On parametric phenomena in correspondence theory

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Abstract

I survey a number of recent results which study the syntactic shape of modal formulas' first-order frame correspondents on different relation semantics and establish systematic links between the correspondents of given modal formula across these different semantic environments. It has been observed that, for instance, in some cases, correspondents remains syntactically unchanged under such semantic shifts; in other cases, correspondents across different semantics can be recovered as parametric instances of the same pattern, with the parameter capturing the embedding between semantic contexts. I discuss some open problems and further directions.

Keywords: Correspondence theory, modal logic, many-valued modal logic, intuitionistic modal logic, non-distributive logics.

Unified correspondence. In recent years, the research programme of unified correspondence (see, e.g., [9] [14]) has developed Sahlqvist correspondence and canonicity results for large families of non-classical logics. In [10] we formulate a general definition of the classes of Sahlqvist and inductive formulas which applies uniformly to any logic algebraically captured by a class of lattice expansions (LE-logics). This definition is based solely on the order-theoretic properties of the algebraic interpretation of the connectives and, when projected onto specific LE-logics, captures or generalises many existing definitions in the literature. A further feature of unified correspondence, is a division of labour which separates the computation of a first-order correspondent for a modal axiom into two phases: a correspondent is first computed and expressed as a conjunction of quasi-inequalities in a extended modal language interpreted on perfect lattice expansions. This is independent of any particular choice of relational semantics for the logic. The second phase of the computation translates this algebraic correspondent by applying the standard translation associated with a particular choice of relational semantics linked with the algebraic semantics via a suitable duality.

Parametric correspondence. This and other recent work in correspondence theory, has focused mostly on the (modal) propositional side of the correspon-

dence theoretic equation, i.e., on identifying classes of (modal) propositional formulas for which first-order correspondents exists. However, beyond the fact of their existence, the properties of the first-order correspondents themselves have received little attention. This situation has changed recently, with a number of results studying the syntactic shape of first-order correspondents on particular relation semantics and, more specifically, comparing the shape of the first-order correspondent of given modal formula across different relational semantics. It has been observed, for instance, that, in some cases, correspondents remains syntactically unchanged under such semantic shifts; in other cases, correspondents across different semantics can be recovered as parametric instances of the same pattern, with the parameter capturing the embedding between semantic contexts. This talk will survey a range of phenomena and results of this type, some of which I discuss below.

Many-valued modal logic. The first result of this type was proven in the context of the correspondence theory of many-valued modal logic (see, e.g., [12,3]). Specifically, considering many-valued modal logics with complete Heyting algebras as truth value spaces, [4] shows that the first-order correspondent of any modal Sahlqvist formula over many-valued Kripke frames is syntactically identical to its first-order correspondent on (crisp) Kripke frames. Of course, the many-valued correspondent should be read as a formula of many-valued predicate logic and therefore its semantic meaning generalizes that of the classical correspondent. Apart from purely theoretical interest, this result has great practical utility as it means that the correspondents of well-known modal axioms can be directly transferred to the many-valued versions of the logics they axiomatize.

Non-distributive modal logics with polarity-based semantics. The next important instance of parametric correspondence which I would like to discuss comes from the study of non-distributive modal logics [11]. These logics can be given semantics based on polarities, or formal contexts, in the sense of formal concept analysis (FCA) [13] and given an epistemic interpretation [8,7] where formulas denote formal categories. In [8,7] it is observed that, over this semantics, the classical S5 axioms of epistemic logic have first-order correspondents which are, in a sense, dual to their well-known correspondents on Kripke frames. Moreover, it is argued that the epistemic meanings of these axioms are preserved in a modified form in this new context. This observation is extensively developed in [6] where the first-order correspondents of modal reduction principles [18] are captured as inclusions of relational compositions, both in the setting of Kripke frames and in that of polarity based frames. It is established that one my pass from the Kripke frame correspondent, thus expressed, to the correspondent on polarity-based frames, roughly speaking, by reversing the inclusion and parameterising the relational compositions with the incidence relation of the polarity. Mathematically, this is underpinned by the embedding of the class of Kripke frames into the class of enriched polarities.

Non-distributive modal logics with graph-based semantics. Alternatively, non-distributive logics may be given graph-based semantics [11,5] based on Ploščica's [15] representation of general lattices. Here too, the correspondents of modal reduction principles may be obtained by appropriately transforming their correspondents on Kripke frames.

Intuitionistic modal logic. In the case of intuitionistic modal logic with semantics based on [19], the correspondents of inductive (and hence also Sahlqvist) formulas can be transparently obtained from the classical correspondents. This is mainly due to the strong interaction conditions between the pre-order of the intuitionistic Kripke frame and the additional accessibility relation used to interpret modalities. In the case of Fisher-Servi's semantics for intuitionistic modal logic [16,17], the interaction conditions between the relations are weaker, and the relationship between correspondence in the classical case and in this setting is somewhat more involved.

Relativized correspondence. When imposing restrictions on the class of Kripke frames considered, more modal formulas become first order definable, in a phenomenon known as relativized correspondence. For instance, all modal reduction principles are first-order definable over transitive frames [18], while all modal formulas have first-order correspondents over S5 frames [2] and over Euclidean frames [1]. Relativized correspondence has been little studied for modal logics on non-classical propositional base or for many-valued modal logics, but some initial results indicates that an interesting landscape waits to be discovered here. I will conclude by sketching some of these results and directions.

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