frames of indiscernibility
$(\mathbf{ParCon}_1, \mathbf{ParCon}_2)$	All paraconjunctive frames All preconjunctive frames All conjunctive frames
$(\mathbf{Equiv}^p_1, \mathbf{Equiv}^p_2)$	All paraconjunctive frames of indiscernibility All preconjunctive frames of indiscernibility All conjunctive frames of indiscernibility

Table 1

As for the proofs of the corresponding completeness statements, they are not so obvious, especially when the considered PMLs are paraconjunctive [6]. In Sections 5 and 6, we adapt the ordinary canonical model construction to the context of our parametrized relational semantics.

5 Completeness: the general case

From now on in this section, we will assume that $(\mathbf{L}_1, \mathbf{L}_2)$ is a consistent PML. Let $(W^g_1, W^g_2, R^g_1, R^g_2)$ be the 4-tuple where for all $i \in \{1, 2\}$,

- W^g_i is the set of all maximal $(\mathbf{L}_1, \mathbf{L}_2)$-consistent sets of i-formulas,
- $R^g_i : \wp(W^g_{\underline{i}}) \longrightarrow \wp(W^g_i \times W^g_i)$ is such that for all $A \in \wp(W^g_{\underline{i}})$ and for all $s_i, t_i \in W^g_i$, $s_i R^g_i(A) t_i$ if and only if for all $\underline{i}$-formulas φ, if $\widehat{\varphi} = A$ then $[\varphi]s_i \subseteq t_i$,

[6] The problem with paraconjunctive PMLs is that the operation of intersection — which is used in conjunctive frames for the interpretation of the modalities — is not modally definable. See [1] and [22] for investigations about the intersection of modalities in epistemic logics and dynamic logics.

where for all $i \in \{1,2\}$ and for all $\underline{i}$-formulas φ, $\widehat{\varphi} = \{u_{\underline{i}} \in W^g_{\underline{i}} : \varphi \in u_{\underline{i}}\}$. Since for all $i \in \{1,2\}$, $\mathbf{L}_i$ is a $(\mathbf{L}_1, \mathbf{L}_2)$-consistent set of i-formulas, by Lemma 4.3, for all $i \in \{1,2\}$, W^g_i is nonempty.

Lemma 5.1 *The* 4*-tuple* $(W^g_1, W^g_2, R^g_1, R^g_2)$ *is a frame.*

The 4-tuple $(W^g_1, W^g_2, R^g_1, R^g_2)$ will be called *general canonical frame for* $(\mathbf{L}_1, \mathbf{L}_2)$. Lemma 5.2 is an immediate consequence of the fact that for all $i \in \{1,2\}$, $\mathbf{Equiv}^g_i$ contains all i-formulas of the form $[\varphi]\psi \to \psi$ and $\langle\varphi\rangle\psi \to [\varphi]\langle\varphi\rangle\psi$.

Lemma 5.2 *If* $(\mathbf{L}_1, \mathbf{L}_2)$ *contains* $(\mathbf{Equiv}^g_1, \mathbf{Equiv}^g_2)$ *then the general canonical frame for* $(\mathbf{L}_1, \mathbf{L}_2)$ *is a frame of indiscernibility.*

Lemma 5.3 is an immediate consequence of Lemma 4.3.

Lemma 5.3 *For all* $i \in \{1,2\}$ *and for all* i*-formulas* φ, ψ, $\widehat{\varphi} = \widehat{\psi}$ *if and only if* $\varphi \leftrightarrow \psi \in \mathbf{L}_i$.

For all $i \in \{1,2\}$, let $V^g_i : \mathcal{L}_i \longrightarrow \wp(W^g_i)$ be such that for all $\varphi \in \mathcal{L}_i$, $V^g_i(\varphi) = \widehat{\varphi}$. The 6-tuple $(W^g_1, W^g_2, R^g_1, R^g_2, V^g_1, V^g_2)$ will be called *general canonical model for* $(\mathbf{L}_1, \mathbf{L}_2)$.

Lemma 5.4 (Truth Lemma: the general case) *The general canonical model for* $(\mathbf{L}_1, \mathbf{L}_2)$ *is a model.*

Proof. The proof that for all $i \in \{1,2\}$, V^g_i satisfies the conditions for $\bot$, $\neg$ and $\vee$ is as expected. We only show that for all $i \in \{1,2\}$, V^g_i satisfies the condition for $[\cdot]$. Let $i \in \{1,2\}$. Let φ be a $\underline{i}$-formula and ψ be a i-formula. Let $s_i \in W^g_i$. We only demonstrate $s_i \in V^g_i([\varphi]\psi)$ if for all $t_i \in W^g_i$, if $s_i R^g_i(V^g_{\underline{i}}(\varphi)) t_i$ then $t_i \in V^g_i(\psi)$, the "only if" direction being left as an exercise for the reader. Suppose $s_i \notin V^g_i([\varphi]\psi)$. We demonstrate there exists $t_i \in W^g_i$ such that $s_i R^g_i(V^g_{\underline{i}}(\varphi)) t_i$ and $t_i \notin V^g_i(\psi)$. Since $s_i \notin V^g_i([\varphi]\psi)$, $[\varphi]\psi \notin s_i$. Let $t^0_i = [\varphi]s_i \cup \{\neg\psi\}$. Notice that $[\varphi]s_i \subseteq t^0_i$ and $\neg\psi \in t^0_i$. By Lemma 4.4, t^0_i is a $(\mathbf{L}_1, \mathbf{L}_2)$-consistent set of i-formulas. Hence, by Lemma 4.3, let t_i be a maximal $(\mathbf{L}_1, \mathbf{L}_2)$-consistent set of i-formulas such that $t^0_i \subseteq t_i$. Since $[\varphi]s_i \subseteq t^0_i$ and $\neg\psi \in t^0_i$, $[\varphi]s_i \subseteq t_i$ and $\neg\psi \in t_i$.

Claim $s_i R^g_i(V^g_{\underline{i}}(\varphi)) t_i$.

Proof. We demonstrate for all $\underline{i}$-formulas φ', if $\widehat{\varphi'} = V^g_{\underline{i}}(\varphi)$ then $[\varphi']s_i \subseteq t_i$. Let φ' be a $\underline{i}$-formula. Suppose $\widehat{\varphi'} = V^g_{\underline{i}}(\varphi)$. We demonstrate $[\varphi']s_i \subseteq t_i$. Let ψ' be a i-formula. Suppose $[\varphi']\psi' \in s_i$. We demonstrate $\psi' \in t_i$. Since $\widehat{\varphi'} = V^g_{\underline{i}}(\varphi)$, by Lemma 5.3, $\varphi' \leftrightarrow \varphi \in \mathbf{L}_{\underline{i}}$. Since $\mathbf{L}_i$ satisfies the closure condition $(\mathbf{RE}_i)$, $[\varphi']\psi' \leftrightarrow [\varphi]\psi' \in \mathbf{L}_i$. Since $[\varphi']\psi' \in s_i$, $[\varphi]\psi' \in s_i$. Thus, $\psi' \in [\varphi]s_i$. Since $[\varphi]s_i \subseteq t_i$, $\psi' \in t_i$. □

Finally, the reader may easily verify that $t_i \notin V^g_i(\psi)$. Here finishes the proof of Lemma 5.4.

Proposition 5.5 is an immediate consequence of Lemmas 4.3, 5.1 and 5.4.

Proposition 5.5 $(\mathbf{K}_1, \mathbf{K}_2)$ *is complete with respect to the class of all frames.*

Proposition 5.6 is an immediate consequence of Lemmas 4.3, 5.2 and 5.4.

Proposition 5.6 $(\mathbf{Equiv}_1^g, \mathbf{Equiv}_2^g)$ *is complete with respect to the class of all frames of indiscernibility.*

6 Completeness: the paraconjunctive case

From now on in this section, we will assume that $(\mathbf{L}_1, \mathbf{L}_2)$ is a consistent paraconjunctive PML. Therefore, for all $i \in \{1, 2\}$, $\mathbf{L}_i$ contains all i-formulas of the form $[\bot]\varphi \to \varphi$ and $\langle\bot\rangle\varphi \to [\bot]\langle\bot\rangle\varphi$. As a result, for all $i \in \{1, 2\}$, $\mathbf{L}_i$ contains all i-formulas of the form $[\bot]\varphi \to [\bot][\bot]\varphi$.

6.1 Preliminaries

Lemmas 6.1 and 6.2 are immediate consequences of the fact that for all $i \in \{1, 2\}$, $\mathbf{L}_i$ contains all i-formulas of the form $[\bot]\varphi \to \varphi$ and $\langle\bot\rangle\varphi \to [\bot]\langle\bot\rangle\varphi$.

Lemma 6.1 *For all $i \in \{1, 2\}$ and for all maximal $(\mathbf{L}_1, \mathbf{L}_2)$-consistent sets s_i of i-formulas, $[\bot]s_i$ is $(\mathbf{L}_1, \mathbf{L}_2)$-consistent.*

Lemma 6.2 *For all $i \in \{1, 2\}$ and for all maximal $(\mathbf{L}_1, \mathbf{L}_2)$-consistent sets s_i, t_i, u_i of i-formulas, if $[\bot]s_i \subseteq t_i$ and $[\bot]s_i \subseteq u_i$ then $[\bot]t_i \subseteq u_i$.*

A readable tip (s_1, s_2) is *paraconjunctive* if for all $i \in \{1, 2\}$, s_i is a maximal $(\mathbf{L}_1, \mathbf{L}_2)$-consistent set of i-formulas such that for all $\underline{i}$-formulas φ, ψ, if $[\bot](\varphi \to \psi) \in s_{\underline{i}}$ then for all i-formulas χ, $[\bot](\langle\psi\rangle\chi \to \langle\varphi\rangle\chi) \in s_i$.

Lemma 6.3 *For all $i \in \{1, 2\}$ and for all maximal $(\mathbf{L}_1, \mathbf{L}_2)$-consistent sets s_i of i-formulas, there exists a maximal $(\mathbf{L}_1, \mathbf{L}_2)$-consistent set $s_{\underline{i}}$ of $\underline{i}$-formulas such that the readable tip (s_1, s_2) is paraconjunctive.*

Proof. Let $i \in \{1, 2\}$ and s_i be a maximal $(\mathbf{L}_1, \mathbf{L}_2)$-consistent set of i-formulas. We demonstrate there exists a maximal $(\mathbf{L}_1, \mathbf{L}_2)$-consistent set $s_{\underline{i}}$ of $\underline{i}$-formulas such that the readable tip (s_1, s_2) is paraconjunctive. Let $s_{\underline{i}}^0$ be the set consisting of the following $\underline{i}$-formulas:

- $\neg[\bot](\varphi \to \psi)$ for each $\varphi, \psi \in \mathcal{L}_{\underline{i}}$ and for each $\chi \in \mathcal{L}_i$ such that $\neg[\bot](\langle\psi\rangle\chi \to \langle\varphi\rangle\chi) \in s_i$,
- $[\bot](\langle\psi'\rangle\chi' \to \langle\varphi'\rangle\chi')$ for each $\varphi', \psi' \in \mathcal{L}_i$ and for each $\chi' \in \mathcal{L}_{\underline{i}}$ such that $[\bot](\varphi' \to \psi') \in s_i$.

Claim $s_{\underline{i}}^0$ is a $(\mathbf{L}_1, \mathbf{L}_2)$-consistent set of $\underline{i}$-formulas.

Proof. For the sake of the contradiction, suppose $s_{\underline{i}}^0$ is not a $(\mathbf{L}_1, \mathbf{L}_2)$-consistent set of $\underline{i}$-formulas. Hence, let m, n in $\mathbb{N}$, φ_1, …, φ_m, ψ_1, …, ψ_m, χ_1', …, χ_n' in $\mathcal{L}_{\underline{i}}$ and φ_1', …, φ_n', ψ_1', …, ψ_n', χ_1, …, χ_m in $\mathcal{L}_i$ be such that

- for all $k \in \{1, \ldots, m\}$, $\neg[\bot](\langle\psi_k\rangle\chi_k \to \langle\varphi_k\rangle\chi_k) \in s_i$,
- for all $l \in \{1, \ldots, n\}$, $[\bot](\varphi_l' \to \psi_l') \in s_i$,

- $\neg(\bigwedge_{k=1\ldots m}\neg[\bot](\varphi_k \to \psi_k) \wedge \bigwedge_{l=1\ldots n}[\bot](\langle\psi'_l\rangle\chi'_l \to \langle\varphi'_l\rangle\chi'_l)) \in \mathbf{L}_{\underline{i}}$.

Thus, $\bigwedge_{l=1\ldots n}[\bot](\langle\psi'_l\rangle\chi'_l \to \langle\varphi'_l\rangle\chi'_l) \to \bigvee_{k=1\ldots m}[\bot](\varphi_k \to \psi_k) \in \mathbf{L}_{\underline{i}}$. Since $\mathbf{L}_i$ satisfies the closure condition $(\mathbf{RI}_i^-)$, $\mathbf{L}_i$ contains the i-formula $\bigwedge_{l=1\ldots n}[\bot](\varphi'_l \to \psi'_l) \to \bigvee_{k=1\ldots m}[\bot](\langle\psi_k\rangle\chi_k \to \langle\varphi_k\rangle\chi_k)$. Since for all $l \in \{1,\ldots,n\}$, $[\bot](\varphi'_l \to \psi'_l) \in s_i$, there exists $k \in \{1,\ldots,m\}$ such that $[\bot](\langle\psi_k\rangle\chi_k \to \langle\varphi_k\rangle\chi_k) \in s_i$. Consequently, there exists $k \in \{1,\ldots,m\}$ such that $\neg[\bot](\langle\psi_k\rangle\chi_k \to \langle\varphi_k\rangle\chi_k) \notin s_i$: a contradiction. □

Thus, by Lemma 4.3, let $s_{\underline{i}}$ be a maximal $(\mathbf{L}_1, \mathbf{L}_2)$-consistent set of $\underline{i}$-formulas such that $s^0_{\underline{i}} \subseteq s_{\underline{i}}$.

Claim The readable tip (s_1, s_2) is paraconjunctive.

Proof. For the sake of the contradiction, suppose the readable tip (s_1, s_2) is not paraconjunctive. Since for all $j \in \{1,2\}$, s_j is a maximal $(\mathbf{L}_1, \mathbf{L}_2)$-consistent set of j-formulas, there exists $j \in \{1,2\}$ and there exists $\underline{j}$-formulas φ, ψ such that $[\bot](\varphi \to \psi) \in s_{\underline{j}}$ and there exists a j-formula χ such that $[\bot](\langle\psi\rangle\chi \to \langle\varphi\rangle\chi) \notin s_j$. Consider the following two cases: $j = i$ and $j = \underline{i}$. In the former case, since $[\bot](\langle\psi\rangle\chi \to \langle\varphi\rangle\chi) \notin s_j$, $\neg[\bot](\langle\psi\rangle\chi \to \langle\varphi\rangle\chi) \in s_i$. Hence, $\neg[\bot](\varphi \to \psi) \in s^0_{\underline{i}}$. Since $s^0_{\underline{i}} \subseteq s_{\underline{i}}$, $\neg[\bot](\varphi \to \psi) \in s_{\underline{i}}$. Thus, $[\bot](\varphi \to \psi) \notin s_{\underline{j}}$: a contradiction. In the latter case, since $[\bot](\varphi \to \psi) \in s_{\underline{j}}$, $[\bot](\varphi \to \psi) \in s_i$. Consequently, $[\bot](\langle\psi\rangle\chi \to \langle\varphi\rangle\chi) \in s^0_{\underline{i}}$. Since $s^0_{\underline{i}} \subseteq s_{\underline{i}}$, $[\bot](\langle\psi\rangle\chi \to \langle\varphi\rangle\chi) \in s_{\underline{i}}$. Hence, $[\bot](\langle\psi\rangle\chi \to \langle\varphi\rangle\chi) \in s_j$: a contradiction. □

Here finishes the proof of Lemma 6.3.

6.2 Paraconjunctive case: first set of completeness results

For a while, let us fix a paraconjunctive readable tip (s_1, s_2). Let $(W^c_1, W^c_2, R^c_1, R^c_2)$ be the 4-tuple where for all $i \in \{1,2\}$,

- W^c_i is the set of all maximal $(\mathbf{L}_1, \mathbf{L}_2)$-consistent sets t_i of i-formulas such that $[\bot]s_i \subseteq t_i$,
- $R^c_i : \wp(W^c_{\underline{i}}) \longrightarrow \wp(W^c_i \times W^c_i)$ is such that for all $A \in \wp(W^c_{\underline{i}})$ and for all $t_i, u_i \in W^c_i$, $t_i R^c_i(A) u_i$ if and only if for all $\underline{i}$-formulas φ, if $\widehat{\varphi} \subseteq A$ then $[\varphi]t_i \subseteq u_i$,

where for all $i \in \{1,2\}$ and for all $\underline{i}$-formulas φ, $\widehat{\varphi} = \{v_{\underline{i}} \in W^c_{\underline{i}} : \varphi \in v_{\underline{i}}\}$. Since for all $i \in \{1,2\}$, s_i is a maximal $(\mathbf{L}_1, \mathbf{L}_2)$-consistent set of i-formulas, by Lemmas 4.3 and 6.1, for all $i \in \{1,2\}$, W^c_i is nonempty. Moreover, by Lemma 6.2, for all $i \in \{1,2\}$ and for all $t_i, u_i \in W^c_i$, $[\bot]t_i \subseteq u_i$.

Lemma 6.4 *For all $i \in \{1,2\}$ and for all i-formulas φ, if $\widehat{\varphi} = W^c_i$ then $[\bot]\varphi \in s_i$.*

Proof. Let $i \in \{1,2\}$ and φ be a i-formula. Suppose $\widehat{\varphi} = W^c_i$. We demonstrate $[\bot]\varphi \in s_i$. For the sake of the contradiction, suppose $[\bot]\varphi \notin s_i$. Let $u^0_i = [\bot]s_i \cup \{\neg\varphi\}$. Notice that $[\bot]s_i \subseteq u^0_i$ and $\neg\varphi \in u^0_i$. By Lemma 4.4, u^0_i is a

$(\mathbf{L}_1, \mathbf{L}_2)$-consistent set of i-formulas. Hence, by Lemma 4.3, let u_i be a maximal $(\mathbf{L}_1, \mathbf{L}_2)$-consistent set of i-formulas such that $u_i^0 \subseteq u_i$. Since $[\bot]s_i \subseteq u_i^0$ and $\neg\varphi \in u_i^0$, $[\bot]s_i \subseteq u_i$ and $\neg\varphi \in u_i$. Thus, $u_i \in W_i^c$ and $\varphi \notin u_i$. Since $\widehat{\varphi} = W_i^c$, $\varphi \in u_i$: a contradiction. □

Lemma 6.5 *For all $i \in \{1, 2\}$ and for all $\underline{i}$-formulas φ, if $\widehat{\varphi} = \emptyset$ then for all $t_i, u_i \in W_i^c$, $[\varphi]t_i \subseteq u_i$.*

Proof. Let $i \in \{1, 2\}$ and φ be a $\underline{i}$-formula. Suppose $\widehat{\varphi} = \emptyset$. We demonstrate for all $t_i, u_i \in W_i^c$, $[\varphi]t_i \subseteq u_i$. Let $t_i, u_i \in W_i^c$. For the sake of the contradiction, suppose $[\varphi]t_i \not\subseteq u_i$. Hence, let ψ be a i-formula such that $[\varphi]\psi \in t_i$ and $\psi \notin u_i$. Thus, $\psi \notin [\bot]t_i$. Consequently, $[\bot]\psi \notin t_i$. Since $[\varphi]\psi \in t_i$, $\langle\bot\rangle\neg\psi \to \langle\varphi\rangle\neg\psi \notin t_i$. Hence, $[\bot](\langle\bot\rangle\neg\psi \to \langle\varphi\rangle\neg\psi) \notin s_i$. Since (s_1, s_2) is paraconjunctive, $[\bot](\varphi \to \bot) \notin s_{\underline{i}}$ Let $w_{\underline{i}}^0 = [\bot]s_{\underline{i}} \cup \{\varphi\}$. Notice that $[\bot]s_{\underline{i}} \subseteq w_{\underline{i}}^0$ and $\varphi \in w_{\underline{i}}^0$. By Lemma 4.4, $w_{\underline{i}}^0$ is a $(\mathbf{L}_1, \mathbf{L}_2)$-consistent set of $\underline{i}$-formulas. Hence, by Lemma 4.3, let $w_{\underline{i}}$ be a maximal $(\mathbf{L}_1, \mathbf{L}_2)$-consistent set of $\underline{i}$-formulas such that $w_{\underline{i}}^0 \subseteq w_{\underline{i}}$. Since $[\bot]s_{\underline{i}} \subseteq w_{\underline{i}}^0$ and $\varphi \in w_{\underline{i}}^0$, $[\bot]s_{\underline{i}} \subseteq w_{\underline{i}}$ and $\varphi \in w_{\underline{i}}$. Thus, $w_{\underline{i}} \in W_{\underline{i}}^c$. Since $\widehat{\varphi} = \emptyset$, $\varphi \notin w_{\underline{i}}$: a contradiction. Here finishes the proof of Lemma 6.5. □

Lemma 6.6 *The 4-tuple $(W_1^c, W_2^c, R_1^c, R_2^c)$ is a paraconjunctive frame.*

Proof. By Lemma 6.5, for all $i \in \{1, 2\}$, for all $t_i, u_i \in W_i^c$ and for all $\underline{i}$-formulas φ, if $\widehat{\varphi} = \emptyset$ then $[\varphi]t_i \subseteq u_i$. Hence, for all $i \in \{1, 2\}$ and for all $t_i, u_i \in W_i^c$, $t_i R_i^c(\emptyset) u_i$. Thus, for all $i \in \{1, 2\}$, $R_i^c(\emptyset) = W_i^c \times W_i^c$. Moreover, for all $i \in \{1, 2\}$ and for all $A, B \in \wp(W_{\underline{i}}^c)$, if $A \subseteq B$ then for all $\underline{i}$-formulas φ, if $\widehat{\varphi} \subseteq A$ then $\widehat{\varphi} \subseteq B$. Consequently, for all $i \in \{1, 2\}$ and for all $A, B \in \wp(W_{\underline{i}}^c)$, if $A \subseteq B$ then for all $t_i, u_i \in W_i^c$, if $t_i R_i^c(B) u_i$ then $t_i R_i^c(A) u_i$. Hence, for all $i \in \{1, 2\}$ and for all $A, B \in \wp(W_{\underline{i}}^c)$, if $A \subseteq B$ then $R_i^c(A) \supseteq R_i^c(B)$. □

The 4-tuple $(W_1^c, W_2^c, R_1^c, R_2^c)$ will be called *paraconjunctive canonical frame for $(\mathbf{L}_1, \mathbf{L}_2)$ determined by (s_1, s_2)*. Lemma 6.7 is an immediate consequence of the fact that for all $i \in \{1, 2\}$, $\mathbf{Equiv}_i^p$ contains all i-formulas of the form $[\varphi]\psi \to \psi$ and $\langle\varphi\rangle\psi \to [\varphi]\langle\varphi\rangle\psi$.

Lemma 6.7 *If $(\mathbf{L}_1, \mathbf{L}_2)$ contains $(\mathbf{Equiv}_1^p, \mathbf{Equiv}_2^p)$ then the paraconjunctive canonical frame for $(\mathbf{L}_1, \mathbf{L}_2)$ determined by (s_1, s_2) is a paraconjunctive frame of indiscernibility.*

For all $i \in \{1, 2\}$, let $V_i^c : \mathcal{L}_i \longrightarrow \wp(W_i^c)$ be such that for all $\varphi \in \mathcal{L}_i$, $V_i^c(\varphi) = \widehat{\varphi}$. The 6-tuple $(W_1^c, W_2^c, R_1^c, R_2^c, V_1^c, V_2^c)$ will be called *paraconjunctive canonical model for $(\mathbf{L}_1, \mathbf{L}_2)$ determined by (s_1, s_2)*.

Lemma 6.8 (Truth Lemma: the paraconjunctive case) *The paraconjunctive canonical model for $(\mathbf{L}_1, \mathbf{L}_2)$ determined by (s_1, s_2) is a model.*

Proof. The proof that for all $i \in \{1, 2\}$, V_i^c satisfies the conditions for $\bot$, $\neg$ and $\vee$ is as expected. We only show that for all $i \in \{1, 2\}$, V_i^c satisfies the condition for $[\cdot]$. Let $i \in \{1, 2\}$. Let φ be a $\underline{i}$-formula and ψ be a i-formula. Let $t_i \in W_i^c$. We only demonstrate $t_i \in V_i^c([\varphi]\psi)$ if for all $u_i \in W_i^c$, if $t_i R_i^c(V_{\underline{i}}^c(\varphi)) u_i$ then $u_i \in V_i^c(\psi)$, the "only if" direction being left as an exercise for the reader. Suppose $t_i \notin V_i^c([\varphi]\psi)$. We demonstrate there exists $u_i \in W_i^c$ such that $t_i R_i^c(V_{\underline{i}}^c(\varphi))u_i$ and $u_i \notin V_i^c(\psi)$. Since $t_i \notin V_i^c([\varphi]\psi)$, $[\varphi]\psi \notin t_i$. Let $u_i^0 = [\varphi]t_i \cup \{\neg\psi\}$. Notice that $[\varphi]t_i \subseteq u_i^0$ and $\neg\psi \in u_i^0$. By Lemma 4.4, u_i^0 is a $(\mathbf{L}_1, \mathbf{L}_2)$-consistent set of i-formulas. Hence, by Lemma 4.3, let u_i be a maximal $(\mathbf{L}_1, \mathbf{L}_2)$-consistent set of i-formulas such that $u_i^0 \subseteq u_i$. Since $[\varphi]t_i \subseteq u_i^0$ and $\neg\psi \in u_i^0$, $[\varphi]t_i \subseteq u_i$ and $\neg\psi \in u_i$.

Claim $u_i \in W_i^c$.

Proof. For the sake of the contradiction, suppose $u_i \notin W_i^c$. Hence, $[\bot]s_i \not\subseteq u_i$. Thus, let χ be a i-formula such that $[\bot]\chi \in s_i$ and $\chi \notin u_i$. Since $\mathbf{L}_i$ contains all i-formulas of the form $[\bot]\chi' \to [\bot][\bot]\chi'$, $[\bot][\bot]\chi \in s_i$. Consequently, $[\bot]\chi \in [\bot]s_i$. Since $[\bot]s_i \subseteq t_i$, $[\bot]\chi \in t_i$. Since $\mathbf{L}_{\underline{i}}$ contains the $\underline{i}$-formula $\bot \to \varphi$ and $\mathbf{L}_{\underline{i}}$ satisfies the closure condition $(\mathbf{Gen}_{\underline{i}})$, $\mathbf{L}_{\underline{i}}$ contains the $\underline{i}$-formula $[\bot](\bot \to \varphi)$. Since $\mathbf{L}_i$ satisfies the closure condition $(\mathbf{RI}_i^-)$, $\mathbf{L}_i$ contains the i-formula $[\bot]([\bot]\chi \to [\varphi]\chi)$ [7]. Since $\mathbf{L}_i$ contains all i-formulas of the form $[\bot]\phi \to \phi$, $\mathbf{L}_i$ contains the i-formula $[\bot]\chi \to [\varphi]\chi$. Since $[\bot]\chi \in t_i$, $[\varphi]\chi \in t_i$. Hence, $\chi \in [\varphi]t_i$. Since $[\varphi]t_i \subseteq u_i$, $\chi \in u_i$: a contradiction. □

Claim $t_i R_i^c(V_{\underline{i}}^c(\varphi))u_i$.

Proof. For the sake of the contradiction, suppose not $t_i R_i^c(V_{\underline{i}}^c(\varphi))u_i$. Thus, there exists a $\underline{i}$-formula φ' such that $\widehat{\varphi'} \subseteq V_{\underline{i}}^c(\varphi)$ and $[\varphi']t_i \not\subseteq u_i$. Consequently, let χ be a i-formula such that $[\varphi']\chi \in t_i$ and $\chi \notin u_i$. Since $\widehat{\varphi'} \subseteq V_{\underline{i}}^c(\varphi)$, for all $v_{\underline{i}} \in W_{\underline{i}}^c$, if $\varphi' \in v_{\underline{i}}$ then $\varphi \in v_{\underline{i}}$. Hence, for all $v_{\underline{i}} \in W_{\underline{i}}^c$, $\varphi' \to \varphi \in v_{\underline{i}}$. Thus, by Lemma 6.4, $[\bot](\varphi' \to \varphi) \in s_{\underline{i}}$. Since (s_1, s_2) is paraconjunctive, $[\bot]([\varphi']\chi \to [\varphi]\chi) \in s_i$. Since $\mathbf{L}_i$ contains all i-formulas of the form $[\bot]\phi \to [\bot][\bot]\phi$, $[\bot][\bot]([\varphi']\chi \to [\varphi]\chi) \in s_i$. Consequently, $[\bot]([\varphi']\chi \to [\varphi]\chi) \in t_i$. Since $\mathbf{L}_i$ contains all i-formulas of the form $[\bot]\phi \to \phi$, $[\varphi']\chi \to [\varphi]\chi \in t_i$. Since $[\varphi']\chi \in t_i$, $[\varphi]\chi \in t_i$. Hence, $\chi \in [\varphi]t_i$. Since $[\varphi]t_i \subseteq u_i$, $\chi \in u_i$: a contradiction. □

Finally, the reader may easily verify that $u_i \notin V_i^c(\psi)$. Here finishes the proof of Lemma 6.8.

Proposition 6.9 is an immediate consequence of Lemmas 4.3, 6.3, 6.6 and 6.8.

Proposition 6.9 $(\mathbf{ParCon}_1, \mathbf{ParCon}_2)$ *is complete with respect to the class of all paraconjunctive frames.*

Proposition 6.10 is an immediate consequence of Lemmas 4.3, 6.3, 6.7 and 6.8.

Proposition 6.10 $(\mathbf{Equiv}_1^p, \mathbf{Equiv}_2^p)$ *is complete with respect to the class of all paraconjunctive frames of indiscernibility.*

[7] Here, we are using the closure condition $(\mathbf{RI}_i^-)$ with $m = 0$ and $n = 1$.

6.3 Paraconjunctive case: second set of completeness results

As for the completeness of $(\mathbf{ParCon}_1, \mathbf{ParCon}_2)$ with respect to the class of all conjunctive frames and the completeness of $(\mathbf{Equiv}_1^p, \mathbf{Equiv}_2^p)$ with respect to the class of all conjunctive frames of indiscernibility, we will show in Propositions 6.11 and 6.19 that every paraconjunctive frame is the bounded morphic image of a conjunctive frame and every paraconjunctive frame of indiscernibility is the bounded morphic image of a conjunctive frame of indiscernibility [8].

Proposition 6.11 *Let (W_1, W_2, R_1, R_2) be a paraconjunctive frame. There exists a conjunctive frame (W_1', W_2', R_1', R_2') and a couple (f_1, f_2) of surjective functions $f_1 : W_1' \longrightarrow W_1$ and $f_2 : W_2' \longrightarrow W_2$ such that for all (W_1, W_2, R_1, R_2)-valuations (V_1, V_2), there exists a (W_1', W_2', R_1', R_2')-valuation (V_1', V_2') such that (f_1, f_2) is a bounded morphism from $(W_1', W_2', R_1', R_2', V_1', V_2')$ to $(W_1, W_2, R_1, R_2, V_1, V_2)$.*

Proof. For all $i \in \{1, 2\}$, let

- $\det_i : \wp(W_{\underline{i}}) \times W_i \times W_i \longrightarrow \wp(W_i)$ be such that for all $A \in \wp(W_{\underline{i}})$ and for all $t_i, u_i \in W_i$, if $t_i R_i(A) u_i$ then $\det(A, t_i, u_i) = \emptyset$ else $\det(A, t_i, u_i) = W_i$ [9].

For all $i \in \{1, 2\}$, let Λ_i be the set of all $\tau_i : \wp(W_{\underline{i}}) \times W_{\underline{i}} \longrightarrow \wp(W_i)$ such that for all $A \in \wp(W_{\underline{i}})$, $\{t_{\underline{i}} \in W_{\underline{i}} : \tau_i(A, t_{\underline{i}}) \neq \emptyset\}$ is finite [10]. Let (W_1', W_2', R_1', R_2') be the 4-tuple where for all $i \in \{1, 2\}$,

- $W_i' = W_i \times \Lambda_i$,
- $R_i' : \wp(W_{\underline{i}}') \longrightarrow \wp(W_i' \times W_i')$ is such that for all $A' \in \wp(W_{\underline{i}}')$ and for all $(t_i, \tau_i), (u_i, \mu_i) \in W_i'$, $(t_i, \tau_i) R_i'(A')(u_i, \mu_i)$ if and only if for all $A \in \wp(W_{\underline{i}})$ [11],
  - · if $A' \cap (A \times \Lambda_{\underline{i}}) \neq \emptyset$ then $\bigoplus_i \{\tau_i(A, v_{\underline{i}}) \oplus_i \mu_i(A, v_{\underline{i}}) : v_{\underline{i}} \in A\} = \det_i(A, t_i, u_i)$,
  - · for all $(v_{\underline{i}}, \nu_{\underline{i}}) \in A' \cap (A \times \Lambda_{\underline{i}})$, $\tau_i(A, v_{\underline{i}}) \oplus_i \mu_i(A, v_{\underline{i}}) = \emptyset$.

Claim 6.12 *For all $i \in \{1, 2\}$ and for all $A' \in \wp(W_{\underline{i}}')$, $R_i'(A') = \bigcap\{R_i'(\{(v_{\underline{i}}, \nu_{\underline{i}})\}) : (v_{\underline{i}}, \nu_{\underline{i}}) \in A'\}$.*

Proof. Let $i \in \{1, 2\}$ and $A' \in \wp(W_{\underline{i}}')$. The proof that $R_i'(A') \subseteq \bigcap\{R_i'(\{(v_{\underline{i}}, \nu_{\underline{i}})\}) : (v_{\underline{i}}, \nu_{\underline{i}}) \in A'\}$ being a simple application of the definitions, it is left as an exercise for the reader. We demonstrate $R_i'(A') \supseteq \bigcap\{R_i'(\{(v_{\underline{i}}, \nu_{\underline{i}})\}) : (v_{\underline{i}}, \nu_{\underline{i}}) \in A'\}$. For the sake of the contradiction, suppose $R_i'(A') \not\supseteq \bigcap\{R_i'(\{(v_{\underline{i}}, \nu_{\underline{i}})\}) : (v_{\underline{i}}, \nu_{\underline{i}}) \in A'\}$. Hence, there exists $(t_i, \tau_i), (u_i, \mu_i) \in W_i'$ such that not $(t_i, \tau_i) R_i'(A')(u_i, \mu_i)$ and for all $(v_{\underline{i}}, \nu_{\underline{i}}) \in A'$, $(t_i, \tau_i) R_i'(\{(v_{\underline{i}}, \nu_{\underline{i}})\})(u_i, \mu_i)$. Thus, for all $(v_{\underline{i}}, \nu_{\underline{i}}) \in A'$ and for all $A \in \wp(W_{\underline{i}})$,

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[8] The sketch of an alternative proof of Proposition 6.11 is presented in the Appendix.

[9] Notice that for all $A \in \wp(W_{\underline{i}})$ and for all $t_i, u_i \in W_i$, $\det(A, t_i, u_i) = \emptyset$ if and only if $t_i R_i(A) u_i$.

[10] Notice that for all $\tau_i, \mu_i \in \Lambda_i$ and for all $A \in \wp(W_{\underline{i}})$, $\{v_{\underline{i}} \in A : \tau_i(A, v_{\underline{i}}) \neq \mu_i(A, v_{\underline{i}})\}$ is finite.

[11] Here, $\oplus_i$ is the operation of symmetric difference in $\wp(W_i)$. Moreover, $(v_{\underline{i}}^1, \ldots, v_{\underline{i}}^N)$ being the list of all $v_{\underline{i}} \in A$ such that $\tau_i(A, v_{\underline{i}}) \neq \mu_i(A, v_{\underline{i}})$, $\bigoplus_i \{\tau_i(A, v_{\underline{i}}) \oplus_i \mu_i(A, v_{\underline{i}}) : v_{\underline{i}} \in A\}$ denotes $\tau_i(A, v_{\underline{i}}^1) \oplus_i \mu_i(A, v_{\underline{i}}^1) \oplus_i \ldots \oplus_i \tau_i(A, v_{\underline{i}}^N) \oplus_i \mu_i(A, v_{\underline{i}}^N)$.

- if $\{(v_{\underline{i}},\nu_{\underline{i}})\} \cap (A \times \Lambda_{\underline{i}}) \neq \emptyset$ then $\bigoplus_i \{\tau_i(A,w_{\underline{i}}) \oplus_i \mu_i(A,w_{\underline{i}}) : w_{\underline{i}} \in A\} = \det_i(A,t_i,u_i)$,
- for all $(w_{\underline{i}},\omega_{\underline{i}}) \in \{(v_{\underline{i}},\nu_{\underline{i}})\} \cap (A \times \Lambda_{\underline{i}})$, $\tau_i(A,w_{\underline{i}}) \oplus_i \mu_i(A,w_{\underline{i}}) = \emptyset$.

Consequently, for all $A \in \wp(W_{\underline{i}})$,

- if $A' \cap (A \times \Lambda_{\underline{i}}) \neq \emptyset$ then $\bigoplus_i \{\tau_i(A,w_{\underline{i}}) \oplus_i \mu_i(A,w_{\underline{i}}) : w_{\underline{i}} \in A\} = \det_i(A,t_i,u_i)$,
- for all $(w_{\underline{i}},\omega_{\underline{i}}) \in A' \cap (A \times \Lambda_{\underline{i}})$, $\tau_i(A,w_{\underline{i}}) \oplus_i \mu_i(A,w_{\underline{i}}) = \emptyset$.

Hence, $(t_i,\tau_i)R'_i(A')(u_i,\mu_i)$: a contradiction. □

Claim 6.13 is an immediate consequence of Claim 6.12.

Claim 6.13 *The 4-tuple (W'_1,W'_2,R'_1,R'_2) is a conjunctive frame.*

Let $f_1 : W'_1 \longrightarrow W_1$ be such that for all $(v_1,\nu_1) \in W'_1$, $f_1(v_1,\nu_1) = v_1$ and $f_2 : W'_2 \longrightarrow W_2$ be such that for all $(v_2,\nu_2) \in W'_2$, $f_2(v_2,\nu_2) = v_2$.

Claim 6.14 *f_1 and f_2 are surjective.*

Let (V_1,V_2) be a (W_1,W_2,R_1,R_2)-valuation. For all $i \in \{1,2\}$, let $V'_i : \mathcal{L}_i \longrightarrow \wp(W'_i)$ be such that for all $\varphi \in \mathcal{L}_i$, $V'_i(\varphi) = f_i^{-1}[V_i(\varphi)]$. Notice that for all $i \in \{1,2\}$ and for all $\varphi \in \mathcal{L}_i$, $V'_i(\varphi) = V_i(\varphi) \times \Lambda_i$.

Claim 6.15 *For all $(t_i,\tau_i),(u_i,\mu_i) \in W'_i$ and for all $\underline{i}$-formulas φ, if $(t_i,\tau_i)R'_i(V'_{\underline{i}}(\varphi))(u_i,\mu_i)$ then $t_iR_i(V_{\underline{i}}(\varphi))u_i$.*

Proof. Let $(t_i,\tau_i),(u_i,\mu_i) \in W'_i$ and φ be a $\underline{i}$-formula. Suppose $(t_i,\tau_i)R'_i(V'_{\underline{i}}(\varphi))(u_i,\mu_i)$. For the sake of the contradiction, suppose not $t_iR_i(V_{\underline{i}}(\varphi))u_i$. Hence, $V_{\underline{i}}(\varphi) \neq \emptyset$. Thus, $V'_{\underline{i}}(\varphi) \cap (V_{\underline{i}}(\varphi) \times \Lambda_{\underline{i}}) \neq \emptyset$. Since $(t_i,\tau_i)R'_i(V'_{\underline{i}}(\varphi))(u_i,\mu_i)$, $\bigoplus_i \{\tau_i(V_{\underline{i}}(\varphi),v_{\underline{i}}) \oplus_i \mu_i(V_{\underline{i}}(\varphi),v_{\underline{i}}) : v_{\underline{i}} \in V_{\underline{i}}(\varphi)\} = \det_i(V_{\underline{i}}(\varphi),t_i,u_i)$. Moreover, for all $(v_{\underline{i}},\nu_{\underline{i}}) \in V'_{\underline{i}}(\varphi) \cap (V_{\underline{i}}(\varphi) \times \Lambda_{\underline{i}})$, $\tau_i(V_{\underline{i}}(\varphi),v_{\underline{i}}) \oplus_i \mu_i(V_{\underline{i}}(\varphi),v_{\underline{i}}) = \emptyset$. Consequently, for all $v_{\underline{i}} \in V_{\underline{i}}(\varphi)$, $\tau_i(V_{\underline{i}}(\varphi),v_{\underline{i}}) \oplus_i \mu_i(V_{\underline{i}}(\varphi),v_{\underline{i}}) = \emptyset$. Hence, $\bigoplus_i \{\tau_i(V_{\underline{i}}(\varphi),v_{\underline{i}}) \oplus_i \mu_i(V_{\underline{i}}(\varphi),v_{\underline{i}}) : v_{\underline{i}} \in V_{\underline{i}}(\varphi)\} = \emptyset$. Since $\bigoplus_i \{\tau_i(V_{\underline{i}}(\varphi),v_{\underline{i}}) \oplus_i \mu_i(V_{\underline{i}}(\varphi),v_{\underline{i}}) : v_{\underline{i}} \in V_{\underline{i}}(\varphi)\} = \det_i(V_{\underline{i}}(\varphi),t_i,u_i)$, $\det_i(V_{\underline{i}}(\varphi),t_i,u_i) = \emptyset$. Thus, $t_iR_i(V_{\underline{i}}(\varphi))u_i$: a contradiction. □

Claim 6.16 *For all $(t_i,\tau_i) \in W'_i$, for all $u_i \in W_i$ and for all $\underline{i}$-formulas φ, if $t_iR_i(V_{\underline{i}}(\varphi))u_i$ then there exists $\mu_i \in \Lambda_i$ such that $(t_i,\tau_i)R'_i(V'_{\underline{i}}(\varphi))(u_i,\mu_i)$.*

Proof. Let $(t_i,\tau_i) \in W'_i$, $u_i \in W_i$ and φ be a $\underline{i}$-formula. Suppose $t_iR_i(V_{\underline{i}}(\varphi))u_i$. We demonstrate there exists $\mu_i \in \Lambda_i$ such that $(t_i,\tau_i)R'_i(V'_{\underline{i}}(\varphi))(u_i,\mu_i)$. Indeed, we are looking for $\mu_i : \wp(W_{\underline{i}}) \times W_{\underline{i}} \longrightarrow \wp(W_i)$ such that for all $A \in \wp(W_{\underline{i}})$,

($\mathbf{C_1}$) $\{w_{\underline{i}} \in W_{\underline{i}} : \mu_i(A,w_{\underline{i}}) \neq \emptyset\}$ is finite,

($\mathbf{C_2}$) if $V'_{\underline{i}}(\varphi) \cap (A \times \Lambda_{\underline{i}}) \neq \emptyset$ then $\bigoplus_i \{\tau_i(A,w_{\underline{i}}) \oplus_i \mu_i(A,w_{\underline{i}}) : w_{\underline{i}} \in A\} = \det_i(A,t_i,u_i)$,

($\mathbf{C_3}$) for all $(w_{\underline{i}},\omega_{\underline{i}}) \in V'_{\underline{i}}(\varphi) \cap (A \times \Lambda_{\underline{i}})$, $\tau_i(A,w_{\underline{i}}) \oplus_i \mu_i(A,w_{\underline{i}}) = \emptyset$.

For all $A \in \wp(W_{\underline{i}})$, let $\mu_i^A : W_{\underline{i}} \longrightarrow \wp(W_i)$ be defined as follows:

Case "$A \subseteq V_{\underline{i}}(\varphi)$": for all $w_{\underline{i}} \in W_{\underline{i}}$, let $\mu_i^A(w_{\underline{i}}) = \tau_i(A,w_{\underline{i}})$,

Case "$A \not\subseteq V_{\underline{i}}(\varphi)$": let $w_{\underline{i}}^A \in W_{\underline{i}}$ be such that $w_{\underline{i}}^A \in A$ and $w_{\underline{i}}^A \notin V_{\underline{i}}(\varphi)$ and for all $w_{\underline{i}} \in W_{\underline{i}}$,

- if $w_{\underline{i}} \neq w_{\underline{i}}^A$ then let $\mu_i^A(w_{\underline{i}}) = \tau_i(A, w_{\underline{i}})$,
- otherwise, let $\mu_i^A(w_{\underline{i}}) = \tau_i(A, w_{\underline{i}}) \oplus_i \det_i(A, t_i, u_i)$.

Let $\mu_i : \wp(W_{\underline{i}}) \times W_{\underline{i}} \longrightarrow \wp(W_i)$ be such that for all $A \in \wp(W_{\underline{i}})$ and for all $w_{\underline{i}} \in W_{\underline{i}}$, $\mu_i(A, w_{\underline{i}}) = \mu_i^A(w_{\underline{i}})$. Now, we just have to verify that for all $A \in \wp(W_{\underline{i}})$, $(\mathbf{C_1})$, $(\mathbf{C_2})$ and $(\mathbf{C_3})$ hold. Let $A \in \wp(W_{\underline{i}})$. Concerning $(\mathbf{C_1})$, it holds, seeing that $\mu_i^A(w_{\underline{i}}) = \tau_i(A, w_{\underline{i}})$ for every $w_{\underline{i}} \in W_{\underline{i}}$ except when $A \not\subseteq V_{\underline{i}}(\varphi)$ and $w_{\underline{i}} = w_{\underline{i}}^A$. About $(\mathbf{C_2})$, suppose $V_{\underline{i}}'(\varphi) \cap (A \times \Lambda_{\underline{i}}) \neq \emptyset$ and consider the following two cases: $A \subseteq V_{\underline{i}}(\varphi)$ and $A \not\subseteq V_{\underline{i}}(\varphi)$. In the former case, since $t_i R_i(V_{\underline{i}}(\varphi)) u_i$, $t_i R_i(A) u_i$. Hence, $\det_i(A, t_i, u_i) = \emptyset$. Since $A \subseteq V_{\underline{i}}(\varphi)$, for all $w_{\underline{i}} \in W_{\underline{i}}$, $\mu_i^A(w_{\underline{i}}) = \tau_i(A, w_{\underline{i}})$. Thus, for all $w_{\underline{i}} \in W_{\underline{i}}$, $\tau_i(A, w_{\underline{i}}) \oplus_i \mu_i(A, w_{\underline{i}}) = \emptyset$. Consequently, $\bigoplus_i \{\tau_i(A, w_{\underline{i}}) \oplus_i \mu_i(A, w_{\underline{i}}) : w_{\underline{i}} \in A\} = \emptyset$. Since $\det_i(A, t_i, u_i) = \emptyset$, $(\mathbf{C_2})$ holds. In the latter case, $\mu_i^A(w_{\underline{i}}) = \tau_i(A, w_{\underline{i}})$ for every $w_{\underline{i}} \in W_{\underline{i}}$ except when $w_{\underline{i}} = w_{\underline{i}}^A$. Hence, $\bigoplus_i \{\tau_i(A, w_{\underline{i}}) \oplus_i \mu_i(A, w_{\underline{i}}) : w_{\underline{i}} \in A\} = \tau_i(A, w_{\underline{i}}^A) \oplus_i \mu_i(A, w_{\underline{i}}^A)$. Since $\mu_i^A(w_{\underline{i}}^A) = \tau_i(A, w_{\underline{i}}^A) \oplus_i \det_i(A, t_i, u_i)$, $(\mathbf{C_2})$ holds. As for $(\mathbf{C_3})$, it holds, seeing that for all $w_{\underline{i}} \in W_{\underline{i}}$, if $w_{\underline{i}} \in A$ and $w_{\underline{i}} \in V_{\underline{i}}(\varphi)$ then $\mu_i^A(w_{\underline{i}}) = \tau_i(A, w_{\underline{i}})$. □

Claim 6.17 *(V_1', V_2') is a (W_1', W_2', R_1', R_2')-valuation.*

Proof. The proof that for all $i \in \{1, 2\}$, V_i' satisfies the conditions for $\bot$, $\neg$ and $\vee$ is as expected. The proof that for all $i \in \{1, 2\}$, V_i' satisfies the condition for $[\cdot]$ being a simple application of Claims 6.15 and 6.16, it is left as an exercise for the reader. □

Claim 6.18 is an immediate consequence of Claims 6.15 and 6.16.

Claim 6.18 *(f_1, f_2) is a bounded morphism from $(W_1', W_2', R_1', R_2', V_1', V_2')$ to $(W_1, W_2, R_1, R_2, V_1, V_2)$.*

Here finishes the proof of Proposition 6.11. □

Proposition 6.19 *Let (W_1, W_2, R_1, R_2) be a paraconjunctive frame of indiscernibility. There exists a conjunctive frame of indiscernibility (W_1', W_2', R_1', R_2') and a couple (f_1, f_2) of surjective functions $f_1 : W_1' \longrightarrow W_1$ and $f_2 : W_2' \longrightarrow W_2$ such that for all (W_1, W_2, R_1, R_2)-valuations (V_1, V_2), there exists a (W_1', W_2', R_1', R_2')-valuation (V_1', V_2') such that (f_1, f_2) is a bounded morphism from $(W_1', W_2', R_1', R_2', V_1', V_2')$ to $(W_1, W_2, R_1, R_2, V_1, V_2)$.*

Proof. For all $i \in \{1, 2\}$, let

- $\det_i : \wp(W_{\underline{i}}) \times W_i \times W_i \longrightarrow \wp(W_i)$ be such that for all $A \in \wp(W_{\underline{i}})$ and for all $t_i, u_i \in W_i$, $\det(A, t_i, u_i) = [t_i]_{R_i(A)} \oplus_i [u_i]_{R_i(A)}$ where $[t_i]_{R_i(A)}$ and $[u_i]_{R_i(A)}$ are the equivalence classes of t_i and u_i modulo $R_i(A)$ [12].

[12] Notice that for all $A \in \wp(W_{\underline{i}})$ and for all $t_i, u_i \in W_i$, $\det(A, t_i, u_i) = \emptyset$ if and only if $t_i R_i(A) u_i$.

Now, the rest of the proof is similar to the corresponding rest of the proof of Proposition 6.11, the main difference being that one has to verify here that the considered frames are frames of indiscernibility, an exercise that we leave for the reader. □

Proposition 6.20 is an immediate consequence of Propositions 3.2, 3.13, 6.9 and 6.11.

Proposition 6.20 ($\mathbf{ParCon}_1, \mathbf{ParCon_2}$) *is complete with respect to the class of all preconjunctive frames and the class of all conjunctive frames.*

Proposition 6.21 is an immediate consequence of Propositions 3.2, 3.13, 6.10 and 6.19.

Proposition 6.21 ($\mathbf{Equiv}_1^p, \mathbf{Equiv}_2^p$) *is complete with respect to the class of all preconjunctive frames of indiscernibility and the class of all conjunctive frames of indiscernibility.*

7 A short case study: social epistemic logic

Formalizing epistemic reasoning in social networks, Seligman *et al.* [28] introduce relational structures of the form $(S, A, \equiv, \triangleright)$ such as the ones considered in our introduction. They also introduce a modal language $\mathcal{L}_{\mathbf{SEL}}$ — the *language of social epistemic logic* — consisting of one type of formulas: state-agent formulas — to be interpreted by sets of state-agent couples. Such formulas (typically denoted φ, ψ, etc) are constructed over the Boolean connectives and the modal connectives K and F as follows:

- $\varphi ::= p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid K\varphi \mid F\varphi$,

p ranging over a countably infinite set of atomic formulas [13]. The formula $K\varphi$ (read "I know that φ holds") is true in a state-agent couple (s, a) of some model if the formula φ is true in every state-agent couple (t, a) such that $s \equiv_a t$ whereas the formula $F\varphi$ ("for all my friends, φ holds") is true in a state-agent couple (s, a) of some model if the formula φ is true in every state-agent couple (s, b) such that $a \triangleright_s b$. See [3,27] and [10, Chapter 5].

In the relational structures of the form $(S, A, \equiv, \triangleright)$ considered in our introduction, assuming that $\equiv$ is also a function associating an equivalence relation $\equiv_B$ on S to every $B \in \wp(A)$ in such a way that for all $B \in \wp(A)$, $\equiv_B = \bigcap\{\equiv_a\colon\ a \in B\}$ and $\triangleright$ is also a function associating a binary relation $\triangleright_T$ on A to every $T \in \wp(S)$ in such a way that for all $T \in \wp(S)$, $\triangleright_T = \bigcap\{\triangleright_s :\ s \in T\}$, the language defined in Section 2 can be interpreted as explained in Section 3. A $\mathcal{L}_{\mathbf{SEL}}$-formula φ is said to be $\mathcal{L}_{\mathbf{PML}}$*-definable in a class* $\mathcal{C}$ *of frames* if there exists $\varphi' \in \mathcal{L}_{\mathbf{PML}}$ such that for all $\mathcal{C}$-frames $(S, A, \equiv, \triangleright)$, $(S, A, \equiv, \triangleright) \models \varphi$ (in the sense of [28]) if and only if $(S, A, \equiv, \triangleright) \models \varphi'$ (in the sense of Section 3). A $\mathcal{L}_{\mathbf{PML}}$-formula φ is said to be $\mathcal{L}_{\mathbf{SEL}}$*-definable in a class* $\mathcal{C}$ *of frames* if there

[13] Indeed, the modal language introduced by Seligman *et al.* contains as well nominals used to give names to agents. In this section, however, we only consider its nominal-free fragment.

exists $\varphi' \in \mathcal{L}_{\mathbf{SEL}}$ such that for all $\mathcal{C}$-frames $(S, A, \equiv, \triangleright)$, $(S, A, \equiv, \triangleright) \models \varphi$ (in the sense of Section 3) if and only if $(S, A, \equiv, \triangleright) \models \varphi'$ (in the sense of [28]).

Let $\mathcal{C}_0$ be the class of all frames $(S, A, \equiv, \triangleright)$ such that for all $a \in A$, $\equiv_a$ is an equivalence relation and for all $s \in S$, $\triangleright_s$ is irreflexive and symmetric [14]. In a $\mathcal{C}_0$-frame $(S, A, \equiv, \triangleright)$, for all $a, b \in A$, we say that a and b are *strong friends* if for all $s \in S$, $a \triangleright_s b$. In a $\mathcal{C}_0$-frame $(S, A, \equiv, \triangleright)$, for all $a \in A$, we say that *a is aware of her friends* if for all $s, t \in S$ and for all $b \in A$, if $s \equiv_a t$ then $a \triangleright_s b$ if and only if $a \triangleright_t b$. One may easily verify that for all $\mathcal{C}_0$-frames $(S, A, \equiv, \triangleright)$,

- $(S, A, \equiv, \triangleright) \models [\top_1]_2 \bot_2$ if and only if no agent has strong friends,
- $(S, A, \equiv, \triangleright) \models \langle \top_1 \rangle_2 \top_2$ if and only if every agent has strong friends,
- $(S, A, \equiv, \triangleright) \models p_2 \wedge_2 \langle p'_1 \rangle_2 q_2 \rightarrow_2 \langle \langle p_2 \rangle_1 p'_1 \rangle_2 q_2$ if and only if every agent is aware of her friends.

Moreover, the elementary conditions "no agent has strong friends", "every agent has strong friends" and "every agent is aware of her friends" correspond to no $\mathcal{L}_{\mathbf{SEL}}$-formula in $\mathcal{C}_0$ [15]. Hence,

Proposition 7.1 *The $\mathcal{L}_{\mathbf{PML}}$-formulas $[\top_1]_2 \bot_2$, $\langle \top_1 \rangle_2 \top_2$ and $p_2 \wedge_2 \langle p'_1 \rangle_2 q_2 \rightarrow_2 \langle \langle p_2 \rangle_1 p'_1 \rangle_2 q_2$ are not $\mathcal{L}_{\mathbf{SEL}}$-definable in $\mathcal{C}_0$.*

Problem 7.2 *We do not know if there exists $\mathcal{L}_{\mathbf{SEL}}$-formulas which are not $\mathcal{L}_{\mathbf{PML}}$-definable in $\mathcal{C}_0$.*

8 Conclusion

What has been done in this paper? Within the context of a two-typed parametrized modal language, we have defined a PML as a couple whose components are sets of formulas containing, in their respective types, all propositional tautologies and closed, in their respective types, under modus ponens and uniform substitution. Assuming the normality condition, these components also contain the distribution axiom and are closed under the generalization rule. We have axiomatically introduced different two-typed PMLs and we have given the proofs of their completeness with respect to appropriate classes of relational structures by means of an adaptation of the

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[14] Although the choice of reflexive, symmetric and transitive relations between states for the epistemic modalities is standard, the choice of irreflexive and symmetric relations between agents for the friendship modalities is debatable. In this paper, we simply follow Seligman *et al.* [28] in their assumptions about the friendship modalities.

[15] For instance, in order to show that the elementary conditions "no agent has strong friends" and "every agent has strong friends" correspond to no $\mathcal{L}_{\mathbf{SEL}}$-formula in $\mathcal{C}_0$, it suffices to consider the frames $(S', A', \equiv', \triangleright')$ and $(S'', A'', \equiv'', \triangleright'')$ where $S' = \{1\}$, $A' = \{a, b, c, d\}$, $\triangleright'_1 = \{(a, b), (b, a), (c, d), (d, c)\}$, $S'' = \{2\}$, $A'' = \{a, b, c, d\}$ and $\triangleright''_2 = \{(a, c), (b, d), (c, a), (d, b)\}$. Let $(S, A, \equiv, \triangleright)$ be their disjoint union. By induction on $\varphi \in \mathcal{L}_{\mathbf{SEL}}$, one may easily verify that $(S, A, \equiv, \triangleright) \models \varphi$ if and only if $(S', A', \equiv', \triangleright') \models \varphi$ and $(S'', A'', \equiv'', \triangleright'') \models \varphi$: a contradiction with the obvious fact that no agent has strong friends in $(S, A, \equiv, \triangleright)$ whereas every agent has strong friends both in $(S', A', \equiv', \triangleright')$ and in $(S'', A'', \equiv'', \triangleright'')$.

ordinary canonical model construction. The operation of intersection — which is used in conjunctive frames for the interpretation of the modalities — being not modally definable, these proofs of completeness are not so obvious when the considered PMLs are paraconjunctive.

What challenges remain? Much of this paper revolves around one goal: the definition of a modal language interpreted over relational structures including different types of entities as well as different kinds of relationships between them. As far as we are aware, the modal languages realizing that goal are scarce, even if the first ones were proposed about 25 years ago within the context of spatial logics and arrow logics [16,26,34]. Therefore, much remains to be done. For instance,

- to investigate the computability of the membership problem in PMLs (filtration method, tableaux-based approach, etc),
- to develop the model theory of PMLs (bisimulations, saturated models, etc),
- to construct the duality theory of PMLs (Boolean algebras with operators, general frames, etc),
- to elaborate the correspondence theory of PMLs (Chagrova's Theorem, Sahlqvist Correspondence Theorem, etc),
- to show how a multi-typed parametrized modal language can be used for solving the formalization problems facing those who have to take into account relationships such as the following ones: "trusts", "is a friend of", etc.

Other avenues of research might consist in considering that frames are 4-tuples of the form $(W_1, W_2, \tau_1, \tau_2)$ where for all $i \in \{1, 2\}$, W_i is a nonempty set and $\tau_i : \wp(W_{\underline{i}}) \longrightarrow \wp(\wp(W_i))$ is such that for all $i \in \{1, 2\}$ and for all $A \in \wp(W_{\underline{i}})$, $\tau_i(A)$ is a topology on W_i. In that case, a *valuation on a frame* $(W_1, W_2, \tau_1, \tau_2)$ will be a couple (V_1, V_2) of functions $V_1 : \mathcal{L}_1 \longrightarrow \wp(W_1)$ and $V_2 : \mathcal{L}_2 \longrightarrow \wp(W_2)$ such that for all $i \in \{1, 2\}$,

- $V_i([\varphi]\psi) = \{s_i \in W_i : \exists \mathcal{O}_i \in \tau_i(V_{\underline{i}}(\varphi))\ (s_i \in \mathcal{O}_i\ \&\ \mathcal{O}_i \subseteq V_i(\psi))\}$,

among other conditions. As a result, $V_i(\langle\varphi\rangle\psi) = \{s_i \in W_i : \forall \mathcal{O}_i \in \tau_i(V_{\underline{i}}(\varphi))\ (s_i \in \mathcal{O}_i \Rightarrow \mathcal{O}_i \cap V_i(\psi) \neq \emptyset)\}$. Further investigations are needed for obtaining the PML that will completely axiomatize the validities thus defined.

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Appendix

Alternative proof of Proposition 6.11. The proof of Proposition 6.11 included in the body of the paper may seem unnecessarily complicated. Its good point is that it can be easily converted into a proof of Proposition 6.19. Now, we present the sketch of a simpler proof of Proposition 6.11 that, unfortunately, does not seem to be easily convertible into a proof of Proposition 6.19. For all $i \in \{1,2\}$, let Λ_i be the set of all $\tau_i : \wp(W_{\underline{i}}) \times W_{\underline{i}} \longrightarrow \{0,1\}$. Let (W_1', W_2', R_1', R_2') be the 4-tuple where for all $i \in \{1,2\}$,

- $W_i' = W_i \times \Lambda_i$,
- $R_i' : \wp(W_{\underline{i}}') \longrightarrow \wp(W_i' \times W_i')$ is such that for all $A' \in \wp(W_{\underline{i}}')$ and for all $(t_i, \tau_i), (u_i, \mu_i) \in W_i'$, $(t_i, \tau_i) R_i'(A')(u_i, \mu_i)$ if and only if for all $A \in \wp(W_{\underline{i}})$,
  - · if $A' \cap (A \times \Lambda_{\underline{i}}) \neq \emptyset$ then $t_i R_i(A) u_i$ if and only if for all $v_{\underline{i}} \in A$, $\tau_i(A, v_{\underline{i}}) = \mu_i(A, v_{\underline{i}})$,
  - · for all $(v_{\underline{i}}, \nu_{\underline{i}}) \in A' \cap (A \times \Lambda_{\underline{i}})$, $\tau_i(A, v_{\underline{i}}) = \mu_i(A, v_{\underline{i}})$.

The reader may easily verify that

- for all $i \in \{1,2\}$ and for all $A' \in \wp(W_{\underline{i}}')$, $R_i'(A') = \bigcap\{R_i'(\{(v_{\underline{i}}, \nu_{\underline{i}})\}) : (v_{\underline{i}}, \nu_{\underline{i}}) \in A'\}$,
- the 4-tuple (W_1', W_2', R_1', R_2') is a conjunctive frame.

Let $f_1 : W_1' \longrightarrow W_1$ be such that for all $(v_1, \nu_1) \in W_1'$, $f_1(v_1, \nu_1) = v_1$ and $f_2 : W_2' \longrightarrow W_2$ be such that for all $(v_2, \nu_2) \in W_2'$, $f_2(v_2, \nu_2) = v_2$. The reader may easily verify that f_1 and f_2 are surjective. Let (V_1, V_2) be a (W_1, W_2, R_1, R_2)-valuation. For all $i \in \{1,2\}$, let $V_i' : \mathcal{L}_i \longrightarrow \wp(W_i')$ be such that for all $\varphi \in \mathcal{L}_i$, $V_i'(\varphi) = f_i^{-1}[V_i(\varphi)]$. Notice that for all $i \in \{1,2\}$ and for all $\varphi \in \mathcal{L}_i$, $V_i'(\varphi) = V_i(\varphi) \times \Lambda_i$. Finally, the reader may easily verify that

- for all $(t_i, \tau_i), (u_i, \mu_i) \in W_i'$ and for all $\underline{i}$-formulas φ, if $(t_i, \tau_i) R_i'(V_{\underline{i}}'(\varphi))(u_i, \mu_i)$ then $t_i R_i(V_{\underline{i}}(\varphi)) u_i$,
- for all $(t_i, \tau_i) \in W_i'$, for all $u_i \in W_i$ and for all $\underline{i}$-formulas φ, if $t_i R_i(V_{\underline{i}}(\varphi)) u_i$ then there exists $\mu_i \in \Lambda_i$ such that $(t_i, \tau_i) R_i'(V_{\underline{i}}'(\varphi))(u_i, \mu_i)$,
- (V_1', V_2') is a (W_1', W_2', R_1', R_2')-valuation,
- (f_1, f_2) is a bounded morphism from $(W_1', W_2', R_1', R_2', V_1', V_2')$ to $(W_1, W_2, R_1, R_2, V_1, V_2)$.

□