Provability Logics of Hierarchies

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Abstract

Provability logic is a framework to investigate the provability behavior of the mathematical theories. More precisely, it studies the relationship between a mathematical theory T and a modal logic L via the provability interpretations that read the modality as a provability predicate of T. In this paper, we will extend this relationship from one single theory to a hierarchy of theories capturing the philosophical intuition of the hierarchy of meta-theories one may use to talk about the theories themselves. More precisely, using the modal language with infinitely many modalities, $\{\Box_n\}_{n=0}^{\infty}$, we will first define the hierarchical counterparts of the classical modal logics $\mathsf{K4}$, $\mathsf{KD4}$ and GL. Then, we will show that they are sound and complete with respect to their provability interpretations in the class of all hierarchies, the hierarchies of consistent theories and the constant hierarchies, respectively. We will also show that none of the extensions of the hierarchical counterpart of $\mathsf{KD45}$ has a provability interpretation.

Keywords: provability logic, provability interpretation, Solovay's completeness theorems.

1 Introduction

Provability logic is a framework that identifies the key modal aspects of the provability predicates of the mathematical theories. This modal approach was roughly initiated by Gödel's short note [11] on the interpretation of the Brouwerian constructions as the usual classical proofs. It then gained power when Löb [16] identified the modal properties of a provability predicate required in the usual proof of Gödel's second incompleteness theorem. He also added his own key generalization, the well-known Löb's axiom $\Box(\Box A \to A) \to \Box A$ that is valid under all provability interpretations interpreting $\Box$ as a provability predicate for a strong enough theory T. Finding such non-trivial modal formulas asked for the characterization of all such formulas, the provability logic of the theory T, where the key step was taken by Solovay in his seminal paper [20]. He invented the internalization technique embedding Kripke frames into the formal arithmetic in order to prove that the provability logic of Peano arithmetic is GL, the logic $\mathsf{K4}$ plus the Löb's axiom. Inspired by this seminal work,

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a series of deep investigations were initiated to study the different layers of the provability behavior of the theories, from first-order provability logics [14] and interpretability logics [22,14] to the provability logics of intuitionistic theories and the bimodal and polymodal provability logics addressing more than one provability predicates at the same time [7,10,19,15,13]. In this paper, we are taking a similar route as in the latter to employ a polymodal language to reflect the provability behavior of a hierarchy of theories. The most well-known provability logic in this sense is GLP, introduced by Japaridze in [15] alongside its elaborate Solovay-style arithmetical completeness theorem (also see [8]) and studied extensively later, from many different angles, from topological semantics [6] to computational complexity [18]. However, our motivation and hence our setting is somewhat different. To explain the motivation, let us come back to the original Gödel's interpretation of Brouwerian constructions.

To have a formal language for classical informal provability, in [11], Gödel proposed the modal logic S4. The axioms are all valid under the intuitive interpretation of $\Box$ as the informal provability predicate. The axiom $(K) : \Box(A \to B) \to (\Box A \to \Box B)$ states that the provability predicate is closed under modus ponens. The axiom $(4) : \Box A \to \Box\Box A$ states that "the provability of a provable statement is also provable" which seems a reasonable condition to have and finally $(T) : \Box A \to A$ states that the proofs are all sound. However, as Gödel observed himself, S4 is not sound with respect to the formal provability interpretation that reads $\Box$ as Pr_T, for some strong enough theory T. Because, $\mathsf{S4} \vdash \neg\Box\bot \wedge \Box\neg\Box\bot$ and hence the formula should be valid under the provability interpretation while its interpretation $\neg\mathrm{Pr}_T(\bot) \wedge \mathrm{Pr}_T(\neg\mathrm{Pr}_T(\bot))$ contradicts with Gödel's own second incompleteness theorem. Having that observation, one may wonder if there is any formalization for the intuitive provability interpretation.

To find the source of the mismatch between the formal and the informal provability interpretations, one should look into the role of the nested modalities. Nested modalities intuitively capture the nested use of the provability predicates to express the statements such as "the provability of p", "the provability of "the provability of p"" and so on. These different layers of provability predicates naturally refer to different layers of theories, meta-theories, meta-meta-theories and so on. But the usual provability interpretation reads all of them as the provability predicate for a fixed theory. Philosophically speaking, there is no reason to assume that all the layers of our meta-theories are the same. Quite the contrary, in the actual practice of proof theory, sometimes we need to have more powerful meta-theories to investigate the behavior of the theory itself. For instance, in the aforementioned problematic formula $\neg\Box\bot \wedge \Box\neg\Box\bot$, observing that the inner box refers to a theory T while the outer box refers to its meta-theory U, transforms the contradictory interpretation of the formula to $\neg\mathrm{Pr}_T(\bot) \wedge \mathrm{Pr}_U(\neg\mathrm{Pr}_T(\bot))$ which simply states the safe and intuitive claim that T is consistent and its consistency is provable in its meta-theory U.

Having that observation, [1] proposed using a hierarchy of theories to formalize the different layers of meta-theories instead of using just one theory for all the levels. Following that approach and using some natural classes of the hierarchies of the arithmetical theories, we found some natural interpretations for some modal logics such as $\mathsf{K4}$, $\mathsf{KD4}$ and $\mathsf{S4}$ and hence a formalization for Brouwer-Heyting-Kolmogorov interpretation.

This framework extension suggests a reverse problem of characterizing the provability behavior of a given class of hierarchies of theories rather than providing a provability interpretation for a given modal logic. The present paper is devoted to this problem. We employ the polymodal language $\mathcal{L}_\infty$ with infinitely many modalities $\{\Box_n\}_{n=0}^{\infty}$ to capture the different layers of the meta-theories' hierarchy. However, our polymodal approach deviates from the usual polymodal approach by making the syntactical restrictions to avoid using the lower boxes over themselves or the higher ones. This captures the intuition that a theory can not refer to itself or its meta-theories. Using this restriction, the modal logics naturally avoids GL-style principles to transparently reflect the provability behavior of the hierarchies rather than the somewhat peculiar behavior of the single theories. Employing this restriction, we will introduce the hierarchical counterparts of the logics $\mathsf{K4}$, $\mathsf{KD4}$, $\mathsf{S4}$, $\mathsf{KD45}$, $\mathsf{S5}$ and GL, denoted by $\mathsf{K4}_\infty$, $\mathsf{KD4}_\infty$, $\mathsf{S4}_\infty$, $\mathsf{KD45}_\infty$, $\mathsf{S5}_\infty$ and GL_∞, respectively. Then, we will introduce the provability interpretation for some of these new logics with respect to the hierarchies of theories. We will see that $\mathsf{K4}_\infty$ is sound and complete with respect to the class of all hierarchies, while $\mathsf{KD4}_\infty$ and GL_∞ capture all consistent and constant hierarchies, respectively. We will also show that no extension of $\mathsf{KD45}_\infty$, including $\mathsf{S5}_\infty$ has a provability interpretation. To prove the completeness results, unfortunately, it seems impossible to imitate Solovay's technique directly. However, we will present a reduction method that reduces the required completeness to Solovay's theorem. It is also possible to develop a similar result for $\mathsf{S4}_\infty$ but its technique is beyond what we employ in this paper. The logics and their connection to hierarchies were introduced in the unpublished preprint [2], where we used the results in [1] to provide the required completeness theorems. In this paper, we present a somewhat different presentation of the systems and a self-contained direct completeness proofs independent of the results in [1].

2 Preliminaries

In this section, we will recall some basic preliminary facts about the modal logic GL, its sequent-style proof system and its provability interpretations. Let $\mathcal{L} = \{\wedge, \vee, \bot, \to, \Box\}$ be the language of modal logics. We use $\neg A$ and $\top$ as abbreviations for $A \to \bot$ and $\bot \to \bot$, respectively. The only modal logic we work with in this paper is Gödel-Löb logic GL defined as the smallest set of formulas in $\mathcal{L}$ containing all classical tautologies, the axioms $(K) : \Box(A \to B) \to (\Box A \to \Box B)$, $(4) : \Box A \to \Box\Box A$ and $(L) : \Box(\Box A \to A) \to \Box A$ and closed under the rules $(MP) : A, A \to B \vdash B$ and $(NC) : A \vdash \Box A$.

By a sequent over $\mathcal{L}$, we mean an expression in the form $S = \Gamma \Rightarrow \Delta$, where Γ and Δ are multisets of formulas in $\mathcal{L}$. Define **GGL** as the system consisting of the rules depicted in Figure 1. **GGL** is equivalent to the system defined in

$$\frac{}{A \Rightarrow A}\,Ax \qquad \frac{}{\bot \Rightarrow}\,L\bot$$

$$\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\,Lc \qquad \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}\,Rc$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\,Lw \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}\,Rw$$

$$i \in \{0,1\}\ \frac{\Gamma, A_i \Rightarrow \Delta}{\Gamma, A_0 \wedge A_1 \Rightarrow \Delta}\,L\wedge \qquad \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}\,R\wedge$$

$$\frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}\,L\vee \qquad i \in \{0,1\}\ \frac{\Gamma \Rightarrow A_i, \Delta}{\Gamma \Rightarrow A_0 \vee A_1, \Delta}\,R\vee$$

$$\frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta}\,L\to \qquad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta}\,R\to$$

$$\frac{\Gamma, \Box\Gamma, \Box A \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\,GL$$

Fig. 1. The sequent calculus **GGL**.

[12] and hence is complete for GL. Later, for some technical reasons, we will extend the language $\mathcal{L}$ by a sequence of fresh atomic variables $Q = \{q_n\}_{n=0}^{\infty}$. We will denote this language, the logic and the sequent system for it by $\mathcal{L}(Q)$, $\mathsf{GL}(Q)$ and $\mathbf{GGL}(Q)$, respectively.

The second ingredient we need is the arithmetical theories and the provability interpretation they provide for the logic GL. We only recall some important points and for the rest refer the reader to [4]. Let $\mathcal{L}_{\mathrm{PA}} = \{\leq, s, +, \cdot, \exp, 0\}$ be the usual language of Peano arithmetic augmented with the symbol exp with the intended meaning $\exp(n) = 2^n$. The expressions $\forall x \leq t\, \phi(x)$ and $\exists x \leq t\, \phi(x)$ abbreviate $\forall x(x \leq t \to \phi(x))$ and $\exists x(x \leq t \wedge \phi(x))$, respectively. The occurrence of the quantifiers in these formulas are called bounded. By Σ_1, we mean the least class of formulas in $\mathcal{L}_{\mathrm{PA}}$ containing the atomic formulas and their negations and closed under conjunction, disjunction, bounded quantifiers and existential quantifiers. By the abuse of notation, we extend Σ_1 to include any formula logically equivalent to a formula in Σ_1. The formulas in Σ_1 describe recursively enumerable sets and vice verse. By $I\Sigma_1$, we mean a basic quantifier-free theory defining the symbols of the language [4], extended by the induction axiom $\phi(0) \wedge \forall x(\phi(x) \to \phi(s(x))) \to \forall x \phi(x)$, where $\phi(x) \in \Sigma_1$. The theory $I\Sigma_1$ enjoys Σ_1-completeness, meaning that for any sentence $\phi \in \Sigma_1$, if $\mathbb{N} \vDash \phi$ then $I\Sigma_1 \vdash \phi$. A theory T is called Σ_1-sound if $T \vdash \phi$ implies $\mathbb{N} \vDash \phi$, for any sentence $\phi \in \Sigma_1$.

One of the interesting properties of $I\Sigma_1$ is its power to formalize a basic amount of meta-mathematics. Let $\ulcorner \phi \urcorner$ be one of the natural Gödel numberings for the formulas of $\mathcal{L}_{\mathrm{PA}}$ and set $\mathrm{Pr}(x) \in \Sigma_1$ as a predicate satisfying

(i) $I\Sigma_1 \vdash \phi$ iff $\mathbb{N} \vDash \mathrm{Pr}(\ulcorner \phi \urcorner)$,

(ii) $I\Sigma_1 \vdash \mathrm{Pr}(\ulcorner \phi \to \psi \urcorner) \to (\mathrm{Pr}(\ulcorner \phi \urcorner) \to \mathrm{Pr}(\ulcorner \psi \urcorner))$, (*iii*) (formalized Σ_1-completeness) $I\Sigma_1 \vdash \phi \to \mathrm{Pr}(\ulcorner \phi \urcorner)$, for any $\phi \in \Sigma_1$.

For such a predicate, see [4]. We fix this predicate throughout the paper as the provability predicate for $I\Sigma_1$. Now, let T be a recursively enumerable theory over $\mathcal{L}_{\mathrm{PA}}$ extending $I\Sigma_1$. By a provability predicate for T, we mean a formula $\mathrm{Pr}_T(x) \in \Sigma_1$ such that:

(*i*) $T \vdash \phi$ iff $\mathbb{N} \vDash \mathrm{Pr}_T(\ulcorner \phi \urcorner)$,

(*ii*) $I\Sigma_1 \vdash \mathrm{Pr}_T(\ulcorner \phi \to \psi \urcorner) \to (\mathrm{Pr}_T(\ulcorner \phi \urcorner) \to \mathrm{Pr}_T(\ulcorner \psi \urcorner))$,

(*iii*) $I\Sigma_1 \vdash \mathrm{Pr}(\ulcorner \phi \urcorner) \to \mathrm{Pr}_T(\ulcorner \phi \urcorner)$.

For simplicity, we usually write $\mathrm{Pr}_T(\phi)$ for $\mathrm{Pr}_T(\ulcorner \phi \urcorner)$. In this paper, we only work with recursively enumerable theories T extending $I\Sigma_1$. Our provability predicates also formally reflect this fact as the part (*iii*) demands. For the future reference, to address (*iii*), if Pr_T is clear from the context, we say that T is extending $I\Sigma_1$, provably in $I\Sigma_1$. It is easy to see that any provability predicate Pr_T satisfies the following conditions:

(*i*) $T \vdash \phi$ iff $I\Sigma_1 \vdash \mathrm{Pr}_T(\phi)$,

(*ii*) $I\Sigma_1 \vdash \mathrm{Pr}_T(\phi \to \psi) \to (\mathrm{Pr}_T(\phi) \to \mathrm{Pr}_T(\psi))$,

(*iii*) $I\Sigma_1 \vdash \mathrm{Pr}_T(\phi) \to \mathrm{Pr}_T(\mathrm{Pr}_T(\phi))$.

The first is a consequence of Σ_1-completeness of $I\Sigma_1$ and the third is a consequence of formalized Σ_1-completeness of $I\Sigma_1$ together with the fact that T extends $I\Sigma_1$, provably in $I\Sigma_1$. It is routine to see that these conditions imply $I\Sigma_1 \vdash \mathrm{Pr}_T(\mathrm{Pr}_T(\phi) \to \phi) \to \mathrm{Pr}_T(\phi)$, for any sentence $\phi \in \mathcal{L}_{\mathrm{PA}}$ and specifically, the formalized Gödel's second incompleteness theorem, i.e., $I\Sigma_1 \vdash \mathrm{Pr}_T(\neg \mathrm{Pr}_T(\bot)) \to \mathrm{Pr}_T(\bot)$. Denoting $\neg \mathrm{Pr}_T(\bot)$ by Cons_T, we have $I\Sigma_1 \vdash \mathrm{Cons}_T \to \neg \mathrm{Pr}_T(\mathrm{Cons}_T)$ [4].

Definition 2.1 By an arithmetical substitution σ, we mean a function assigning arithmetical sentences to the atomic formulas of $\mathcal{L}$. Let $T \supseteq I\Sigma_1$ be a theory, Pr_T be a provability predicate for T and $A \in \mathcal{L}$ be a modal formula. Then, by $A^{\mathrm{Pr}_T, \sigma}$, we mean an arithmetical sentence resulting by substituting the atoms of A according to σ and interpreting its boxes as Pr_T.

In the following, we will present the uniform version of Solovay's characterization of GL [20] investigated in [17,3,21,9,5]. For a clear exposition and the generality we use here, see [4].

Theorem 2.2 *(Uniform Solovay's Theorem) If* $\mathsf{GL} \vdash A$*, then* $I\Sigma_1 \vdash A^{\mathrm{Pr}_T, \sigma}$*, for any arithmetical theory* $T \supseteq I\Sigma_1$*, any provability predicate* Pr_T *for* T *and any arithmetical substitution* σ*. Conversely, for any* Σ_1*-sound theory* $T \supseteq I\Sigma_1$*, there is a provability predicate* Pr_T *and an arithmetical substitution* $*$ *such that for any modal formula* $A \in \mathcal{L}$*, if* $T \vdash A^{\mathrm{Pr}_T, *}$ *then* $\mathsf{GL} \vdash A$*.*

Notice the uniformity in the completeness part of the theorem that provides one arithmetical substitution for all modal formulas. This property will play a crucial role in our completeness results later in Section 5.

3 Hierarchical Modal Logics

In this section, we first introduce a polymodal language to reflect the provability predicates for a hierarchy of theories rather than just one theory. Then, we will introduce the hierarchical counterparts of some basic modal logics.

Definition 3.1 Let $\mathcal{L}_\infty = \{\wedge, \vee, \bot, \to\} \cup \{\Box_n\}_{n=0}^{\infty}$ be a modal language with infinitely many modalities. The set of formulas in this language, also denoted by $\mathcal{L}_\infty$, is defined as the least set of expressions containing the atomic formulas and $\bot$ and closed under all propositional operations and the following operation: If $A \in \mathcal{L}_\infty$ and n is *strictly greater than* the index of any box occurring in A, then $\Box_n A \in \mathcal{L}_\infty$. By the rank of $A \in \mathcal{L}_\infty$, denoted by $r(A)$, we mean the greatest index of the boxes occurring in A. If there is none, then set $r(A) = -1$. Finally, for a multiset Γ of formulas in $\mathcal{L}_\infty$, define $r(\Gamma)$ as the maximum of the ranks of its elements.

Notice the difference between the formulas in $\mathcal{L}_\infty$ and the usual polymodal formulas. In the former case, we impose a syntactic restriction that only allows a box in a formula if its index is greater than all the indices of the boxes lying in its scope. For instance, the expression $\Box_1(\Box_0 p \to p)$ is a formula in $\mathcal{L}_\infty$ with rank one, while the expression $\Box_1\Box_1 p$ is not a formula. From now on, we implicitly assume that any polymodal formula used in this paper belongs to the set $\mathcal{L}_\infty$. For instance, whenever we consider an axiom, we only allow the substitutions that result in a formula in $\mathcal{L}_\infty$. As an example, in the axiom $\Box_1 p \to \Box_1 p$, the formula p can be substituted by $\Box_0 q \to r$ but not $\Box_2 q$.

Definition 3.2 Consider the following set of axioms:

(H) $\Box_n A \to \Box_{n+1} A$,

(K_∞) $\Box_n(A \to B) \to (\Box_n A \to \Box_n B)$,

(4_∞) $\Box_n A \to \Box_{n+1}\Box_n A$,

(D_∞) $\neg\Box_n \bot$,

(T_∞) $\Box_n A \to A$,

(5_∞) $\neg\Box_n A \to \Box_{n+1}\neg\Box_n A$,

(L_∞) $\Box_{n+1}(\Box_n A \to A) \to \Box_n A$.

Let $\mathcal{A}$ be a set of these axioms. Define the set of $L(\mathcal{A})$-proofs as the least set of finite sequences of formulas containing the sequences with length one of classical tautologies over the language $\mathcal{L}_\infty$ or instances of the axioms in $\mathcal{A}$ and closed under the following two rules:

(MP) If $\{A_i\}_{i=1}^m$ and $\{B_j\}_{j=1}^l$ are $L(\mathcal{A})$-proofs such that $A_m = D$ and $B_l = D \to E$, then $\{C_k\}_{k=1}^{m+l+1}$ is an $L(\mathcal{A})$-proof, where $C_k = A_k$, for $1 \leq k \leq m$, $C_k = B_{k-m}$, for $m+1 \leq k \leq m+l$ and $C_{m+l+1} = E$,

(NC_∞) If $\{A_i\}_{i=1}^m$ is an $L(\mathcal{A})$-proof such that $A_m = D$ and $r(A_i) < n$, for any $1 \leq i \leq m$, then $\{B_k\}_{k=1}^{m+1}$ is an $L(\mathcal{A})$-proof, where $B_k = A_k$, for $1 \leq k \leq m$ and $B_{m+1} = \Box_n D$, By the rank of an $L(\mathcal{A})$-proof, we mean the maximum of the ranks of the formulas it contains. If an $L(\mathcal{A})$-proof ends with the formula A, we call it an $L(\mathcal{A})$-proof for A. If there exists an $L(\mathcal{A})$-proof for A, we write $L(\mathcal{A}) \vdash A$. For any (not necessarily finite) set $\Gamma \cup \{A\} \subseteq \mathcal{L}_\infty$, by $L(\mathcal{A}) \vdash \Gamma \Rightarrow A$, we mean the existence of a finite set $\Delta \subseteq \Gamma$ such that $L(\mathcal{A}) \vdash \bigwedge \Delta \to A$. We denote the following $L(\mathcal{A})$'s by their usual modal terminology: $\mathsf{K4}_\infty = L(H, K_\infty, 4_\infty)$, $\mathsf{KD4}_\infty = L(H, K_\infty, 4_\infty, D_\infty)$, $\mathsf{GL}_\infty = L(H, K_\infty, 4_\infty, L_\infty)$, $\mathsf{KD45}_\infty = L(H, K_\infty, 4_\infty, D_\infty, 5_\infty)$, and $\mathsf{S5}_\infty = L(H, K_\infty, 4_\infty, T_\infty, 5_\infty)$.

The only point to clarify is the deviation of (NC_∞) from the usual necessitation rule. To explain, assume we already provided a proof for a statement A using formulas with maximum rank $n-1$. This argument, as it refers to the meta-theories up to the level $n-1$, must live in a higher meta-theory. Hence, it is reasonable to conclude "the provability of A" in the level n or higher, i.e., $\Box_m A$, for $m \geq n$. Note that even if $\Box_k A \in \mathcal{L}_\infty$, for some $k \leq n-1$, we can not use (NC_∞) to conclude $\Box_k A$ as the whole proof lives in the level n or higher. In this sense, our necessitation is a global operation depending on the whole proof. One may wonder if it is the case that any provable formula A has a proof with rank bounded by $r(A)$. To prove this form of analyticity, we need to design cut-free sequent calculi for our logics which is beyond the scope of this paper, see [2]. However, we use an indirect method to prove it for $\mathbf{K4}_\infty$ in Section 7.

4 Provability Models

The canonical notion of model for the introduced hierarchical modal logics must consist of a classical model to interpret the box-free formulas and a hierarchy of theories to interpret the boxes.

Definition 4.1 A provability model is a tuple $\mathfrak{M} = (M, \{T_n\}_{n=0}^\infty, \{\mathrm{Pr}_n\}_{n=0}^\infty)$, where M is a model of $I\Sigma_1$, $\{T_n\}_{n=0}^\infty$ is a hierarchy of recursively enumerable arithmetical theories, all extending $I\Sigma_1$ and $\{\mathrm{Pr}_n\}_{n=0}^\infty$ is a sequence of provability predicates such that for any $n \geq 0$, Pr_n is a provability predicate for T_n and $T_n \subseteq T_{n+1}$, provably in $I\Sigma_1$, i.e., $I\Sigma_1 \vdash \mathrm{Pr}_n(\phi) \to \mathrm{Pr}_{n+1}(\phi)$, for any arithmetical sentence ϕ. We denote M, T_n and Pr_n, by $|\mathfrak{M}|$, $T_n^{\mathfrak{M}}$ and $\mathrm{Pr}_n^{\mathfrak{M}}$, respectively.

Remark 4.2 Here are some remarks. First, we assume that all theories extend the basic theory $I\Sigma_1$ and $M \vDash I\Sigma_1$ as we want our theories and our model to have the power to implement and understand the basic meta-mathematical theorems, respectively. As long as the base theory is powerful enough, the choice of $I\Sigma_1$ is immaterial. Secondly, from now on, as a theory is uniquely determined by its provability predicate, by dropping $\{T_n\}_{n=0}^\infty$, we only use the pair $\mathfrak{M} = (M, \{\mathrm{Pr}_n\}_{n=0}^\infty)$ to denote a provability model.

Definition 4.3 (i) The class of all provability models is denoted by $\mathbf{PrM}$.

(ii) A provability model $(M, \{\mathrm{Pr}_n\}_{n=0}^\infty)$ is called *consistent*, if for any $n \geq 0$, the model M thinks that T_n is consistent and $T_{n+1} \vdash \mathrm{Cons}(T_n)$, i.e., $M \vDash$

Cons(T_n) and $M \vDash \mathrm{Pr}_{n+1}(\mathrm{Cons}(T_n))$. The class of all consistent provability models is denoted by **Cons**.

(*iii*) A provability model $(M, \{\mathrm{Pr}_n\}_{n=0}^{\infty})$ is *constant*, if for any n and m, $(M, \{\mathrm{Pr}_n\}_{n=0}^{\infty})$ thinks that $T_n = T_m$, i.e., $M \vDash \mathrm{Pr}_m(\phi) \leftrightarrow \mathrm{Pr}_n(\phi)$ and $M \vDash \mathrm{Pr}_0(\mathrm{Pr}_m(\phi) \leftrightarrow \mathrm{Pr}_n(\phi))$, for any sentence $\phi \in \mathcal{L}_{\mathrm{PA}}$. The class of all constant provability models is denoted by **Cst**.

Definition 4.4 By an arithmetical substitution, we mean a function assigning an arithmetical sentence to any atomic formula of $\mathcal{L}_\infty$. If $\mathfrak{M} = (M, \{\mathrm{Pr}_n\}_{n=0}^{\infty})$ is a provability model, $A \in \mathcal{L}_\infty$ is a formula and σ is an arithmetical substitution, then by $A^{\mathfrak{M},\sigma}$, we mean the arithmetical sentence resulting from substituting the atomic formulas in A according to σ and interpreting $\Box_n$ in A as Pr_n. If Γ is a set of formulas, by $\Gamma^{\mathfrak{M},\sigma}$, we mean the set $\{A^{\mathfrak{M},\sigma} \mid A \in \Gamma\}$.

Definition 4.5 Let $\mathfrak{M}$ be a provability model and $A \in \mathcal{L}_\infty$ be a formula. Then, A is satisfied in the model $\mathfrak{M}$, denoted by $\mathfrak{M} \vDash A$, if $|\mathfrak{M}| \vDash A^{\mathfrak{M},\sigma}$, for any arithmetical substitution σ. Moreover, if $\Gamma \cup \{A\}$ is a (not necessarily finite) set of formulas, $\mathcal{C}$ is a class of provability models and σ is an arithmetical substitution, we write $\mathcal{C} \vDash \Gamma^\sigma \Rightarrow A^\sigma$ when $|\mathfrak{M}| \vDash \bigwedge \Gamma^{\mathfrak{M},\sigma}$ implies $|\mathfrak{M}| \vDash A^{\mathfrak{M},\sigma}$, for any $\mathfrak{M} \in \mathcal{C}$ and we write $\mathcal{C} \vDash \Gamma \Rightarrow A$ if $\mathcal{C} \vDash \Gamma^\sigma \Rightarrow A^\sigma$, for any arithmetical substitution σ.

Example 4.6 Define $T_0 = I\Sigma$ and $T_{n+1} = T_n + \mathrm{Cons}(T_n)$, for any $n \geq 0$ and set $\mathrm{Pr}_0 = \mathrm{Pr}$ and $\mathrm{Pr}_{n+1}(\phi) = \mathrm{Pr}_n(\mathrm{Cons}_{T_n} \to \phi)$. Then, the pair $(\mathbb{N}, \{\mathrm{Pr}_n\}_{n=0}^{\infty})$ is clearly a consistent provability model. To have an example of satisfaction of a formula in a provability model, note that $(\mathbb{N}, \{T_n\}_{n=0}^{\infty}) \vDash \Box_{n+1}(\neg\Box_n p \vee \neg\Box_n \neg p)$, as for any arithmetical substitution σ, as $T_{n+1} \vdash \neg\mathrm{Pr}_n(\bot)$, $I\Sigma_1 \subseteq T_{n+1}$ and Pr_n is a provability predicate, we have $T_{n+1} \vdash \neg\mathrm{Pr}_n(p^\sigma) \vee \neg\mathrm{Pr}_n(\neg p^\sigma)$ which implies $\mathbb{N} \vDash \mathrm{Pr}_{n+1}(\neg\mathrm{Pr}_n(p^\sigma) \vee \neg\mathrm{Pr}_n(\neg p^\sigma))$.

Lemma 4.7 *Let $(L, \mathcal{C})$ be one of the pairs $(\mathsf{K4}_\infty, \mathbf{PrM})$, $(\mathsf{KD4}_\infty, \mathbf{Cons})$, or $(\mathsf{GL}_\infty, \mathbf{Cst})$. If A has an L-proof with rank n, then $|\mathfrak{M}| \vDash A^{\mathfrak{M},\sigma}$ and $|\mathfrak{M}| \vDash \mathrm{Pr}_m(A^{\mathfrak{M},\sigma})$, for any provability model $\mathfrak{M} \in \mathcal{C}$, any $m > n$ and any arithmetical substitution σ.*

Proof. Fix σ and $\mathfrak{M} = (M, \{\mathrm{Pr}_n\}_{n=0}^{\infty}) \in \mathcal{C}$ and denote $D^{\mathfrak{M},\sigma}$ by D^σ, for any $D \in \mathcal{L}_\infty$. Now, use a structural induction on the set of L-proofs to prove the claim. If A is a classical tautology or an instance of the axioms (H), (K_∞), or (4_∞), we first show $I\Sigma_1 \vdash A^\sigma$. The case for the classical tautology, (H) and (K_∞) are easy. For (4_∞), we have $A = \Box_{n-1}B \to \Box_n\Box_{n-1}B$, for some B. Therefore, $A^\sigma = \mathrm{Pr}_{n-1}(B^\sigma) \to \mathrm{Pr}_n(\mathrm{Pr}_{n-1}(B^\sigma))$. As $\mathrm{Pr}_{n-1} \in \Sigma_1$, by the formalized Σ_1-completeness, we have $I\Sigma_1 \vdash \mathrm{Pr}_{n-1}(B^\sigma) \to \mathrm{Pr}(\mathrm{Pr}_{n-1}(B^\sigma))$. Finally, as $I\Sigma \subseteq T_n$ provably in $I\Sigma_1$, we have $I\Sigma_1 \vdash \mathrm{Pr}_{n-1}(B^\sigma) \to \mathrm{Pr}_n(\mathrm{Pr}_{n-1}(B^\sigma))$. Now, as $I\Sigma_1 \vdash A^\sigma$ for a classical tautology or an instance of the axioms (H), (K_∞) or (4_∞), we have $M \vDash A^\sigma$ as $M \vDash I\Sigma_1$. On the other hand, as $I\Sigma_1 \subseteq T_m$, we have $T_m \vdash A^\sigma$. By Σ_1-completeness, we reach $I\Sigma_1 \vdash \mathrm{Pr}_m(A^\sigma)$ which implies $M \vDash \mathrm{Pr}_m(A^\sigma)$, again by $M \vDash I\Sigma_1$.
For the axiom (D_∞), we have $A = \neg\Box_n\bot$ and $\mathfrak{M} \in \mathbf{Cons}$. Hence, $M \vDash$

$\neg\mathrm{Pr}_n(\bot)$ and $M \vDash \mathrm{Pr}_{n+1}(\neg\mathrm{Pr}_n(\bot))$, by definition. As $m \geq n+1$ and the hierarchy is increasing, provably in $I\Sigma_1$, we have $I\Sigma_1 \vdash \mathrm{Pr}_{n+1}(\neg\mathrm{Pr}_n(\bot)) \to \mathrm{Pr}_m(\neg\mathrm{Pr}_n(\bot))$ and hence, we reach $M \vDash \mathrm{Pr}_m(\neg\mathrm{Pr}_n(\bot))$.
For the axiom (L_∞), we have $A = \Box_n(\Box_{n-1}B \to B) \to \Box_{n-1}B$, for some B and $\mathfrak{M} \in \mathbf{Cst}$. Let $\phi = \mathrm{Pr}_{n-1}(B^\sigma) \to B^\sigma$. Then, $M \vDash \mathrm{Pr}_n(\phi) \leftrightarrow \mathrm{Pr}_{n-1}(\phi)$ and $M \vDash \mathrm{Pr}_0(\mathrm{Pr}_n(\phi) \leftrightarrow \mathrm{Pr}_{n-1}(\phi))$ as $\mathfrak{M}$ is constant. Therefore, as $T_0 \subseteq T_m$, provably in $I\Sigma_1$, we have $M \vDash \mathrm{Pr}_m(\mathrm{Pr}_n(\phi) \leftrightarrow \mathrm{Pr}_{n-1}(\phi))$. Hence, M thinks that A^σ and $\mathrm{Pr}_m(A^\sigma)$ are equivalent to $\mathrm{Pr}_{n-1}(\mathrm{Pr}_{n-1}(B^\sigma) \to B^\sigma) \to \mathrm{Pr}_{n-1}(B^\sigma)$ and $\mathrm{Pr}_m(\mathrm{Pr}_{n-1}(\mathrm{Pr}_{n-1}(B^\sigma) \to B^\sigma) \to \mathrm{Pr}_{n-1}(B^\sigma))$, respectively. However, as Pr_{n-1} is a provability predicate, we have $I\Sigma_1 \vdash \mathrm{Pr}_{n-1}(\mathrm{Pr}_{n-1}(B^\sigma) \to B^\sigma) \to \mathrm{Pr}_{n-1}(B^\sigma)$ and hence $M \vDash \mathrm{Pr}_{n-1}(\mathrm{Pr}_{n-1}(B^\sigma) \to B^\sigma) \to \mathrm{Pr}_{n-1}(B^\sigma)$. Moreover, we have $I\Sigma_1 \vdash \mathrm{Pr}(\mathrm{Pr}_{n-1}(\mathrm{Pr}_{n-1}(B^\sigma) \to B^\sigma) \to \mathrm{Pr}_{n-1}(B^\sigma))$, by formalized Σ_1-completeness. As $I\Sigma_1 \subseteq T_m$, provably in $I\Sigma_1$, we finally reach $I\Sigma_1 \vdash \mathrm{Pr}_m(\mathrm{Pr}_{n-1}(\mathrm{Pr}_{n-1}(B^\sigma) \to B^\sigma) \to \mathrm{Pr}_{n-1}(B^\sigma))$ which implies $M \vDash \mathrm{Pr}_m(\mathrm{Pr}_{n-1}(\mathrm{Pr}_{n-1}(B^\sigma) \to B^\sigma) \to \mathrm{Pr}_{n-1}(B^\sigma))$.
For the rules, if A is the result of the modus ponens rule over the L-proofs for B and $B \to A$, by the induction hypothesis, we have $M \vDash B^\sigma$, $M \vDash \mathrm{Pr}_m(B^\sigma)$, $M \vDash B^\sigma \to A^\sigma$, and $M \vDash \mathrm{Pr}_m(B^\sigma \to A^\sigma)$. Hence, $M \vDash A^\sigma$ and $M \vDash \mathrm{Pr}_m(A^\sigma)$, as Pr_m is a provability predicate. If A is a consequence of the rule (NC_∞), then $A = \Box_n B$, for some B and n is greater than the rank of all formulas in the proof prior to B. By the induction hypothesis, we have $M \vDash \mathrm{Pr}_n(B^\sigma)$. Moreover, by the formalized Σ_1-completeness, we have $I\Sigma_1 \vdash \mathrm{Pr}_n(B^\sigma) \to \mathrm{Pr}(\mathrm{Pr}_n(B^\sigma))$. As $I\Sigma_1 \subseteq T_m$, provably in $I\Sigma_1$, we reach $I\Sigma_1 \vdash \mathrm{Pr}_n(B^\sigma) \to \mathrm{Pr}_m(\mathrm{Pr}_n(B^\sigma))$ which proves $M \vDash \mathrm{Pr}_m(\mathrm{Pr}_n(B^\sigma))$, as $M \vDash I\Sigma_1$. □

Theorem 4.8 *(Soundness Theorem)*

(i) If $\mathsf{K4}_\infty \vdash \Gamma \Rightarrow A$ *then* $\mathbf{PrM} \vDash \Gamma \Rightarrow A$.

(ii) If $\mathsf{KD4}_\infty \vdash \Gamma \Rightarrow A$ *then* $\mathbf{Cons} \vDash \Gamma \Rightarrow A$.

(iii) If $\mathsf{GL}_\infty \vdash \Gamma \Rightarrow A$ *then* $\mathbf{Cst} \vDash \Gamma \Rightarrow A$.

Proof. We only prove the case of $\mathsf{K4}_\infty$. The rest are similar. If $\mathsf{K4}_\infty \vdash \Gamma \Rightarrow A$, then, there exists a finite set $\Delta \subseteq \Gamma$ such that $\mathsf{K4}_\infty \vdash \bigwedge \Delta \to A$. Then, by Lemma 4.7, for any $\mathfrak{M} \in \mathbf{PrM}$ and any arithmetical substitution σ, we have $|\mathfrak{M}| \vDash \bigwedge \Delta^{\mathfrak{M},\sigma} \to A^{\mathfrak{M},\sigma}$. Therefore, $\mathfrak{M} \vDash \Gamma \Rightarrow A$. □

5 Completeness Results

In this section, we will provide the completeness results for the provability interpretations we have provided before.

5.1 Logics $\mathsf{K4}_\infty$ and $\mathsf{KD4}_\infty$

For the completeness of $\mathsf{K4}_\infty$ and $\mathsf{KD4}_\infty$, our strategy is using a translation between GL and $\mathsf{K4}_\infty$ to reduce the completeness to uniform Solovay's theorem.

Definition 5.1 Let $Q = \{q_n\}_{n=0}^\infty$ be a sequence of fresh atomic formulas occurring nowhere in the formulas of $\mathcal{L}$. Define the translation function

$t : \mathcal{L}_\infty \to \mathcal{L}(Q)$ as follows: $\bot^t = \bot$, $p^t = p$, for any atomic formula p, $(B \circ C)^t = B^t \circ C^t$, for any $\circ \in \{\wedge, \vee, \to\}$ and $(\Box_n B)^t = \Box(\bigwedge_{i=0}^n q_i \to B^t)$.

The translation t is a syntactical way to connect the provability interpretations of $\mathsf{K4}_\infty$ to that of GL by interpreting a (suitable) *hierarchy of theories* as *a base theory* extended by a sequence of formulas. The following lemma provides the connection we are seeking. The proof is the main machinery of the present paper and shall be given in a separate Section 7.

Lemma 5.2 *(Reduction Lemma) If* $\mathsf{GL}(Q) \vdash A^t$ *then* $\mathsf{K4}_\infty \vdash A$.

Theorem 5.3 *(Uniform Strong Completeness) Let* $(L, \mathcal{C})$ *be one of the pairs* $(\mathsf{K4}_\infty, \mathbf{PrM})$ *or* $(\mathsf{KD4}_\infty, \mathbf{Cons})$. *Then, for any* Σ_1*-sound recursively enumerable arithmetical theory* $T \supseteq I\Sigma_1$, *there exist a hierarchy of theories* $\{T_n\}_{n=0}^\infty$, *all extending* T, *a hierarchy of provability predicates* $\{\mathrm{Pr}_n\}_{n=0}^\infty$ *for* $\{T_n\}_{n=0}^\infty$ *and an arithmetical substitution* τ *such that* $T_n \subseteq T_{n+1}$, *provably in* $I\Sigma_1$, *for any* n *and for any set (not necessarily finite)* $\Gamma \cup \{A\}$ *of formulas in* $\mathcal{L}_\infty$, *if* $\{(M, \{\mathrm{Pr}_n\}_{n=0}^\infty) \in \mathcal{C} \mid M \vDash T\} \vDash \Gamma^\tau \Rightarrow A^\tau$, *then* $L \vdash \Gamma \Rightarrow A$.

Proof. We first prove the claim for $L = \mathsf{K4}_\infty$. Let Pr_T and $*$ be the provability predicate and the substitution that the uniform Solovay's theorem, Theorem 2.2, provides. Therefore, $T \vdash B^{\mathrm{Pr}_T, *}$ iff $\mathsf{GL}(Q) \vdash B$, for any formula $B \in \mathcal{L}(Q)$. For any $n \geq 0$, set $T_n = T + \{q_i^*\}_{i=0}^n$ with the provability predicate $\mathrm{Pr}_n(\phi) = \mathrm{Pr}_T(\bigwedge_{i=0}^n q_i^* \to \phi)$. We claim that $\tau = *$ together with the hierarchies $\{T_n\}_{n=0}^\infty$ and $\{Pr_n\}_{n=0}^\infty$ satisfies the properties the theorem claims. First, note that by definition, $T \subseteq T_n$ and $T_n \subseteq T_{n+1}$, provably in $I\Sigma_1$, for any n. Then, let M be an arbitrary model of T and fix $\mathfrak{M}_M = (M, \{\mathrm{Pr}_n\}_{n=0}^\infty)$. Recall that the translation between $\mathsf{K4}_\infty$ and $\mathsf{GL}(Q)$ interprets $\Box_m C$ as $\Box(\bigwedge_{i=0}^m q_i \to C^t)$. Therefore, it is easy to see that $D^{\mathfrak{M}_M, \tau} = (D^t)^{\mathrm{Pr}_T, *}$, for any $D \in \mathcal{L}_\infty$. Now, if for any $M \vDash T$, we have $M \vDash \Gamma^{\mathfrak{M}_M, \tau} \Rightarrow A^{\mathfrak{M}_M, \tau}$, we reach $M \vDash (\Gamma^t)^{\mathrm{Pr}_T, *} \Rightarrow (A^t)^{\mathrm{Pr}_T, *}$ which implies $T \cup (\Gamma^t)^{\mathrm{Pr}_T, *} \vdash (A^t)^{\mathrm{Pr}_T, *}$. Hence, there is a finite $\Delta \subseteq \Gamma$ such that $T \vdash \bigwedge(\Delta^t)^{\mathrm{Pr}_T, *} \to (A^t)^{\mathrm{Pr}_T, *}$. Note that $\Delta^t \cup \{A^t\} \subseteq \mathcal{L}(Q)$. Hence, by uniform Solovay's theorem, $\mathsf{GL}(Q) \vdash \bigwedge \Delta^t \to A^t$. Finally, by using Lemma 5.2, we reach $\mathsf{K4}_\infty \vdash \bigwedge \Delta \to A$ and hence $\mathsf{K4}_\infty \vdash \Gamma \Rightarrow A$.
For $L = \mathsf{KD4}_\infty$, let $\Pi = \{\neg \Box_n \bot, \Box_{n+1} \neg \Box_n \bot\}_{n \in \mathbb{N}}$. Then, it is easy to see that a provability model $\mathfrak{M}$ satisfies all the elements of Π iff it is consistent. Therefore, if $\{(M, \{\mathrm{Pr}_n\}_{n=0}^\infty) \in \mathbf{Cons} \mid M \vDash T\} \vDash \Gamma^\tau \Rightarrow A^\tau$, we can conclude $\{(M, \{\mathrm{Pr}_n\}_{n=0}^\infty) \in \mathbf{PrM} \mid M \vDash T\} \vDash \Pi^\tau, \Gamma^\tau \Rightarrow A^\tau$. Hence, by the first part we have $\mathsf{K4}_\infty \vdash \Gamma \cup \Pi \Rightarrow A$. Since $\mathsf{KD4}_\infty$ proves all formulas in Π, we finally reach $\mathsf{KD4}_\infty \vdash \Gamma \Rightarrow A$. □

Corollary 5.4 *(Strong Completeness)*

(i) If $\mathbf{PrM} \vDash \Gamma \Rightarrow A$ *then* $\mathsf{K4}_\infty \vdash \Gamma \Rightarrow A$.

(ii) If $\mathbf{Cons} \vDash \Gamma \Rightarrow A$ *then* $\mathsf{KD4}_\infty \vdash \Gamma \Rightarrow A$.

5.2 Logic GL_∞

For the logic GL_∞, the canonical strategy is reducing the completeness of GL_∞ directly to Solovay's result. For that purpose, we need the forgetful translation

$f : \mathcal{L}_\infty \to \mathcal{L}$ that keeps the atomic formulas and the propositional connectives intact and maps $\Box_n$ to $\Box$.

Lemma 5.5 *If* $\mathsf{GL} \vdash A^f$ *then* $\mathsf{GL}_\infty \vdash A$.

Proof. We prove the following, where the part (iii) is our main claim. The other two are required to prove (iii).

(i) For any $A \in \mathcal{L}_\infty$ and any natural numbers $m, n > r(A)$, $\mathsf{GL}_\infty \vdash \Box_m A \leftrightarrow \Box_n A$.

(ii) For any $A, B \in \mathcal{L}_\infty$, if $A^f = B^f$, then $\mathsf{GL}_\infty \vdash A \leftrightarrow B$.

(iii) If $\mathsf{GL} \vdash A^f$ then $\mathsf{GL}_\infty \vdash A$.

For (i), it is enough to show that if $n > r(A)$, we have $\mathsf{GL}_\infty \vdash \Box_n A \leftrightarrow \Box_{n+1} A$. The direction $\Box_n A \to \Box_{n+1} A$ is an instance of the axiom (H). For the other direction, by (L_∞), we have $\mathsf{GL}_\infty \vdash \Box_{n+1}(\Box_n A \to A) \to \Box_n A$. Using (K_∞), it is easy to see that $\mathsf{GL}_\infty \vdash \Box_{n+1} A \to \Box_{n+1}(\Box_n A \to A)$. which implies $\mathsf{GL}_\infty \vdash \Box_{n+1} A \to \Box_n A$.
For (ii), use induction on the structure of A. The atomic and propositional cases are straightforward. For the modal case, assume $A = \Box_m C$ which implies $B^f = A^f = \Box C^f$. Hence, there must be a formula $D \in \mathcal{L}_\infty$ such that $B = \Box_n D$ and $C^f = D^f$. By induction hypothesis, $\mathsf{GL}_\infty \vdash C \leftrightarrow D$. Therefore, for a large enough k, we can use (NC_∞) to prove $\mathsf{GL}_\infty \vdash \Box_k(C \leftrightarrow D)$. Hence, by (K_∞), we have $\mathsf{GL}_\infty \vdash \Box_k C \leftrightarrow \Box_k D$. By (i), we have $\mathsf{GL}_\infty \vdash \Box_k C \leftrightarrow \Box_m C$ and $\mathsf{GL}_\infty \vdash \Box_k D \leftrightarrow \Box_n D$. Hence, $\mathsf{GL}_\infty \vdash \Box_m C \leftrightarrow \Box_n D$ which means $\mathsf{GL}_\infty \vdash A \leftrightarrow B$.
For (iii), first consider the translation $g : \mathcal{L} \to \mathcal{L}_\infty$ as follows: $\bot^g = \bot$, $p^g = p$, for any atomic formula p, $(B \circ C)^g = B^g \circ C^g$, for any $\circ \in \{\wedge, \vee, \to\}$ and $(\Box B)^g = \Box_n B^g$, where $n = r(B^g) + 1$. It is clear that $(B^g)^f = B$. Now, we show that if $\mathsf{GL} \vdash B$, then there exists a formula $B' \in \mathcal{L}_\infty$ such that $\mathsf{GL}_\infty \vdash B'$ and $B'^f = B$. For that purpose, use induction on the length of the proof of B in GL. If B is a classical tautology, set $B' = B^g$. It is easy to see that B^g is also a classical tautology. For the axiom (K), if $B = \Box(C \to D) \to (\Box C \to \Box D)$, then set $B' = \Box_n(C^g \to D^g) \to (\Box_n C^g \to \Box_n D^g)$, where $n = max\{r(C^g), r(D^g)\} + 1$. The proof for the other axioms are similar. For the modus ponens, if $\mathsf{GL} \vdash C$ and $\mathsf{GL} \vdash C \to B$, by induction hypothesis, there are formulas C', C'', and B' such that $C'^f = C$, $C''^f = C$, $B'^f = B$, $\mathsf{GL}_\infty \vdash C'$ and $\mathsf{GL}_\infty \vdash C'' \to B'$. By part (ii), we have $\mathsf{GL}_\infty \vdash C' \leftrightarrow C''$. Hence, $\mathsf{GL}_\infty \vdash B'$. For necessitation, we must have $B = \Box C$ and $\mathsf{GL} \vdash C$. By induction hypothesis, there exists C' such that $\mathsf{GL}_\infty \vdash C'$ and $C'^f = C$. Then, for a large enough n, by (NC_∞), we have $\mathsf{GL}_\infty \vdash \Box_n C'$.
Now, it is easy to prove (iii). If $\mathsf{GL} \vdash A^f$, then, there exists a formula B' such that $\mathsf{GL}_\infty \vdash B'$ and $B'^f = A^f$. Hence, by part (ii), we have $\mathsf{GL}_\infty \vdash A \leftrightarrow B'$. Hence, $\mathsf{GL}_\infty \vdash A$. □

Theorem 5.6 *(Uniform Strong Completeness) For any Σ_1-sound recursively enumerable arithmetical theory $T \supseteq I\Sigma_1$, there is a provability predicate Pr_T and an arithmetical substitution τ such that for any set (not necessarily finite)*

$\Gamma \cup \{A\}$ *of formulas in* $\mathcal{L}_\infty$*, if* $\{(M, \{\mathrm{Pr}_T\}_{n=0}^\infty) \mid M \vDash T\} \vDash \Gamma^\tau \Rightarrow A^\tau$*, then* $\mathsf{GL}_\infty \vdash \Gamma \Rightarrow A$*.*

Proof. First, notice that for any arithmetical substitution σ, any formula $B \in \mathcal{L}_\infty$ and any $M \vDash T$, if we set $\mathfrak{M}_M = (M, \{\mathrm{Pr}_T\}_{n=0}^\infty)$, then $B^{\mathfrak{M}_M, \sigma} = (B^f)^{\mathrm{Pr}_T, \sigma}$ simply because all provability predicates are equal to Pr_T. Therefore, $\{(M, \{\mathrm{Pr}_T\}_{n=0}^\infty) \mid M \vDash T\} \vDash \Gamma^\tau \Rightarrow A^\tau$ implies $M \vDash (\Gamma^f)^{\mathrm{Pr}_T, \tau} \Rightarrow (A^f)^{\mathrm{Pr}_T, \tau}$, for any $M \vDash T$. Hence, $T \cup (\Gamma^f)^{\mathrm{Pr}_T, \tau} \vdash (A^f)^{\mathrm{Pr}_T, \tau}$. Therefore, there is a finite set $\Delta \subseteq \Gamma$ such that $T \vdash \bigwedge (\Delta^f)^{\mathrm{Pr}_T, \tau} \to (A^f)^{\mathrm{Pr}_T, \tau}$. By Theorem 2.2, we have $\mathsf{GL} \vdash \bigwedge \Delta^f \to A^f$. Finally, by Lemma 5.5, we have $\mathsf{GL}_\infty \vdash \Delta \Rightarrow A$ and hence $\mathsf{GL}_\infty \vdash \Gamma \Rightarrow A$. □

Corollary 5.7 *(Strong Completeness) If* $\mathbf{Cst} \vDash \Gamma \Rightarrow A$ *then* $\mathsf{GL}_\infty \vdash \Gamma \Rightarrow A$*.*

6 The Extensions of $\mathsf{KD45}_\infty$

The logic $\mathsf{S5}_\infty$ is too strong to have a provability interpretation. The axioms (T_∞), (4_∞), and (5_∞) together imply that $\Box_n A \leftrightarrow \Box_{n+1}\Box_n A$ and $\neg\Box_n A \leftrightarrow \Box_{n+1}\neg\Box_n A$ which informally state that the provability in T_n is decidable in T_{n+1} and as T_{n+1} is recursively enumerable, we reach the decidability of T_n that is impossible. In this section, we will prove a stronger version that generalizes the result to $\mathsf{KD45}_\infty$.

Theorem 6.1 *There is no provability model for any extension of the logic* $\mathsf{KD45}_\infty$*. Specially,* $\mathsf{S5}_\infty$ *has no provability model.*

Proof. Assume $(M, \{\mathrm{Pr}_n\}_{n=0}^\infty) \vDash \mathsf{KD45}_\infty$. Then, for any arithmetical substitution σ, we have $M \vDash \neg\mathrm{Pr}_n(p^\sigma) \to \mathrm{Pr}_{n+1}(\neg\mathrm{Pr}_n(p^\sigma))$. Pick an arithmetical substitution that maps p to the arithmetical sentence $\mathrm{Pr}_{n+1}(\bot)$. Hence,

$$M \vDash \neg\mathrm{Pr}_n(\mathrm{Pr}_{n+1}(\bot)) \to \mathrm{Pr}_{n+1}(\neg\mathrm{Pr}_n(\mathrm{Pr}_{n+1}(\bot))). \qquad (*)$$

On the other hand, by the formalized Σ_1-completeness and the fact that $I\Sigma_1 \subseteq T_n$, provably in $I\Sigma_1$, we have $I\Sigma_1 \vdash \mathrm{Pr}_{n+1}(\bot) \to \mathrm{Pr}_n(\mathrm{Pr}_{n+1}(\bot))$ and hence $I\Sigma_1 \vdash \neg\mathrm{Pr}_n(\mathrm{Pr}_{n+1}(\bot)) \to \neg\mathrm{Pr}_{n+1}(\bot)$. Thus, $T_{n+1} \vdash \neg\mathrm{Pr}_n(\mathrm{Pr}_{n+1}(\bot)) \to \neg\mathrm{Pr}_{n+1}(\bot)$. Moreover, by Σ_1-completeness, we have $I\Sigma_1 \vdash \mathrm{Pr}_{n+1}(\neg\mathrm{Pr}_n(\mathrm{Pr}_{n+1}(\bot)) \to \neg\mathrm{Pr}_{n+1}(\bot))$. Therefore, $I\Sigma_1 \vdash \mathrm{Pr}_{n+1}(\neg\mathrm{Pr}_n(\mathrm{Pr}_{n+1}(\bot))) \to \mathrm{Pr}_{n+1}(\neg\mathrm{Pr}_{n+1}(\bot))$. And since $M \vDash I\Sigma_1$, we have $M \vDash \mathrm{Pr}_{n+1}(\neg\mathrm{Pr}_n(\mathrm{Pr}_{n+1}(\bot))) \to \mathrm{Pr}_{n+1}(\neg\mathrm{Pr}_{n+1}(\bot))$. Therefore, using $(*)$, we have $M \vDash \neg\mathrm{Pr}_n(\mathrm{Pr}_{n+1}(\bot)) \to \mathrm{Pr}_{n+1}(\neg\mathrm{Pr}_{n+1}(\bot))$. By the formalized Gödel's second incompleteness theorem, we have $I\Sigma_1 \vdash \neg\mathrm{Pr}_{n+1}(\bot) \to \neg\mathrm{Pr}_{n+1}(\neg\mathrm{Pr}_{n+1}(\bot))$. Therefore, $M \vDash \neg\mathrm{Pr}_n(\mathrm{Pr}_{n+1}(\bot)) \to \mathrm{Pr}_{n+1}(\bot)$. However, the provability model $(M, \{\mathrm{Pr}_n\}_{n=0}^\infty)$ is a model of (D_∞). Hence, $M \vDash \neg\mathrm{Pr}_{n+1}(\bot)$. Hence, $M \vDash \mathrm{Pr}_n(\mathrm{Pr}_{n+1}(\bot))$. Since $T_n \subseteq T_{n+2}$, provably in $I\Sigma_1$, we reach $M \vDash \mathrm{Pr}_{n+2}(\mathrm{Pr}_{n+1}(\bot))$. Again, since the provability model is a model for the logic $\mathsf{KD4}_\infty$, it satisfies the formula $\Box_{n+2}\neg\Box_{n+1}\bot$. Hence, $M \vDash \mathrm{Pr}_{n+2}(\neg\mathrm{Pr}_{n+1}(\bot))$. Therefore, $M \vDash \mathrm{Pr}_{n+2}(\bot)$, which contradicts with an instance of the axiom (D_∞). □

It is worth mentioning that the main reason behind this lack of provability models for $\mathsf{KD45}_\infty$ is the assumption that all the theories in a provability model are recursively enumerable. In the well-known polymodal provability logic GLP, this condition is relaxed [13] and hence having the axiom (5_∞) does not make any problem. However, we believe that the restriction we use is philosophically justified, as the theories for such interpretations must be human understandable.

7 Proof of the Reduction Lemma

The main strategy to prove the reduction lemma, Lemma 5.2, is using a cut-free proof for A^t in $\mathsf{GL}(Q)$ to construct a $\mathsf{K4}_\infty$-proof for A. For that purpose, we need to prove a stronger version of necessitation in $\mathsf{K4}_\infty$. Let us start with this task. Let $n \geq 0$ be a given natural number and set the n-truncation as the function over $\mathcal{L}_\infty$ defined as follows: $\bot^n = \bot$, $p^n = p$, for any atomic formula p, $(B \circ C)^n = B^n \circ C^n$, for any $\circ \in \{\wedge, \vee, \to\}$ and $(\Box_i B)^n = \Box_i B$, for $i < n$ and $(\Box_i B)^n = \top$, for $i \geq n$. Moreover, for any sequence of formulas $\{A_i\}_{i=1}^m$, define $(\{A_i\}_{i=1}^m)^n = \{A_i^n\}_{i=1}^m$. Observe that for any formula A, if $r(A) < n$, then $A^n = A$ and $r(B^n) < n$, for any formula $B \in \mathcal{L}_\infty$.

Lemma 7.1 *If π is a $\mathsf{K4}_\infty$-proof for A, then π^n is a $\mathsf{K4}_\infty$-proof for A^n. Specially, if $\mathsf{K4}_\infty \vdash A$, then A has a $\mathsf{K4}_\infty$-proof with rank $r(A)$.*

Proof. We use structural induction on the set of the $\mathsf{K4}_\infty$-proofs. If A is a classical tautology, as the translation commutes with the propositional connectives, A^n would also be a classical tautology and hence there is nothing to prove. If A is an instance of the axiom (H), then $A = \Box_k B \to \Box_{k+1} B$. If $k+1 < n$, then $A^n = A$ and hence there is nothing to prove. If $n \leq k+1$, then $A^n = (\Box_k B)^n \to \top$ which is a classical tautology. If A is an instance of the axiom (K_∞), then $A = \Box_k(B \to C) \to (\Box_k B \to \Box_k C)$. If $k < n$, we have $A^n = A$ and hence, there is nothing to prove. If $k \geq n$, we have $A^n = \top \to (\top \to \top)$ which is a classical tautology. If A is an instance of the axiom (4_∞), then $A = \Box_k B \to \Box_{k+1} \Box_k B$. The case $k+1 < n$ is trivial. If $n \leq k+1$, we have $A^n = (\Box_k B)^n \to \top$ which is again a classical tautology. For the rules, if the last rule is modus ponens, there is nothing to prove as the n-truncation commutes with implication. If it is the necessitation rule, then $\pi = \{A_i\}_{i=1}^{m+1}$, $A_{m+1} = \Box_k B$ and $r(A_i) < k$, for any $i \leq m$. Set $\pi' = \{A_i\}_{i=1}^m$. If $k < n$, then $r(\pi) = k < n$. Hence, π remains intact under the n-truncation and hence there is nothing to prove. If $k \geq n$, then using the induction hypothesis, π'^n is a $\mathsf{K4}_\infty$-proof. As $(\Box_k B)^n = \top$, the sequence π^n is just π'^n with one classical tautology $\top$ added to its end. Therefore, π^n is clearly a $\mathsf{K4}_\infty$-proof.
For the second part, if $\mathsf{K4}_\infty \vdash A$, then there is a $\mathsf{K4}_\infty$-proof π for A. Set $n = r(A) + 1$. Then, as $r(A) < n$, we have $A^n = A$. By the first part, π^n is a proof for $A^n = A$ and $r(\pi^n) < n = r(A) + 1$. $\Box$

Theorem 7.2 *(Strong Necessitation) Let I and J be some finite sets. Then, if $\mathsf{K4}_\infty \vdash \{\Box_{n_i} A_i\}_{i \in I}, \{\Box_n B_j\}_{j \in J} \Rightarrow A$, where $r(A) < n$ and $n_i < n$, for any $i \in I$, then $\mathsf{K4}_\infty \vdash \{\Box_{n_i} A_i\}_{i \in I} \Rightarrow \Box_n A$.*

Proof. Assume π is a $\mathsf{K4}_\infty$-proof for $\bigwedge_{i\in I}\Box_{n_i}A_i\wedge\bigwedge_{j\in J}\Box_n B_j\to A$. Therefore, by Lemma 7.1, π^n is a $\mathsf{K4}_\infty$-proof for $\bigwedge_{i\in I}(\Box_{n_i}A_i)^n\wedge\bigwedge_{j\in J}(\Box_n B_j)^n\to A^n$. As $r(A), n_i<n$, we have $A^n=A$, $(\Box_{n_i}A_i)^n=\Box_{n_i}A_i$ and $(\Box_n B_j)^n=\top$. Hence, π^n is a $\mathsf{K4}_\infty$-proof for $\bigwedge_{i\in I}\Box_{n_i}A_i\wedge\bigwedge_{j\in J}\top\to A$. As $r(\pi^n)<n$, by necessitation, we have $\mathsf{K4}_\infty\vdash\Box_n(\bigwedge_{i\in I}\Box_{n_i}A_i\wedge\bigwedge_{j\in J}\top\to A)$. Hence, by using the axiom (K_∞), we have $\mathsf{K4}_\infty\vdash\bigwedge_{i\in I}\Box_n\Box_{n_i}A_i\wedge\bigwedge_{j\in J}\Box_n\top\to\Box_n A$. Finally, using the axioms (4_∞) and (H) and the fact that $\mathsf{K4}_\infty\vdash\Box_n\top$, we have $\mathsf{K4}_\infty\vdash\bigwedge_{i\in I}\Box_{n_i}A_i\to\Box_n A$. $\Box$

Now, as we have proved the strong necessitation, we are ready to prove the reduction lemma. Define the set $X\subseteq\mathcal{L}(Q)$ as the least set of modal formulas containing $\bot$ and all the atomic formulas (including the atoms in Q) and closed under the propositional connectives and the following rule: If $A\in X$, then:

- $\Box(\bigwedge_{i=0}^n q_i\to A)\in X$, if n is greater than all the indices of q_j's occurring in A.
- $\Box(\bigwedge_{i=0}^m q_i\wedge\bigwedge_{i=m+1}^n\bot\to A)\in X$, if m is greater than or *equal to* all the indices of q_j's occurring in A and $m<n$.

Set X_0 and X_1 as the sets of all formulas in forms $\Box(\bigwedge_{i=0}^n q_i\to A)$ and $\Box(\bigwedge_{i=0}^m q_i\wedge\bigwedge_{i=m+1}^n\bot\to A)$ in X, respectively. Note that these are the only formulas in the form $\Box B$ in X. It is easy to check that X includes all sub-formulas of formulas D^t, for any $D\in\mathcal{L}_\infty$. The proof is by induction on the structure of D. The only non-trivial case is $D=\Box_n E$. In this case, $D^t=\Box(\bigwedge_{i=0}^n q_i\to E^t)$ in which n is greater than all indices of q_j's occurring in E. Hence, $D^t\in X_0$. It is easy to see that all proper subformulas of D^t also belong to X.

Here are some terminology. An X-proof is a cut-free proof in the system $\mathbf{GGL}(Q)$ consisting only of the formulas in X. By the rank of a formula $A\in X$, denoted by $r(A)$, we mean the greatest number n such that q_n occurs in the formula A. If there is none, set $r(A)=-1$. For any multiset $\Gamma\subseteq X$, by $r(\Gamma)$, we mean the maximum of the ranks of the elements of Γ. An X-proof is called nice, if $r(\Box\Gamma)\leq r(\Box A)$, for any occurrence of the rule

$$\frac{\Gamma,\Box\Gamma,\Box A\Rightarrow A}{\Box\Gamma\Rightarrow\Box A}\,GL$$

in the proof. Note that the equality $r(\Box\Gamma)=r(\Box A)$ is also allowed.

Let n be a natural number and σ_n be the substitution that maps q_i to $\bot$, for $i>n$ and keeps the other atomic formulas intact. It is easy to see that $r(\sigma_n(A))\leq n$, for any formula $A\in X$. Moreover, for a boxed formula $A\in X$, if $r(A)>n$, we have $r(\sigma_n(A))=n$. The latter is easy to prove by checking the two forms of the modal formulas in X.

Denoting the result of applying σ_n on a sequence π by $\sigma_n(\pi)$, we have:

Lemma 7.3 *The sets X and X_1 are closed under σ_n and if $r(A)>n$ and $\Box A\in X_0$, then $\sigma_n(\Box A)\in X_1$. Moreover, if π is a nice X-proof, then so is $\sigma_n(\pi)$.*

Proof. For the first part, to show the closure of X under σ_n, use a simple structural induction. We only explain the box case. If $A\in X_0$, we have

$A = \Box(\bigwedge_{i=0}^{m} q_i \to B)$. If $n \geq m$, then $\sigma_n(A) = A$, as there is no q_i in A such that $i > n$. Therefore, $\sigma_n(A) \in X$. If $n < m$, then $\sigma_n(A) = \Box(\bigwedge_{i=0}^{n} q_i \wedge \bigwedge_{i=n+1}^{m} \bot \to \sigma_n(B))$. By induction hypothesis, $\sigma_n(B) \in X$. As $n \geq r(\sigma_n(B))$, we have $\sigma_n(A) \in X_1$. If $A \in X_1$, we have $A = \Box(\bigwedge_{i=0}^{m} q_i \wedge \bigwedge_{i=m+1}^{k} \bot \to B)$, where $m < k$. If $n \geq m$, we have $\sigma_n(A) = A$, as there is no q_i in B such that $i > n$. Therefore, $\sigma_n(A) \in X$. If $n < m$, then $\sigma_n(A) = \Box(\bigwedge_{i=0}^{n} q_i \wedge \bigwedge_{i=n+1}^{k} \bot \to \sigma_n(B))$. By induction hypothesis, we have $\sigma_n(B) \in X$ which implies $\sigma_n(A) \in X_1$, as $n \geq r(\sigma_n(B))$. Note that our argument shows that $\sigma_n(A) \in X_1$, for $A \in X_1$ and if $r(A) > n$ and $\Box A \in X_0$, then $\sigma_n(\Box A) \in X_1$.
For the second part, note that if π is an X-proof then as proofs are closed under substitutions, by the first part, we know that $\sigma_n(\pi)$ is also an X-proof. For niceness, since π is a nice X-proof, $r(\Box\Gamma) \leq r(\Box A)$, for any occurrence of the rule (GL) in π:

$$\frac{\Gamma, \Box\Gamma, \Box A \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\, GL$$

If $r(\Box A) \leq n$, we also have $r(\Box\Gamma) \leq n$ and hence $\Box\Gamma$ and $\Box A$ have no q_j with $j > n$. Therefore, $\Box\Gamma$, Γ, $\Box A$ and A remain intact and hence there is nothing to prove. If $r(\Box A) > n$, then as $r(\sigma_n(\Box A)) = n$, we have to show that $r(\sigma_n(\Box\Gamma)) \leq n$ which is trivial. □

Lemma 7.4 *If $\Gamma \Rightarrow \Delta$ has an X-proof, then there exists $\Box\Sigma \subseteq X_1$ such that $\Box\Sigma, \Gamma \Rightarrow \Delta$ has a nice X-proof.*

Proof. We use induction on the length of the X-proof of $\Gamma \Rightarrow \Delta$. If the last rule is an axiom, a structural rule or a propositional rule, then the claim is obvious from the induction hypothesis. For the modal rule (GL), we know that $\Box\Gamma \Rightarrow \Box A$ is proved by $\Gamma, \Box\Gamma, \Box A \Rightarrow A$. By the induction hypothesis, there exists $\Box\Sigma \subseteq X_1$ such that $\Box\Sigma, \Gamma, \Box\Gamma, \Box A \Rightarrow A$ has a nice X-proof. Call it π. Note that since $\Box\Sigma \subseteq X_1$, we also have $\Sigma \subseteq X$. Set $r(A) = n$. Divide Γ into two parts, Γ_0 and Γ_1 in a way that $r(\Gamma_0) \leq n$ and $r(\gamma) > n$, for any $\gamma \in \Gamma_1$. Notice that σ_n does not change Γ_0 and A, as their ranks are bounded by n. By Lemma 7.3, we know that $\sigma_n(\pi)$ is a nice X-proof for $\Box\sigma_n(\Sigma), \Gamma_0, \Box\Gamma_0, \sigma_n(\Gamma_1), \Box\sigma_n(\Gamma_1), \Box A \Rightarrow A$. Set $\Sigma' = \sigma_n(\Sigma) \cup \sigma_n(\Gamma_1)$ and note that $\Sigma' \subseteq X$, as X is closed under σ_n, by Lemma 7.3. Hence, by left weakening to add Σ', we have a nice X-proof for $\Sigma', \Box\Sigma', \Gamma_0, \Box\Gamma_0, \Box A \Rightarrow A$. Now, use (GL) to prove $\Box\Sigma', \Box\Gamma_0 \Rightarrow \Box A$. Again, by the left weakening for $\Box\Gamma_1$, we have $\Box\Sigma', \Box\Gamma_0, \Box\Gamma_1 \Rightarrow \Box A$. Therefore, we have provided a proof for $\Box\Sigma', \Box\Gamma \Rightarrow \Box A$. It is clear that this proof is an X-proof. To show that it is nice, notice that the use of (GL) is allowed, as $r(\Box\Gamma_0) \leq n$, by definition, and $r(\Box\Sigma') \leq n$, as $\Box\Sigma' = \sigma_n(\Box\Sigma \cup \Box\Gamma_1)$. Finally, we must show $\Box\Sigma' \subseteq X_1$. First, note that as $\Box\Sigma \subseteq X_1$ and X_1 is closed under σ_n, we have $\sigma_n(\Box\Sigma) \subseteq X_1$. On the other hand, for any formula $\gamma \in \Gamma_1$, if $\Box\gamma \in X_1$, then $\sigma_n(\Box\gamma) \in X_1$ by the closure of X_1 under σ_n and if $\Box\gamma \in X_0$, then as $r(\gamma) > n$, we have $\sigma_n(\Box\gamma) \in X_1$, by Lemma 7.3. □

Define the translation function $s : X \to \mathcal{L}_\infty$ as follows: $\bot^s = \bot$, $p^s = p$ and

$q_i^s = \top$, for any $i \geq 0$, $(B \circ C)^s = B^s \circ C^s$, for any $\circ \in \{\wedge, \vee, \rightarrow\}$ and if $A = \Box(\bigwedge_{i=0}^{n} q_i \rightarrow B)$ then $A^s = \Box_n B^s$ and if $A = \Box(\bigwedge_{i=0}^{m} q_i \wedge \bigwedge_{i=m+1}^{n} \bot \rightarrow B)$ then $A^s = \top$. Here are some basic properties of the translation s. First of all, s is well-defined, meaning that $A^s \in \mathcal{L}_\infty$, for any $A \in X$. To prove, we show the stronger claim that if $A \in X$, then $A^s \in \mathcal{L}_\infty$ and $r(A^s) \leq r(A)$. The proof is by structural induction on X. The only non-trivial case is when $A \in X_0$. If $A = \Box(\bigwedge_{i=0}^{n} q_i \rightarrow B)$, then $n > r(B)$, by definition. By induction hypothesis, $B^s \in \mathcal{L}_\infty$ and $r(B^s) \leq r(B)$. Therefore, $\Box_n B^s \in \mathcal{L}_\infty$ and $r(A^s) = r(\Box_n B^s) = n = r(A)$. Secondly, if $A = \Box B \in X_1$, then B^s is provably equivalent to $\top$ in $\mathsf{K4}_\infty$, because $B = \bigwedge_{i=0}^{m} q_i \wedge \bigwedge_{i=m+1}^{n} \bot \rightarrow C$ has a $\bot$ in its premises, which means that B^s is equivalent to $\top$. Thirdly, note that s is a left inverse for the translation t, meaning that for any $A \in \mathcal{L}_\infty$, we have $(A^t)^s = A$.

Lemma 7.5 *If* $\Gamma \Rightarrow \Delta$ *has a nice* X*-proof, then* $\mathsf{K4}_\infty \vdash \bigwedge \Gamma^s \rightarrow \bigvee \Delta^s$.

Proof. The proof is based on an induction on the length of the nice X-proof. If the last rule is an axiom, a structural rule or a propositional rule, then the claim follows from the induction hypothesis. The reason is that s commutes with the propositional connectives and $\mathsf{K4}_\infty$ proves all propositional tautologies. For the modal rule (GL), if $\Box\Gamma \Rightarrow \Box A$ is proved by $\Box\Gamma, \Gamma, \Box A \Rightarrow A$, by induction hypothesis, we have $\mathsf{K4}_\infty \vdash (\Box\Gamma)^s, \Gamma^s, (\Box A)^s \Rightarrow A^s$. We want to show $\mathsf{K4}_\infty \vdash (\Box\Gamma)^s \Rightarrow (\Box A)^s$. For that purpose, we must investigate the form of $(\Box A)^s$ and the elements of $(\Box\Gamma)^s$. If $\Box A \in X_1$, then by definition $(\Box A)^s = \top$ and hence there is nothing to prove. Therefore, assume $\Box A \in X_0$. Hence, A has the form $A = \bigwedge_{i=0}^{m} q_i \rightarrow B$ and $r(B) < r(A) = m$. As $r(B^s) \leq r(B)$, by Lemma 7.1, the formula $A^s = \bigwedge_{i=0}^{m} \top \rightarrow B^s$ is equivalent to B^s, by a $\mathsf{K4}_\infty$-proof with rank $r(B)$. Therefore, by (NC_∞) and (K_∞) and using the fact that $r(B) < m$, it is easy to see that $(\Box A)^s$ is equivalent to $\Box_m B^s$, provably in $\mathsf{K4}_\infty$. On the other hand, for any $\Box\gamma \in \Box\Gamma \subseteq X$, if $\Box\gamma \in X_1$, then γ^s is equivalent to $\top$, provably in $\mathsf{K4}_\infty$. Moreover, $(\Box\gamma)^s = \top$, by definition. Therefore, we can ignore this kind of boxed formulas in $\Box\Gamma$ and w.l.o.g., assume $\Box\Gamma \subseteq X_0$. Therefore, γ has the form $\gamma = \bigwedge_{i=0}^{n_\gamma} q_i \rightarrow \beta_\gamma$ and hence $r(\gamma) = n_\gamma > r(\beta_\gamma)$. Therefore, $\gamma^s = \bigwedge_{i=0}^{n_\gamma} \top \rightarrow \beta_\gamma^s$ and as $r(\beta_\gamma^s) \leq r(\beta_\gamma)$, by Lemma 7.1, the formulas γ^s and β_γ^s are equivalent with a $\mathsf{K4}_\infty$-proof with rank $r(\beta_\gamma)$ and hence as before, since $r(\beta_\gamma) < n_\gamma$, we know that $(\Box\gamma)^s$ is equivalent to $\Box_{n_\gamma}\beta_\gamma^s$, provably in $\mathsf{K4}_\infty$. Now, the result of the induction hypothesis, i.e., $\mathsf{K4}_\infty \vdash (\Box\Gamma)^s, \Gamma^s, (\Box A)^s \Rightarrow A^s$ implies $\mathsf{K4}_\infty \vdash \{\Box_{n_\gamma}\beta_\gamma^s, \beta_\gamma^s\}_{\gamma\in\Gamma}, \Box_m B^s \Rightarrow B^s$. As the X-proof is nice, we have $m = r(A) = r(\Box A) \geq n_\gamma = r(\gamma) = r(\Box\gamma)$. Split Γ into Γ_1 consisting of $\gamma \in \Gamma$ such that $n_\gamma = m$ and $\Gamma_2 = \Gamma - \Gamma_1$. Hence, $\mathsf{K4}_\infty \vdash \{\Box_m \beta_\gamma^s\}_{\gamma\in\Gamma_1}, \Box_m B^s, \{\Box_{n_\gamma}\beta_\gamma^s\}_{\gamma\in\Gamma_2} \Rightarrow \bigwedge_{\gamma\in\Gamma} \beta_\gamma^s \rightarrow B^s$. As $r(\beta_\gamma) < r(\gamma) \leq m$ and $r(B) < r(A) = m$, we have $r(\beta_\gamma^s) \leq r(\beta_\gamma) < m$ and $r(B^s) \leq r(B) < m$. By strong necessitation, Theorem 7.2, we reach $\mathsf{K4}_\infty \vdash \{\Box_{n_\gamma}\beta_\gamma^s\}_{\gamma\in\Gamma_2} \Rightarrow \Box_m(\bigwedge_{\gamma\in\Gamma} \beta_\gamma^s \rightarrow B^s)$. By a simple application of (K_∞), we will have $\mathsf{K4}_\infty \vdash \{\Box_{n_\gamma}\beta_\gamma^s\}_{\gamma\in\Gamma_2}, \{\Box_m \beta_\gamma^s\}_{\gamma\in\Gamma} \Rightarrow \Box_m B^s$. As $n_\gamma \leq m$, for any $\gamma \in \Gamma$, by (H), $\mathsf{K4}_\infty \vdash \{\Box_{n_\gamma}\beta_\gamma^s\}_{\gamma\in\Gamma} \Rightarrow \Box_m B^s$ which completes the proof. □

Proof. [of Lemma 5.2] If $\mathsf{GL}(Q) \vdash A^t$, there is a cut-free proof of $(\Rightarrow A^t)$ in

$\mathbf{GGL}(Q)$. Therefore, every formula in the proof is a subformula of A^t and hence it is in X. Hence, $(\Rightarrow A^t)$ has an X-proof. By Lemma 7.4, there is a set $\Box\Sigma \subseteq X_1$ such that $\Box\Sigma \Rightarrow A^t$ has a nice X-proof. Then, by Lemma 7.5, $\mathsf{K4}_\infty \vdash \bigwedge(\Box\Sigma)^s \Rightarrow (A^t)^s$. We know that $(A^t)^s = A$. Since $\Box\Sigma \subseteq X_1$, we have $(\Box\sigma)^s = \top$, for any $\sigma \in \Sigma$. Therefore, $\mathsf{K4}_\infty \vdash A$. $\Box$

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