Goldblatt-Thomason-style Characterization for Intuitionistic Inquisitive Logic

Katsuhiko Sano [1]

Faculty of Humanities and Human Sciences, Hokkaido University
Nishi 7 Chome, Kita 10 Jo, Kita-ku
Sapporo, Hokkaido, 060-0810, Japan

Abstract

The purpose of this paper is to investigate a possible characterization of frame definability of intuitionistic inquisitive logic by Ciardelli et al. (2020) in terms of frame constructions such as generated subframes and bounded morphic images. Sano and Virtema (2015, 2019) provided a Goldblatt-Thomason-style characterization for (extended) modal dependence logic with the help of a normal form result for the logic. A key ingredient of establishing the characterization was to show that the ordinary modal logic expanded with positive occurrences of the universal modality and extended modal dependence logic have the same definability over Kripke models. This paper first reviews Goldblatt-Thomason-style characterization for intuitionistic logic from Rodenburg (1986)'s work on intuitionistic correspondence theory. Then we employ a similar strategy to Sano and Virtema (2015, 2019) and provide a Goldblatt-Thomason-style characterization for intuitionistic inquisitive logic.

Keywords: Intuitionistic Logic, Frame Definability, Universal Modality, Goldblatt-Thomason Theorem, Inquisitive Semantics, Inquisitive Logic

1 Introduction

Goldblatt-Thomason Theorem [11] for modal logic enables us to characterize elementary frame class definability in terms of frame construction. To be more specific, it states that an elementary (or first-order definable) frame class $\mathbb{F}$ is definable by a set of modal formulas iff $\mathbb{F}$ is closed under taking bounded morphic images, generated subframes, disjoint unions and $\mathbb{F}$ reflects ultrafilter

[1] `v-sano@let.hokudai.ac.jp`. I would like to thank three anonymous referees for their careful reading of the submission and helpful comments. This paper is based on my talk at Inquisitive Logic 2018 workshop, held at Amsterdam. In particular, I would like to thank Giuseppe Greco, Fan Yang, Vít Punčochář, Wesley Holliday, and Ivano Ciardelli for discussions at the workshop. I also wish to thank Jonni Virtema for our discussion at Sapporo in the spring of 2020. The work of this paper was partially supported by JSPS KAKENHI Grant-in-Aid for Scientific Research (C) Grant Number 19K12113, JSPS KAKENHI Grant-in-Aid for Scientific Research (B) Grant Number 17H02258, and JSPS Core-to-Core Program (A. Advanced Research Networks).

extensions (i.e., the complement of $\mathbb{F}$ is closed under taking ultrafilter extensions). Since then, Goldblatt-Thomason-style (GT-style, for short) characterization has been provided for a rich variety of logics: modal logic with the universal modality [9], hybrid logics [22], graded modal logic [17], modal logic over topological semantics [23], coalgebraic modal logic [12], intuitionistic logic [16], modal dependence logic [19], etc. Let us comment on intuitionistic logic. If we replace the reflection of ultrafilter extensions with the reflection of *prime filter extensions* in frame constructions for Goldblatt-Thomason Theorem for modal logic, we can obtain Rodenburg [16]'s characterization of intuitionistic elementary frame definability.

Inquisitive logic [3,2] (or inquisitive semantics [4]) is a recent theoretical framework for studying both declarative and interrogative sentences in one setting. It often assumes classical logic as a background logic. Then, on the top of classical logic, we add the *inquisitive disjunction* $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, which allows us to formalize a question "Does Taro play tennis?" as $?p := p \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg p$, where p denotes the declarative sentence "Taro plays tennis". Semantically, a formula is evaluated not by a single state but by a set of states (which is called a *team*). This semantic feature is also a core of (propositional) *dependence logic* (cf. [26]), where we can study the notion of *functional dependence* $\mathrm{dep}(q; p)$, "q truth-functionally determines p". In this sense, the semantics for dependence logic is called *team* semantics. Moreover, the recent interaction between the two communities reveal, e.g., that functional dependency $\mathrm{dep}(q; p)$ can be understood as an implication from the question $?q$ to the question $?p$ (see [2] for more detail).

Recently, the ideas of inquisitive logic and dependence logic are generalized also to non-classical logics, i.e., modal logic [24,8,7], (dynamic) epistemic logic [6], intuitionistic logic [13,14,5], substructural logic [15], etc. For modal dependence logic (modal logic extended with atoms for functional dependency), [19] provided a Goldblatt-Thomason-style characterization. A key ingredient for the characterization is that (extended) modal dependence logic (with team semantics) and modal logic expanded with the positive occurrences of the universal modality (with Kripke semantics) have the same definability for frame classes.

While modal dependence logic still assumes classical logic, intuitionistic inquisitive logic [5]'s background logic is intuitionistic. We add the inquisitive disjunction to the syntax of intuitionistic logic, and "lift" the ordinary state-based Kripke semantics for intuitionistic logic to the semantics based on teams (sets of states). Then, we can study questions and dependency also in the intuitionistic setting. As for frame definability, [5] raises the following research question ("[24]" and "[25]" in the citation correspond to [20] and [18] respectively):

> [...] it would also be interesting to look at the issue of frame definability in InqI. [...] Clearly, if a standard formula defines a certain frame class in IPL, then this formula still defines the same class in InqI. At the same time, however, some frame classes which cannot be characterized in IPL can

> now be characterized with the help of inquisitive formulas: for instance, $?p$ characterizes the class of singleton frames. Recent work on frame definability in the context of modal dependence logic (see, e.g., Sano and Virtema [24], [25]) might provide a handle on this question. [5, p.110]

This paper tackles this question and provide a GT-style characterization for intuitionistic inquisitive logic. For this purpose, we follow a similar strategy to [19]. That is, we first study intuitionistic logic with the universal modality A, which was, as far as the author knows, less studied in the literature (e.g., [21] studies the axiomatization of bi-intuitionistic tense logic expanded with the universal modality). Then, we provide GT-style characterizations for a special fragment of intuitionistic logic with the universal modality, which in turn gives us our intended GT-style characterization for intuitionistic inquisitive logic. An important insight is: we can mimic the behavior of inquisitive disjunction $\varphi \mathbin{\vee\!\!\!\vee} \psi$ by A-prefixed disjunction $\mathsf{A}\,\varphi \vee \mathsf{A}\,\psi$ where φ and ψ are intuitionistic formulas.

Our proof of Goldblatt-Thomason-type characterization is based on van Benthem's model-theoretic argument [25], though the original proof by Rodenburg [16] is based on the representation theorem of Heyting algebras. When we try to transfer the idea of Golcblatt-Thomason Theorems for modal dependence logic [19] to our current study, there is a tricky point on the negation. While we need to handle the intuitionistic negation for frame definability, we also need to deal with the classical negation when we use the standard translation to apply the first-order model theory. The results of this paper show that this tricky distinction can be overcome in applying van Benthem's model-theoretic argument [25].

We proceed as follows. Section 2 introduces Kripke semantics for the syntax of intuitionistic logic (the set of formulas is denoted by Form) and four frame constructions, and then reviews Rodenburg's Goldblatt-Thomason Theorem for intuitionistic logic. Section 3 adds the universal modality A to the syntax of intuitionistic logic (the resulting set of formulas is denoted by $\mathsf{Form}(\mathsf{A})$) and introduce the syntactic notion of disjunctive A-clauses, i.e., a formula of the form $\bigvee_{i \in I} \mathsf{A}\,\varphi_i$, where I is finite and φ_is does not contain any occurrences of A, i.e., an intuitionistic formula. We use $\bigvee \mathsf{A}\,\mathsf{Form}$ to denote the set of all disjunctive A-clauses. Section 4 provides two Goldblatt-Thomason-type characterizations of elementary frame definability in terms of $\mathsf{Form}(\mathsf{A})$ and $\bigvee \mathsf{A}\,\mathsf{Form}$ (Theorems 4.2 and 4.3, respectively). Section 5 introduces the *inquisitive disjunction* $\mathbin{\vee\!\!\!\vee}$ to Form (where the resulting set of formulas is denoted by $\mathsf{Form}(\mathbin{\vee\!\!\!\vee})$) and team semantics for it, and then establishes that $\mathsf{Form}(\mathbin{\vee\!\!\!\vee})$ and $\bigvee \mathsf{A}\,\mathsf{Form}$ have the same frame definability. This equi-definability result enables us to provide Goldblatt-Thomason-type Theorem for intuitionistic inquisitive logic (Theorem 5.12). Section 6 explains several directions of further research.

2 Goldblatt-Thomason Theorem for Intuitionistic Logic

2.1 Syntax and Kripke Semantics for Intuitionistic Logic

Let Prop be a set of propositional variables (we mostly assume that Prop is countably infinite). The set Form of all formulas for intuitionistic logic is defined inductively as follows:

$$\mathsf{Form} \ni \varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \quad (p \in \mathsf{Prop}).$$

The negation is defined as $\neg\varphi := \varphi \to \bot$.

We move on to Kripke semantics. We say that $\mathfrak{F} = (W, R)$ is a *Kripke frame* (or simply *frame*) if W is a non-empty set of states and $R \subseteq W \times W$ is reflexive and transitive, i.e., (W, R) is a preorder or a quasi-order. We say that $\mathfrak{M} = (W, R, V)$ is a *Kripke model* (or simply *model*) if (W, R) is a frame and $V : \mathsf{Prop} \to \mathcal{P}(W)$ is a *valuation function* (or simply *valuation*) such that every $V(p)$ is *persistent* (or $V(p)$ is an *upset*) in the following sense: if $w \in V(p)$ and wRv then $v \in V(p)$, for all states $w, v \in W$. For a frame $\mathfrak{F}$ and a model $\mathfrak{M}$ we use $|\mathfrak{F}|$ and $|\mathfrak{M}|$ to mean the underlying domain.

Definition 2.1 Given a model $\mathfrak{M} = (W, R, V)$, a state $w \in W$ and a formula φ, the *satisfaction relation* $\mathfrak{M}, w \Vdash \varphi$ is defined inductively as follows:

$$\begin{array}{lll}
\mathfrak{M}, w \Vdash p & \text{iff} & w \in V(p), \\
\mathfrak{M}, w \nVdash \bot, & & \\
\mathfrak{M}, w \Vdash \varphi \wedge \psi & \text{iff} & \mathfrak{M}, w \Vdash \varphi \text{ and } \mathfrak{M}, w \Vdash \psi, \\
\mathfrak{M}, w \Vdash \varphi \vee \psi & \text{iff} & \mathfrak{M}, w \Vdash \varphi \text{ or } \mathfrak{M}, w \Vdash \psi, \\
\mathfrak{M}, w \Vdash \varphi \to \psi & \text{iff} & \forall\, v\, ((wRv \text{ and } \mathfrak{M}, v \Vdash \varphi) \text{ imply } \mathfrak{M}, v \Vdash \psi).
\end{array}$$

The truth set $[\![\varphi]\!]_{\mathfrak{M}}$ is defined as $\{\, w \in W \mid \mathfrak{M}, w \Vdash \varphi \,\}$. For a set Δ of formulas, we write $\mathfrak{M}, w \Vdash \Delta$ to mean $\mathfrak{M}, w \Vdash \varphi$ for all $\varphi \in \Delta$.

For the negation, we have the following satisfaction clause:

$$\mathfrak{M}, w \Vdash \neg\varphi \text{ iff } \forall\, v\, (wRv \text{ implies } \mathfrak{M}, v \nVdash \varphi).$$

Definition 2.2 Let $\mathfrak{F} = (W, R)$ be a frame and $X \subseteq W$. We define the upward closure $\uparrow X$ of X as the set $\{\, v \in W \mid \exists\, w \in X\, (wRv) \,\}$. We usually write $\uparrow w$ instead of $\uparrow \{\, w \,\}$ for $w \in W$. Given upsets X, $Y \subseteq W$, we define $X \Rightarrow Y := \{\, w \in W \mid \uparrow w \cap X \subseteq Y \,\}$.

For a model $\mathfrak{M}$, it is noted that $[\![\varphi \to \psi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}} \Rightarrow [\![\psi]\!]_{\mathfrak{M}}$. Given a frame (W, R), it is remarked that X is an upset iff $\uparrow X = X$.

By induction on a formula, we can show that the persistency can be extended from propositional variables to formulas.

Proposition 2.3 *The set* $[\![\varphi]\!]_{\mathfrak{M}}$ *is an upset for all formulas* φ.

Definition 2.4 A formula φ is *valid* in a model $\mathfrak{M}$ (notation: $\mathfrak{M} \Vdash \varphi$) if $\mathfrak{M}, w \Vdash \varphi$ for all states $w \in W$, or equivalently, $[\![\varphi]\!]_{\mathfrak{M}} = W$. A set Γ of formulas is *valid* in a frame $\mathfrak{F} = (W, R)$ (notation: $\mathfrak{F} \Vdash \Gamma$) if, for every valuation V, $(\mathfrak{F}, V) \Vdash \varphi$ holds for all formulas $\varphi \in \Gamma$. When Γ is a singleton $\{\, \varphi \,\}$, we simply write $\mathfrak{F} \Vdash \varphi$ to mean $\mathfrak{F} \Vdash \{\varphi\}$. A set Γ of formulas *defines* a class $\mathbb{F}$ of frames if the following equivalence holds: $\mathfrak{F} \Vdash \Gamma$ iff $\mathfrak{F} \in \mathbb{F}$, for all frames $\mathfrak{F}$.

The following table demonstrates frame definability taken from [16].

Formula	Property of R
$p \vee \neg p$	$\forall w, v\,(wRv$ implies $vRw)$
$(p \to q) \vee (q \to p)$	$\forall w, v, u\,((wRv$ and $wRu)$ imply $(vRu$ or $uRv))$
$\neg p \vee \neg\neg p$	$\forall w, v, u\,((wRv$ and $wRu)$ imply $\exists z(vRz$ and $uRz))$

Definition 2.5 Let $\mathcal{L}^1_f$ be the first-order frame language (with equality) which has a binary predicate $x \leqslant y$ (corresponding to a relation R of a Kripke frame (W, R)). Let $\mathcal{L}^1_m$ be the first-order model language which expands $\mathcal{L}^1_f$ with a set $\{\, p(x) \mid p \in \mathsf{Prop} \,\}$ of unary predicates corresponding to Prop. Given any first-order variable x, we define the *standard translation* ST_x from Form to the set of first-order formulas in $\mathcal{L}^1_m$ as follows:

$$\begin{aligned}
\mathrm{ST}_x(p) &:= p(x),\\
\mathrm{ST}_x(\bot) &:= \bot,\\
\mathrm{ST}_x(\varphi \wedge \psi) &:= \mathrm{ST}_x(\varphi) \wedge \mathrm{ST}_x(\psi),\\
\mathrm{ST}_x(\varphi \vee \psi) &:= \mathrm{ST}_x(\varphi) \vee \mathrm{ST}_x(\psi),\\
\mathrm{ST}_x(\varphi \to \psi) &:= \forall y\,(x \leqslant y \wedge \mathrm{ST}_y(\varphi) \to \mathrm{ST}_y(\psi)),
\end{aligned}$$

where y is a fresh variable.

We note that $\mathfrak{M}$ and $\mathfrak{F}$ are regarded as first-order structures of $\mathcal{L}^1_m$ and $\mathcal{L}^1_f$, respectively. In what follows, we keep the symbol "$\models$" for the satisfaction relation for $\mathcal{L}^1_m$ or $\mathcal{L}^1_f$, while we keep "$\Vdash$" for Kripke semantics. By induction on φ, we get the following (see [16, p.7]).

Proposition 2.6 *Let* $\mathfrak{M} = (W, R, V)$ *be a model. For every formula* $\varphi \in \mathsf{Form}$ *and* $w \in W$, $\mathfrak{M}, w \Vdash \varphi$ *iff* $\mathfrak{M} \models \mathrm{ST}_x(\varphi)[w]$.

2.2 Frame Constructions and Rodenburg's Characterization of Intuitionistic Frame Definability

This subsection first introduces four frame constructions: bounded morphic images, generated subframes, disjoint unions, and prime filter extensions. Then, we review Rodenburg [16]'s Goldblatt-Thomason Theorem for intuitionistic logic in terms of the four frame constructions.

Definition 2.7 Let $\mathfrak{F} = (W, R)$ and $\mathfrak{F}' = (W', R')$. A mapping $f : W \to W'$ is a *bounded morphism* from $\mathfrak{F}$ to $\mathfrak{F}'$ if f satisfies the following:

(Forth) For every $w, v \in W$, wRv implies $f(w)R'f(v)$.

(Back) For every $w \in W$ and $b \in W'$, $f(w)R'b$ implies that $f(v) = b$ and wRv for some $v \in W$.

We say that $\mathfrak{F}'$ is a *bounded morphic images* of $\mathfrak{F}$ (notation: $\mathfrak{F} \twoheadrightarrow \mathfrak{F}'$) if there exists a *surjective* bounded morphism from $\mathfrak{F}$ onto $\mathfrak{F}'$. Given any models $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$, a mapping $f : W \to W'$ is a *bounded* *morphism* from $\mathfrak{M}$ to $\mathfrak{M}'$ if f is a bounded morphism from (W, R) to (W', R') and it also satisfies the following:

(Atom) $V(p) = f^{-1}[V'(p)]$ for all propositional variables p.

Definition 2.8 We say that $\mathfrak{F}' = (W', R')$ is a *generated subframe* of $\mathfrak{F} = (W, R)$ (notation: $\mathfrak{F}' \rightarrowtail \mathfrak{F}$) if the following conditions hold: (i) $W' \subseteq W$ is an upset with respect to R, and (ii) $R' = R \cap (W' \times W')$. A model $\mathfrak{M}' = (W', R', V')$ is a *generated submodel* of a model $\mathfrak{M} = (W, R, V)$ if (W', R') is a generated subframe of (W, R) and $V'(p) = V(p) \cap W'$ for all propositional variables p. Given a subset X of the domain of a frame $\mathfrak{F}$ (or a model $\mathfrak{M}$), $\mathfrak{F}_X$ (or $\mathfrak{M}_X$) is the smallest generated subframe (or submodel) whose domain contains X. When X is a singleton $\{ w \}$, we simply write $\mathfrak{F}_w$ and $\mathfrak{M}_w$ to mean $\mathfrak{F}_{\{ w \}}$ and $\mathfrak{M}_{\{ w \}}$, respectively.

By induction on φ, we can easily prove the following (cf. [16, p.5]).

Proposition 2.9 *Let* $\mathfrak{M}' = (W, R, V)$ *be a generated submodel of* $\mathfrak{M}$. *For every formula* $\varphi \in \mathsf{Form}$ *and* $w \in W'$, $\mathfrak{M}', w \Vdash \varphi$ iff $\mathfrak{M}, w \Vdash \varphi$.

Definition 2.10 Given a family $(\mathfrak{F}_i)_{i \in I}$ of frames where $\mathfrak{F}_i = (W_i, R_i)$, the *disjoint union* $\biguplus_{i \in I} \mathfrak{F}_i = (W, R)$ of $(\mathfrak{F}_i)_{i \in I}$ is defined as:

(i) $W := \bigcup_{i \in I} (W_i \times \{ i \})$ and

(ii) $(w, i) R (v, j)$ iff $i = j$ and $w R_i v$.

For a family $(\mathfrak{M}_i)_{i \in I}$ of models where $\mathfrak{M}_i = (W_i, R_i, V_i)$, the *disjoint union* $\biguplus_{i \in I} \mathfrak{M}_i = (W, R, V)$ of $(\mathfrak{M}_i)_{i \in I}$ is defined as follows: (W, R) is the disjoint union of $(W_i, R_i)_{i \in I}$ and $(w, i) \in V(p)$ iff $w \in V_i(p)$ for all $p \in \mathsf{Prop}$.

The following proposition has been already established in [16, Section 2.4].

Proposition 2.11 (i) *If* $\mathfrak{F} \twoheadrightarrow \mathfrak{G}$, *then* $\mathfrak{F} \Vdash \varphi$ *implies* $\mathfrak{G} \Vdash \varphi$ *for all* $\varphi \in \mathsf{Form}$.

(ii) *If* $\mathfrak{F}' \rightarrowtail \mathfrak{F}$, *then* $\mathfrak{F} \Vdash \varphi$ *implies* $\mathfrak{F}' \Vdash \varphi$ *for all* $\varphi \in \mathsf{Form}$.

(iii) *Given a family* $(\mathfrak{F}_i)_{i \in I}$ *of frames, if* $\mathfrak{F}_i \Vdash \varphi$ *for all* $i \in I$, *then* $\biguplus_{i \in I} \mathfrak{F}_i \Vdash \varphi$, *for all* $\varphi \in \mathsf{Form}$.

Now, we move to our final frame construction of *prime filter extensions*.

Definition 2.12 Let $\mathfrak{F} = (W, R)$ be a frame (or preorder) and define

$$\wp^{\uparrow}(W) := \{ X \subseteq W \mid X \text{ is an upset } \}.$$

We say that $\mathcal{F} \subseteq \wp^{\uparrow}(W)$ is a *filter* on W if $X \cap Y \in \mathcal{F}$ iff $X \in \mathcal{F}$ and $Y \in \mathcal{F}$, for every X, $Y \in \wp^{\uparrow}(W)$. A filter $\mathcal{F}$ is *prime* if the following two conditions hold: (i) $\varnothing \notin \mathcal{F}$ and $\mathcal{F} \neq \varnothing$, i.e., $\mathcal{F}$ is *proper*; (ii) $X \cup Y \in \mathcal{F}$ implies $X \in \mathcal{F}$ or $Y \in \mathcal{F}$, for every X, $Y \in \wp^{\uparrow}(W)$.

For a filter $\mathcal{F}$, $X \in \mathcal{F}$ and $X \subseteq Y$ imply $Y \in \mathcal{F}$ for all X, $Y \in \wp^{\uparrow}(W)$, i.e., $\mathcal{F}$ is *upward closed* (with respect to $\subseteq$).

Definition 2.13 The *prime filter extension* $\mathfrak{pe}\,\mathfrak{F} = (\mathrm{Pf}(W), R^{\mathfrak{pe}})$ of a frame $\mathfrak{F} = (W, R)$ is defined as follows: (i) $\mathrm{Pf}(W)$ is the set of all the prime filters on W; (ii) $\mathcal{F}_1 R^{\mathfrak{pe}} \mathcal{F}_2$ iff $\mathcal{F}_1 \subseteq \mathcal{F}_2$. We say that $\mathfrak{pe}\,\mathfrak{M} = (\mathrm{Pf}(W), R^{\mathfrak{pe}}, V^{\mathfrak{pe}})$ is the *prime filter extension* of a model $\mathfrak{M} = (W, R, V)$ if $(\mathrm{Pf}(W), R^{\mathfrak{pe}})$ is the prime filter extension of (W, R), and $\mathcal{F} \in V^{\mathfrak{pe}}(p)$ iff $V(p) \in \mathcal{F}$, for every propositional variable p.

It is noted that $V^{\mathfrak{pe}}(p)$ is clearly an upset with respect to $R^{\mathfrak{pe}}$.

Proposition 2.14 (Rodenburg [16]) (i) *Let* $\mathfrak{M} = (W, R, V)$ *be a model. Then, for any prime filter* $\mathcal{F}$ *on* W, $\mathfrak{pe}\,\mathfrak{M}, \mathcal{F} \Vdash \varphi$ *iff* $[\![\varphi]\!]_{\mathfrak{M}} \in \mathcal{F}$.

(ii) *Given any frame* $\mathfrak{F}$, *if* $\mathfrak{pe}\,\mathfrak{F} \Vdash \varphi$ *then* $\mathfrak{F} \Vdash \varphi$, *for every* $\varphi \in \mathsf{Form}$.

Item (ii) of Proposition 2.14 is from [16, Proposition 14.18.3], but there is no explicit proof of item (i) there, and so, we provide an outline of the argument for item (i).

Proof. (i) By induction on φ. We only deal with the case where φ is of the form $\psi \to \theta$. First, we prove the right-to-left direction. Assume that $[\![\psi \to \theta]\!]_{\mathfrak{M}} \in \mathcal{F}$. Fix any prime filter $\mathcal{F}' \in \mathrm{Pf}(W)$ such that $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathfrak{pe}\,\mathfrak{M}, \mathcal{F}' \Vdash \psi$. Our goal is to show: $\mathfrak{pe}\,\mathfrak{M}, \mathcal{F}' \Vdash \theta$. It follows from $\mathfrak{pe}\,\mathfrak{M}, \mathcal{F}' \Vdash \psi$ and induction hypothesis that $[\![\psi]\!]_{\mathfrak{M}} \in \mathcal{F}'$. Thus, we have that $[\![\psi \to \theta]\!]_{\mathfrak{M}} \cap [\![\psi]\!]_{\mathfrak{M}} \in \mathcal{F}'$ hence $[\![\theta]\!]_{\mathfrak{M}} \in \mathcal{F}'$ since $[\![\psi \to \theta]\!]_{\mathfrak{M}} \cap [\![\psi]\!]_{\mathfrak{M}} \subseteq [\![\theta]\!]_{\mathfrak{M}}$. We can conclude $\mathfrak{pe}\,\mathfrak{M}, \mathcal{F}' \Vdash \theta$ by induction hypothesis. Second, we prove the left-to-right direction by the contrapositive implication and so assume that $[\![\psi \to \theta]\!]_{\mathfrak{M}} \notin \mathcal{F}$. Then, we can find a prime filter $\mathcal{F}'$ such that $\mathcal{F} \subseteq \mathcal{F}'$, $[\![\psi]\!]_{\mathfrak{M}} \in \mathcal{F}'$, and $[\![\theta]\!]_{\mathfrak{M}} \notin \mathcal{F}'$. By induction hypothesis, this implies that $\mathfrak{pe}\,\mathfrak{M}, \mathcal{F}' \nVdash \psi \to \theta$, as desired.

(ii) Fix any frame $\mathfrak{F} = (W, R)$ and formula φ. We prove the contrapositive implication and so assume that $\mathfrak{F} \nVdash \varphi$, i.e., there exists a valuation V and a state $w \in W$ such that $(\mathfrak{F}, V), w \nVdash \varphi$. Put $\mathfrak{M} := (\mathfrak{F}, V)$. Let $\mathcal{F}_w := \{\, X \in \wp^{\uparrow}(W) \mid w \in X \,\}$. It is easy to see that $\mathcal{F}_w$ is a prime filter. Since $w \notin [\![\varphi]\!]_{\mathfrak{M}}$, we get $[\![\varphi]\!]_{\mathfrak{M}} \notin \mathcal{F}_w$. It follows from item (i) that $\mathfrak{pe}\,\mathfrak{M}, \mathcal{F}_w \nVdash \varphi$, i.e., $\mathfrak{pe}\,\mathfrak{F} \nVdash \varphi$. □

Definition 2.15 Let $\mathbb{F}$ be a frame class. We say that $\mathbb{F}$ is *closed under taking bounded morphic images* if $\mathfrak{F} \in \mathbb{F}$ and $\mathfrak{F} \twoheadrightarrow \mathfrak{G}$ imply $\mathfrak{G} \in \mathbb{F}$, for all frames $\mathfrak{F}$ and $\mathfrak{G}$. The class $\mathbb{F}$ is *closed under taking generated subframes* if $\mathfrak{F} \in \mathbb{F}$ and $\mathfrak{G} \rightarrowtail \mathfrak{F}$ imply $\mathfrak{G} \in \mathbb{F}$, for all frames $\mathfrak{F}$ and $\mathfrak{G}$. The class $\mathbb{F}$ is *closed under taking disjoint unions* if, whenever $\mathfrak{F}_i \in \mathbb{F}$ for all $i \in I$, $\biguplus_{i \in I} \mathfrak{F}_i \in \mathbb{F}$ holds, for all families $(\mathfrak{F}_i)_{i \in I}$ of frames. A class $\mathbb{F}$ of frames *reflects prime filter extensions* if $\mathfrak{pe}\,\mathfrak{F} \in \mathbb{F}$ implies $\mathfrak{F} \in \mathbb{F}$, for all frames $\mathfrak{F}$. We say that a class $\mathbb{F}$ of frames is *elementary* (or *first-order definable*) if there exists a set Σ of sentences in $\mathcal{L}^1_f$ such that Σ defines $\mathbb{F}$ in the sense of first-order model theory.

Theorem 2.16 (Rodenburg [16]) *An elementary frame class* $\mathbb{F}$ *is definable by a set of intuitionistic formulas* (*i.e., a subset of* Form) *iff* $\mathbb{F}$ *is closed under taking bounded morphic images, generated subframes, and disjoint unions and* $\mathbb{F}$ *reflects prime filter extensions.*

It is noted that the left-to-right direction is shown by Propositions 2.11 and 2.14 where we do not need to use the assumption that $\mathbb{F}$ is elementary.

Rodenburg proved the right-to-left direction via the representation theorem of Heyting algebras (see a proof given for [16, Theorem 15.3]). We can also prove Theorem 2.16 by van Benthem's model-theoretic argument [25] (the reader can get an idea of it from our proof of Theorems 4.2 and 4.3).

Proposition 2.17 *The following frame properties are undefinable in the syntax of intuitionistic logic.*

(i) *Antisymmetry of R, i.e., $\forall\, x, y\, (xRy$ and yRx imply $x = y)$.*

(ii) $\exists\, x, y\, (xRy$ *and* $x \neq y)$.

(iii) *R is a total relation, i.e., $\forall\, x, y\, (xRy)$.*

(iv) $\forall\, x, y\, (xRy$ *or* $yRx)$.

(v) $\forall\, x, y\, \exists\, z\, (xRz$ *and* $yRz)$.

(vi) $\exists\, y\, \forall\, x\, (xRy)$*, i.e., the existence of the maximum element.*

(vii) $\exists\, y\, \forall\, x\, (yRx)$*, i.e., the existence of the minimum element.*

Proof. For (i), let us consider $\mathfrak{F} = (\mathbb{N}, \leqslant)$ with the ordinary partial order $\leqslant$ and $\mathfrak{G} = (\{0, 1\}, \{0, 1\} \times \{0, 1\})$. Then the mapping sending even and odd numbers to 0 and 1 respectively is a surjective bounded morphism. While $\mathfrak{F}$ is anti-symmetric, $\mathfrak{G}$ is not. Then, Proposition 2.11 (i) implies the desired undefinability. The property (ii) is clearly not closed under generated subframes and we get the undefinability by Proposition 2.11 (ii). The remaining properties from (iii) to (vii) are not closed under disjoint unions. For example, the single point reflexive frame satisfies all the properties from (iii) to (vii) but two copies of it do not satisfy them. Then, Proposition 2.11 (iii) gives us the desired undefinability. □

3 Intuitionistic Logic with the Universal Modality

The set $\mathsf{Form}(\mathsf{A})$ of all formulas of the intuitionistic logic with the *universal modality* A is defined inductively as follows:

$$\mathsf{Form}(\mathsf{A}) \ni \varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \mathsf{A}\,\varphi, \quad (p \in \mathsf{Prop}).$$

The set $\bigvee \mathsf{A}\,\mathsf{Form}$ of *disjunctive* A*-clauses* is defined as follows.

$$\bigvee \mathsf{A}\,\mathsf{Form} \ni \rho ::= \bot \mid \mathsf{A}\,\varphi \mid \rho \vee \rho \quad (\varphi \in \mathsf{Form}).$$

where it is noted that $\varphi \in \mathsf{Form}$ is a formula of the intuitionistic logic and so it does not contain any occurrence of A. For example, $\mathsf{A}\,p \vee \mathsf{A}\,\neg p$ is a disjunctive A-clause. It is clear that $\bigvee \mathsf{A}\,\mathsf{Form} \subseteq \mathsf{Form}(\mathsf{A})$.

Given a model $\mathfrak{M} = (W, R, V)$, a state $w \in W$ and a formula $\varphi \in \mathsf{Form}(\mathsf{A})$, the satisfaction relation $\mathfrak{M}, w \Vdash \varphi$ is defined in the same way as in Definition 2.1 except

$$\mathfrak{M}, w \Vdash \mathsf{A}\,\varphi \;\text{ iff }\; \forall\, v \in W\, (\mathfrak{M}, v \Vdash \varphi).$$

It is easy to see that $[\![\mathsf{A}\,\varphi]\!]_{\mathfrak{M}} = W$ or $[\![\mathsf{A}\,\varphi]\!]_{\mathfrak{M}} = \varnothing$ for all models $\mathfrak{M} = (W, R, V)$. Since W and $\varnothing$ are upsets, we can easily obtain the following.

Proposition 3.1 *Given a model* $\mathfrak{M} = (W, R, V)$ *and a formula* $\varphi \in \mathsf{Form}(\mathsf{A})$, $[\![\varphi]\!]_{\mathfrak{M}}$ *is an upset.*

The reader may wonder if the existential dual E of A is defined as $\mathsf{E}\,\varphi := \neg\,\mathsf{A}\,\neg\varphi$. This is the case as shown in the following (we need to use reflexivity of R).

Proposition 3.2 *Given a model* $\mathfrak{M} = (W, R, V)$ *and a state* $w \in W$ *and a formula* $\varphi \in \mathsf{Form}(\mathsf{A})$, $\mathfrak{M}, w \Vdash \neg\,\mathsf{A}\,\neg\varphi$ iff $\mathfrak{M}, v \Vdash \varphi$ *for some* $v \in W$.

Proof. $\mathfrak{M}, w \Vdash \neg\,\mathsf{A}\,\neg\varphi$ iff $\forall\, v$ (wRv implies $\mathfrak{M}, v \nVdash \mathsf{A}\,\neg\varphi$) iff $\forall\, v$ (wRv implies $\exists\, u\, (\mathfrak{M}, u \nVdash \neg\varphi)$) iff $\forall\, v$ (wRv implies $\exists\, u\, \exists\, x\, (uRx$ and $\mathfrak{M}, x \Vdash \varphi)$) iff $\exists\, u\, \exists\, x$ (uRx and $\mathfrak{M}, x \Vdash \varphi$). The last statement implies $\mathfrak{M}, x \Vdash \varphi$ for some $x \in W$. Moreover, the converse direction of this is trivial by reflexivity of R. ☐

Therefore, we can define $\mathsf{E}\,\varphi := \neg\,\mathsf{A}\,\neg\varphi$ and obtain the following satisfaction clause:

$$\mathfrak{M}, w \Vdash \mathsf{E}\,\varphi \text{ iff } \mathfrak{M}, v \Vdash \varphi \text{ for some } v \in W.$$

Similarly to Form, we define the notions of validity, definability, etc. also for $\mathsf{Form}(\mathsf{A})$ hence also for $\bigvee \mathsf{A}\,\mathsf{Form}$. Some undefinable frame properties in the syntax of intuitionistic logic of Proposition 2.17 become definable with the help of A as follows.

Proposition 3.3 (i) $\mathsf{A}\,p \vee \mathsf{A}\,\neg p$ *defines* $\forall\, x, y\, (xRy)$.

(ii) $\mathsf{A}(p \to q) \vee \mathsf{A}(q \to p)$ *defines* $\forall\, x, y\, (xRy$ *or* $yRx)$.

(iii) $\mathsf{A}\,\neg p \vee \mathsf{A}\,\neg\neg p$ *defines* $\forall\, x, y\, \exists\, z\, (xRz$ *and* $yRz)$.

Proof.

(i) Fix any frame $\mathfrak{F} = (W, R)$. Suppose the frame property $\forall\, x, y\, (xRy)$. To show the validity of $\mathsf{A}\,p \vee \mathsf{A}\,\neg p$, fix any valuation V and any state $w \in W$ such that $\mathfrak{M}, w \nVdash \mathsf{A}\,\neg p$ where $\mathfrak{M} = (\mathfrak{F}, V)$. It follows that we can find states $v, u \in W$ such that vRu and $\mathfrak{M}, u \Vdash p$. To show $\mathfrak{M}, w \Vdash \mathsf{A}\,p$, fix any $a \in W$. Our goal is to show that $\mathfrak{M}, a \Vdash p$. By the supposed property, we get uRa. By $\mathfrak{M}, u \Vdash p$, we can conclude $\mathfrak{M}, a \Vdash p$.

Conversely, suppose that $\mathfrak{F} \Vdash \mathsf{A}\,p \vee \mathsf{A}\,\neg p$. Fix any $x, y \in W$. We show xRy. Define a valuation V such that $V(p) = {\uparrow}\, x$, which is an upset. By the supposition, we get 1) $V(p) = W$ or 2) $[\![\neg p]\!]_{(\mathfrak{F}, V)} = W$. But the case 2) is impossible by reflexivity of R and $V(p) = {\uparrow}\, x$. So, we get case 1), which implies $y \in {\uparrow}\, x$ hence xRy.

(ii) Fix any frame $\mathfrak{F} = (W, R)$. Suppose the property $\forall\, x, y\, (xRy$ or $yRx)$. To show $\mathfrak{F} \Vdash \mathsf{A}(p \to q) \vee \mathsf{A}(q \to p)$, fix any valuation V and any state $w \in W$. Put $\mathfrak{M} = (\mathfrak{F}, V)$ and assume that $\mathfrak{M}, w \nVdash \mathsf{A}(p \to q)$. This implies that we can find states v and u such that vRu, $\mathfrak{M}, u \Vdash p$ and $\mathfrak{M}, u \nVdash q$. We prove that $\mathfrak{M}, w \Vdash \mathsf{A}(q \to p)$. So, fix any a and b such that aRb and $\mathfrak{M}, b \Vdash q$. Our goal is to show $\mathfrak{M}, b \Vdash p$. By $\mathfrak{M}, b \Vdash q$ and $\mathfrak{M}, u \nVdash q$, bRu fails. By the supposed frame property, we get uRb. By $\mathfrak{M}, u \Vdash p$, we conclude $\mathfrak{M}, b \Vdash p$.

Conversely, suppose that $\mathfrak{F} \Vdash \mathsf{A}(p \to q) \vee \mathsf{A}(q \to p)$. Fix any $x, y \in W$. We show that xRy or yRx. Define a valuation V such that $V(p) = {\uparrow} x$ and $V(q) = {\uparrow} y$. By the supposition, we can derive ${\uparrow} x \cap V(p) \subseteq V(q)$ or ${\uparrow} y \cap V(q) \subseteq V(p)$, which implies xRy or yRx, as required.

(iii) Fix any frame $\mathfrak{F} = (W, R)$. Suppose that $\forall x, y \exists z\,(xRz \text{ and } yRz)$ and fix any valuation V and state $w \in W$ such that $\mathfrak{M}, w \nVdash \mathsf{A}\,\neg p$, where $\mathfrak{M} = (\mathfrak{F}, V)$. It follows that we can find states v and u such that vRu and $\mathfrak{M}, u \Vdash p$. We show $\mathfrak{M}, w \Vdash \mathsf{A}\,\neg\neg p$. So, fix any x. We show $\mathfrak{M}, x \Vdash \neg\neg p$. Moreover, fix any y such that xRy. Our goal now is $\mathfrak{M}, y \nVdash \neg p$. By applying the supposed frame property for states u and y, we can find a state z such that uRz and yRz. It follows from $\mathfrak{M}, u \Vdash p$ that $\mathfrak{M}, z \Vdash p$. Together with yRz, we can conclude that $\mathfrak{M}, y \nVdash \neg p$.

Conversely, suppose that $\mathfrak{F} \Vdash \mathsf{A}\,\neg p \vee \mathsf{A}\,\neg\neg p$. We show that $\forall x, y \exists z\,(xRz \text{ and } yRz)$. Fix any $x, y \in W$. Define a valuation V such that $V(p) := {\uparrow} x$. Put $\mathfrak{M} = (\mathfrak{F}, V)$. We show that $\mathfrak{M}, x \nVdash \mathsf{A}\,\neg p$, i.e., there exists v such that $\mathfrak{M}, v \Vdash \neg p$. This holds since xRx and $x \in V(p) = {\uparrow} x$. Buy the supposition, we get $\mathfrak{M}, x \Vdash \mathsf{A}\,\neg\neg p$. Thus, $\mathfrak{M}, y \Vdash \neg\neg p$ holds. It follows that $\mathfrak{M}, y \nVdash \neg p$ by yRy. Therefore, we can find a state z such that yRz and $z \in V(p)$, i.e., xRz, as desired.

□

We remark that $\bigvee \mathsf{A}\,\mathsf{Form}$-definable frame class is *not* closed under taking disjoint unions because $\mathsf{A}\,p \vee \mathsf{A}\,\neg p$ defines $\forall x, y\,(xRy)$ (it is remarked that, when we assume antisymmetry, the same formula defines $\#W = 1$, i.e., the cardinality of the domain is 1). Therefore, $\mathsf{Form}(\mathsf{A})$-definable frame class is also *not* closed under taking disjoint unions.

Proposition 3.4 (i) *Let f be a surjective bounded morphism from $\mathfrak{M} = (W, R, V)$ to $\mathfrak{M}' = (W', R', V')$. Then, for every formula $\varphi \in \mathsf{Form}(\mathsf{A})$ and $w \in W$, $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}', f(w) \Vdash \varphi$.*

(ii) *Given any model $\mathfrak{M} = (W, R, V)$, the following equivalence holds: for every formula $\varphi \in \mathsf{Form}(\mathsf{A})$, $\mathfrak{pe}\,\mathfrak{M}, \mathcal{F} \Vdash \varphi$ iff $[\![\varphi]\!]_{\mathfrak{M}} \in \mathcal{F}$.*

Proof. We only show the case where φ is of the form $\mathsf{A}\,\psi$ for both items.

(i) Since the right-to-left direction is easy, we focus on the converse direction. Suppose that $\mathfrak{M}, w \Vdash \mathsf{A}\,\psi$. To show $\mathfrak{M}', f(w) \Vdash \mathsf{A}\,\psi$, fix any state $v' \in W'$. Since f is surjective, there exists $v \in W$ such that $f(v) = v'$. By our supposition, $\mathfrak{M}, v \Vdash \psi$ hence $\mathfrak{M}', f(v) \Vdash \psi$, which is our goal.

(ii) Recall that $[\![\mathsf{A}\,\psi]\!]_{\mathcal{M}} = W$ or $\varnothing$. First, we prove the right-to-left direction and so assume that $[\![\mathsf{A}\,\psi]\!]_{\mathfrak{M}} \in \mathcal{F}$. Since $\varnothing \notin \mathcal{F}$, $[\![\mathsf{A}\,\psi]\!]_{\mathfrak{M}} = W$. It also follows that $[\![\psi]\!]_{\mathfrak{M}} = W \in \mathcal{F}'$ for all prime filters $\mathcal{F}'$. By induction hypothesis, we get $\mathfrak{pe}\,\mathfrak{M}, \mathcal{F} \Vdash \mathsf{A}\,\psi$, as required. Second, we prove the left-to-right direction. Suppose that $[\![\mathsf{A}\,\psi]\!]_{\mathfrak{M}} \notin \mathcal{F}$. Then $[\![\mathsf{A}\,\psi]\!]_{\mathfrak{M}} \neq W$ and so $[\![\mathsf{A}\,\psi]\!]_{\mathfrak{M}} = \varnothing$. It follows that $\mathfrak{M}, w \nVdash \psi$ for some $w \in W$. So, $W \nsubseteq [\![\psi]\!]_{\mathfrak{M}}$. Then we can find a prime filter $\mathcal{F}'$ such that $W \in \mathcal{F}'$ but $[\![\psi]\!]_{\mathfrak{M}} \notin \mathcal{F}'$. By induction hypothesis, we obtain $\mathfrak{pe}\,\mathfrak{M}, \mathcal{F}' \nVdash \psi$ hence $\mathfrak{pe}\,\mathfrak{M}, \mathcal{F} \nVdash \mathsf{A}\,\psi$, as desired. □

Proposition 3.5 (i) *If* $\mathfrak{F} \twoheadrightarrow \mathfrak{G}$ *and* $\mathfrak{F} \Vdash \varphi$ *then* $\mathfrak{G} \Vdash \varphi$ *for all* $\varphi \in \mathsf{Form}(\mathsf{A})$.

(ii) *If* $\mathfrak{pe}\,\mathfrak{F} \Vdash \varphi$, *then* $\mathfrak{F} \Vdash \varphi$ *for all* $\varphi \in \mathsf{Form}(\mathsf{A})$.

(iii) *If* $\mathfrak{G} \rightarrowtail \mathfrak{F}$, *then* $\mathfrak{F} \Vdash \rho$ *implies* $\mathfrak{G} \Vdash \rho$ *for all disjunctive* A*-clauses.*

Proof. Items (i) and (ii) follow from Proposition 3.4 (i) and (ii), respectively. Let us prove item (iii). Let $\rho = \bigvee_{i \in I} \mathsf{A}\,\varphi_i$ where $\varphi_i \in \mathsf{Form}$, i.e., an intuitionistic formula. Assume that $\mathfrak{G}$ is a generated subframe of $\mathfrak{F}$. Suppose also that $\mathfrak{F} \Vdash \bigvee_{i \in I} \mathsf{A}\,\varphi_i$. Our goal is to show $\mathfrak{G} \Vdash \bigvee_{i \in I} \mathsf{A}\,\varphi_i$. Fix any valuation V and $w \in |\mathfrak{G}|$. We show $(\mathfrak{G}, V), w \Vdash \bigvee_{i \in I} \mathsf{A}\,\varphi_i$, i.e., we show that there exists $i \in I$ such that $[\![\varphi_i]\!]_{(\mathfrak{G},V)} = |\mathfrak{G}|$. Because V is also a valuation on $|\mathfrak{F}|$, we get from the supposition that $[\![\varphi_i]\!]_{(\mathfrak{F},V)} = |\mathfrak{F}|$ for some $i \in I$. Fix such $i \in I$. Let us prove that $|\mathfrak{G}| \subseteq [\![\varphi_i]\!]_{(\mathfrak{G},V)}$. Fix any $v \in |\mathfrak{G}|$. Since $(\mathfrak{F}, V), v \Vdash \varphi_i$ iff $(\mathfrak{G}, V), v \Vdash \varphi_i$ by Proposition 2.9, we conclude from $[\![\varphi_i]\!]_{(\mathfrak{F},V)} = |\mathfrak{F}|$ that $v \in [\![\varphi_i]\!]_{(\mathfrak{G},V)}$. □

Proposition 3.6 *Let* $\mathfrak{F} = (W, R)$ *be a frame and* ρ *a disjunctive* A*-clause. If* $\mathfrak{F}_X \Vdash \rho$ *for all finite* $X \subseteq W$ *then* $\mathfrak{F} \Vdash \rho$, *where recall that* $\mathfrak{F}_Z$ *is the generated subframe of* $\mathfrak{F}$ *by* Z.

Proof. Let $\rho = \bigvee_{i \in I} \mathsf{A}\,\varphi_i$ where $\varphi_i \in \mathsf{Form}$. We prove the contrapositive implication and so assume that $(\mathfrak{F}, V), w \nVdash \bigvee_{i \in I} \mathsf{A}\,\varphi_i$ for some valuation V and state $w \in |\mathfrak{F}|$. Our goal is to show: there exists some finite $X \subseteq |\mathfrak{F}|$ such that $\mathfrak{F}_X \nVdash \bigvee_{i \in I} \mathsf{A}\,\varphi_i$. By assumption, for every choice $i \in I$, there exists $v_i \in |\mathfrak{F}|$ such that $(\mathfrak{F}, V), v_i \nVdash \varphi_i$. Put $X := \{\, v_i \mid i \in I \,\}$ and consider the finitely generated subframe $\mathfrak{F}_X$ of $\mathfrak{F}$ by the finite generator X. Let $V \upharpoonright |\mathfrak{F}_X|$ be a valuation V restricted to the domain $|\mathfrak{F}_X|$. For each $i \in I$, we have $(\mathfrak{F}_X, V \upharpoonright |\mathfrak{F}_X|), v_i \nVdash \varphi_i$ by $(\mathfrak{F}, V), w \nVdash \bigvee_{i \in I} \mathsf{A}\,\varphi_i$ (by Proposition 2.9). This allows us to conclude $\mathfrak{F}_X \nVdash \bigvee_{i \in I} \mathsf{A}\,\varphi_i$. □

Definition 3.7 We say that a class $\mathbb{F}$ of frames *reflects finitely generated subframes* if, for every frame $\mathfrak{F} = (W, R)$, whenever $\mathfrak{F}_X \in \mathbb{F}$ for all finite $X \subseteq W$, it holds that $\mathfrak{F} \in \mathbb{F}$.

Proposition 3.8 (i) *Each of antisymmetry and* $\exists\, y\, \forall\, x\, (xRy)$ *is not definable by any subset of* $\mathsf{Form}(\mathsf{A})$.

(ii) *Each of* $\exists\, x, y\, (xRy$ *and* $x \neq y)$ *and* $\exists\, y\, \forall\, x\, (yRx)$ *is not definable by any set of disjunctive* A*-clauses.*

Proof. For (i), it suffices to show that $\exists\, y\, \forall\, x\, (xRy)$ is not definable by any subset of $\mathsf{Form}(\mathsf{A})$ since we can use the same argument as in the proof of Proposition 2.17 for antisymmetry with the help of Proposition 3.5 (i). Consider $(\mathbb{N}, \leqslant)$, where $\leqslant$ is the ordinary partial ordering. Since $\wp^{\uparrow}(\mathbb{N}) = \{\, \varnothing \,\} \cup \{\, \uparrow n \mid n \in \mathbb{N} \,\}$, all the prime filters consist of $\{\, \uparrow n \mid n \in \mathbb{N} \,\}$ and $\mathcal{F}_n := \{\, X \in \wp^{\uparrow}(\mathbb{N}) \mid n \in X \,\}$ $= \{\, \uparrow 0, \uparrow 1, \ldots, \uparrow n \,\}$ $(n \in \mathbb{N})$. Then, it is easy to see that $(\mathrm{Pf}(\mathbb{N}), \leqslant^{\mathfrak{pe}})$ satisfies $\exists\, y\, \forall\, x\, (xRy)$ ($\{\, \uparrow n \mid n \in \mathbb{N} \,\}$ is a maximum element) but $(\mathbb{N}, \leqslant)$ does not. Thus, Proposition 3.5 (ii) implies the intended undefinability.

For (ii), we only prove that $\exists\, y\, \forall\, x\, (yRx)$ is undefinable by any subset of $\bigvee \mathsf{A}\,\mathsf{Form}$, since the other property $\exists\, x, y\, (xRy$ and $x \neq y)$ is undefinable by the same argument given in the proof of Proposition 2.17 with the help of

Proposition 3.5 (iii). Consider the set of all integers $(\mathbb{Z}, \leqslant)$ with the ordinary partial ordering $\leqslant$. Then, all finitely generated subframes of $(\mathbb{Z}, \leqslant)$ satisfy $\exists y \forall x\,(yRx)$ but the original frame does not. Then Proposition 3.6 implies the undefinability in $\bigvee \mathsf{A}\,\mathsf{Form}$. □

4 Characterizing Elementary Frame Definability by Intuitionistic Logic with the Universal Modality

We employ van Benthem [25]'s purely model-theoretic argument for characterizing elementary frame definability of both $\mathsf{Form}(\mathsf{A})$ and $\bigvee \mathsf{A}\,\mathsf{Form}$.

Definition 4.1 Let Γ be a set of formulas, $\mathfrak{M}$ a model and $\mathbb{F}$ a frame class. We say that Γ is *satisfiable* in $\mathfrak{M}$ if there exists a state w in $\mathfrak{M}$ such that $\mathfrak{M}, w \Vdash \Gamma$ and that Γ is *finitely satisfiable* in $\mathfrak{M}$ if every finite subset of Γ is satisfiable in $\mathfrak{M}$. The set Γ is *satisfiable* in $\mathbb{F}$ if there exists a frame $\mathfrak{F} \in \mathbb{F}$ and a valuation V on $\mathfrak{F}$ such that Γ is satisfiable in $(\mathfrak{F}, V)$, and Γ is *finitely satisfiable* in $\mathbb{F}$ if every finite subset of Γ is satisfiable in $\mathbb{F}$.

In what follows in this section, we use some notions from first-order model theory such as (finite) satisfiability, compactness, elementary extension and ω-saturation, and so, the reader is unfamiliar with those is referred to [1].

4.1 Goldblatt-Thomason Theorem for $\mathsf{Form}(\mathsf{A})$

Theorem 4.2 *For any elementary frame class $\mathbb{F}$, the following are equivalent:*

(i) *$\mathbb{F}$ is definable by a subset of $\mathsf{Form}(\mathsf{A})$,*

(ii) *$\mathbb{F}$ is closed under taking bounded morphic images and it reflects prime filter extensions.*

If we replace "prime fitter extensions" with "ultrafilter extensions", we can obtain Gargov and Goranko [9]'s Goldblatt-Thomason-type characterization for modal logic with the universal modality. So, Theorem 4.2 can be regarded as the intuitionistic version of their result.

Proof. The direction from (i) to (ii) is due to Proposition 3.5. So, we focus on the direction from (ii) to (i) and so let us assume (ii). We show that $\mathbb{F}$ is defined by $\mathsf{Log}(\mathbb{F}) := \{\, \varphi \in \mathsf{Form}(\mathsf{A}) \mid \mathbb{F} \Vdash \varphi \,\}$. That is, we show that, for every frame $\mathfrak{F} = (W, R)$, $\mathfrak{F} \in \mathbb{F}$ iff $\mathfrak{F} \Vdash \mathsf{Log}(\mathbb{F})$. Let us fix any frame $\mathfrak{F} = (W, R)$. When $\mathfrak{F} \in \mathbb{F}$, it is easy to see that $\mathfrak{F} \Vdash \mathsf{Log}(\mathbb{F})$. Conversely, we suppose that $\mathfrak{F} \Vdash \mathsf{Log}(\mathbb{F})$. The rest of this proof is devoted to establishing $\mathfrak{F} \in \mathbb{F}$. Let us expand our syntax with a (possibly uncountably infinite) set $\{\, p_A \mid A \in \wp^{\uparrow}(W) \,\}$. Remark that we can still keep the supposition $\mathfrak{F} \Vdash \mathsf{Log}(\mathbb{F})$ even if we regard $\mathsf{Log}(\mathbb{F})$ as a set of formulas in the expanded language. Moreover, let us define $\Delta_{\mathfrak{F}}$ as the set of all the following formulas:

$$\mathsf{A}(p_{A\cap B} \leftrightarrow (p_A \wedge p_B)),\ \mathsf{A}(p_{A\cup B} \leftrightarrow (p_A \vee p_B)),\ \mathsf{A}(p_{A\Rightarrow B} \leftrightarrow (p_A \to p_B)),\ \mathsf{A}(p_{\varnothing} \leftrightarrow \bot),$$

where A, $B \in \wp^{\uparrow}(W)$ and recall that $A \Rightarrow B := \{\, w \in W \mid {\uparrow} w \cap A \subseteq B \,\}$. An underlying idea of $\Delta_{\mathfrak{F}}$ is to provide a complete enough description of the frame $\mathfrak{F}$ in terms of the propositional variables $\{\, p_A \mid A \in \wp^{\uparrow}(W) \,\}$.

We are going to show that $\Delta_{\mathfrak{F}}$ is finitely satisfiable in $\mathbb{F}$. So, let us fix any finite $\Delta' \subseteq \Delta_{\mathfrak{F}}$ and suppose for contradiction that $\bigwedge \Delta'$ is unsatisfiable in $\mathbb{F}$, i.e., for all frames $\mathfrak{G} \in \mathbb{F}$, valuations U and states v in $\mathfrak{G}$, we have $(\mathfrak{G}, U), v \nVdash \bigwedge \Delta'$. This implies $\mathbb{F} \Vdash \neg \bigwedge \Delta'$ (note that $\neg$ here is the intuitionistic negation). By our assumption of $\mathfrak{F} \Vdash \mathsf{Log}(\mathbb{F})$, we get $\mathfrak{F} \Vdash \neg \bigwedge \Delta'$. This means that $\bigwedge \Delta'$ is unsatisfiable in $\mathfrak{F}$. But Δ' is clearly satisfiable in $\mathfrak{F}$, a contradiction. Therefore, $\Delta_{\mathfrak{F}}$ is finitely satisfiable in $\mathbb{F}$.

Since $\mathbb{F}$ is elementary, we can deduce from finite satisfiability of $\Delta_{\mathfrak{F}}$ that $\Delta_{\mathfrak{F}}$ is satisfiable in $\mathbb{F}$ by compactness. Thus, there exists a frame $\mathfrak{G} \in \mathbb{F}$, a valuation U on $\mathfrak{G}$, and a state v in $\mathfrak{G}$ such that $(\mathfrak{G}, U), v \Vdash \Delta_{\mathfrak{F}}$. Since all the elements of $\Delta_{\mathfrak{F}}$ are A-prefixed, we get $(\mathfrak{G}, U) \Vdash \Delta_{\mathfrak{F}}$. For an ω-saturated elementary extension $(\mathfrak{G}^*, U^*)$ of $(\mathfrak{G}, U)$ such that $\mathfrak{G}^* \in \mathbb{F}$ (since $\mathbb{F}$ is elementary), we also have $(\mathfrak{G}^*, U^*) \Vdash \Delta_{\mathfrak{F}}$. Now we define a mapping $f : |\mathfrak{G}^*| \to |\mathfrak{pe}\,\mathfrak{F}|$ by: $f(s) := \{ X \mid (\mathfrak{G}^*, U^*), s \Vdash p_X \}$. For this mapping f, the following claim holds.

Claim 1 f is a surjective bounded morphism from $\mathfrak{G}^*$ to $\mathfrak{pe}\,\mathfrak{F}$.

This claim implies $\mathfrak{pe}\,\mathfrak{F} \in \mathbb{F}$ because $\mathfrak{G}^* \in \mathbb{F}$ and $\mathfrak{G}^* \twoheadrightarrow \mathfrak{pe}\,\mathfrak{F}$. Moreover, since $\mathbb{F}$ reflects prime filter extensions, we obtain $\mathfrak{F} \in \mathbb{F}$, as desired.

So, let us provide a proof of the claim below (a basic idea of the proof is from [16, p.132, Lemma 15.2]) to finish the proof of this theorem.

(Proof of Claim) We show that $f : |\mathfrak{G}^*| \to |\mathfrak{pe}\,\mathfrak{F}|$ is a surjective bounded morphism. Let S be the binary relation of $\mathfrak{G}^*$.

(Well-defined) We show that $f(s)$ is a prime filter. First, we check that $\varnothing \notin f(s)$ and $f(s) \neq \varnothing$. We have $\varnothing \notin f(s)$ because $p_\varnothing \leftrightarrow \bot$ is valid on $(\mathfrak{G}^*, U^*)$ and $\bot$ is unsatisfiable in $(\mathfrak{G}^*, U^*)$. As for $f(s) \neq \varnothing$, it suffices to note that we can derive from $(\mathfrak{G}^*, U^*) \Vdash \Delta_{\mathfrak{F}}$ that $(\mathfrak{G}^*, U^*), s \Vdash p_W$ hence $W \in f(s)$. The other conditions for prime filter are also established by $(\mathfrak{G}^*, U^*) \Vdash \Delta_{\mathfrak{F}}$.

(Forth) Suppose that sSs'. We prove that $f(s) R^{\mathfrak{pe}} f(s')$, i.e., $f(s) \subseteq f(s')$. Fix any $X \in f(s)$. Then we have $(\mathfrak{G}^*, U^*), s \Vdash p_X$. We want to show that $(\mathfrak{G}^*, U^*), s' \Vdash p_X$. Since the persistency is the first-order condition, the set $U^*(p_X)$ is an upset with respect to S. Therefore, we can conclude $(\mathfrak{G}^*, U^*), s' \Vdash p_X$.

(Back) Fix any $s \in |\mathfrak{G}^*|$ and $\mathcal{F} \in |\mathfrak{pe}\,\mathfrak{F}|$ such that $f(s) R^{\mathfrak{pe}} \mathcal{F}$, i.e., $f(s) \subseteq \mathcal{F}$. We establish that there exists $s' \in |\mathfrak{G}^*|$ such that sSs' and $f(s') = \mathcal{F}$. We need to use ω-saturation here. Let us put a type

$$\Gamma(x) := \{ p_X(x) \mid X \in \mathcal{F} \} \cup \{ \neg p_X(x) \mid X \notin \mathcal{F} \} \cup \{ \underline{s} \leqslant x \}$$

of first-order formulas, where $\underline{s}$ denotes the corresponding constant symbol to s, "$\neg$" of $\neg p_X(x)$ is the classical negation since we are considering the first-order language $\mathcal{L}^1_m$. Now we show that $\Gamma(x)$ is finitely satisfiable in $(\mathfrak{G}^*, U^*)$ in the sense of the first-order model theory. Fix any $\Gamma'(x) := \{ p_{X_1}(x), \ldots, p_{X_n}(x), \neg p_{Y_1}(x), \ldots, \neg p_{Y_n}(x), \underline{s} \leqslant x \}$ where $X_i \in \mathcal{F}$ and $Y_j \notin \mathcal{F}$, and suppose for contradiction that $\Gamma'(x)$ is not satisfiable in

$(\mathfrak{G}^*, U^*)$. It follows that, for every $s' \in |\mathfrak{G}^*|$,

$$(\mathfrak{G}^*, U^*) \models (\underline{s} \leqslant x \wedge \bigwedge_{1\leqslant i\leqslant n} p_{X_i}(x)) \to \bigvee_{1\leqslant j\leqslant m} p_{Y_j}(x))[s'].$$

Hence we get:

$$(\mathfrak{G}^*, U^*) \models \forall x\, ((\underline{s} \leqslant x \wedge \bigwedge_{1\leqslant i\leqslant n} p_{X_i}(x)) \to \bigvee_{1\leqslant j\leqslant m} p_{Y_j}(x)).$$

By shifting our semantics, this implies from Proposition 2.6 that

$$(\mathfrak{G}^*, U^*), s \Vdash \bigwedge_{1\leqslant i\leqslant n} p_{X_i} \to \bigvee_{1\leqslant j\leqslant m} p_{Y_j}$$

where "$\to$" is intuitionistic. It also follows from $(\mathfrak{G}^*, U^*) \Vdash \Delta_{\mathfrak{F}}$ that

$$(\mathfrak{G}^*, U^*), s \Vdash p_{\bigcap_{1\leqslant i\leqslant n} X_i} \to p_{\bigcup_{1\leqslant j\leqslant m} Y_j}.$$

Hence $(\mathfrak{G}^*, U^*), s \Vdash p_{X\Rightarrow Y}$, where $X := \bigcap_{1\leqslant i\leqslant n} X_i$ and $Y := \bigcup_{1\leqslant j\leqslant m} Y_j$. Thus, $X \Rightarrow Y \in f(s)$. Since $X \in f(s)$, we get $Y = \bigcup_{1\leqslant j\leqslant m} Y_j \in f(s)$, which implies $Y_j \in f(s) \subseteq \mathcal{F}$ for some $1 \leqslant j \leqslant m$. This is a contradiction with $Y_j \notin \mathcal{F}$ for all indices j.

Therefore, we have shown that $\Gamma(x)$ is finitely satisfiable in $(\mathfrak{G}^*, U^*)$ in the sense of the first-order model theory. This implies that $\Gamma(x)$ is satisfiable in $(\mathfrak{G}^*, U^*)$ by ω-saturation. Thus, fix a solution s' of $\Gamma(x)$ at $(\mathfrak{G}^*, U^*)$. It is easy to see that sSs'. We can also establish $f(s') = \mathcal{F}$ as follows. By $(\mathfrak{G}^*, U^*) \models \Gamma(x)[s']$, it follows that $(\mathfrak{G}^*, U^*) \models p_X(x)[s']$ implies $X \in \mathcal{F}$ and that $(\mathfrak{G}^*, U^*) \not\models p_X(x)[s']$ implies $X \notin \mathcal{F}$. Therefore, $f(s') = \mathcal{F}$.

(Onto) Fix any prime filter $\mathcal{F} \in |\mathfrak{pe}\,\mathfrak{F}|$. Let us put a type

$$\Gamma(x) := \{\, p_X(x) \mid X \in \mathcal{F} \,\} \cup \{\, \neg p_X(x) \mid X \notin \mathcal{F} \,\}$$

of first-order formulas. Similarly to our argument for (Back), we can prove that $\Gamma(x)$ is satisfiable in $(\mathfrak{G}^*, U^*)$ hence $f(s') = \mathcal{F}$ for some $s' \in |\mathfrak{G}^*|$.

This finishes establishing that f is a surjective bounded morphism. ⊣

Therefore, we conclude that $\mathsf{Log}(\mathbb{F})$ defines $\mathbb{F}$. □

4.2 Goldblatt-Thomason Theorem for $\bigvee \mathsf{A}\,\mathsf{Form}$

Theorem 4.3 *For any elementary frame class $\mathbb{F}$, the following are equivalent:*

(i) *$\mathbb{F}$ is definable by a set of disjunctive A-clauses, i.e., a subset of $\bigvee \mathsf{A}\,\mathsf{Form}$.*

(ii) *$\mathbb{F}$ is closed under taking bounded morphic images and generated subframes and $\mathbb{F}$ reflects finitely generated subframes and prime filter extensions.*

If we replace "prime filter extensions" with "ultrafilter extension", we can obtain [20,19]'s Goldblatt-Thomason-style characterization for (extended) modal dependence logic. Therefore, Theorem 4.3 is an intuitionistic variant of the GT-style characterization in [20,19].

Proof. The direction from (i) to (ii) is established by Propositions 3.5 and 3.6 by $\bigvee \mathsf{A}\,\mathsf{Form} \subseteq \mathsf{Form}(\mathsf{A})$. So, we prove the converse direction. Assume (ii). Let us define $\mathsf{Log}_{\vee\mathsf{A}}(\mathbb{F}) := \{\, \rho \in \bigvee \mathsf{A}\,\mathsf{Form} \mid \mathbb{F} \Vdash \rho \,\}$. We show that $\mathsf{Log}_{\vee\mathsf{A}}(\mathbb{F})$ defines $\mathbb{F}$. Let us fix any frame $\mathfrak{F} = (W, R)$. We need to establish the following equivalence: $\mathfrak{F} \in \mathbb{F}$ iff $\mathfrak{F} \Vdash \mathsf{Log}_{\vee\mathsf{A}}(\mathbb{F})$. The left-to-right direction is easy to show, and so, we focus on showing the right-to-left direction. Suppose that $\mathfrak{F} \Vdash \mathsf{Log}_{\vee\mathsf{A}}(\mathbb{F})$. Our goal is to show $\mathfrak{F} \in \mathbb{F}$. The rest of the proof is devoted to showing it. Since $\mathbb{F}$ reflects finitely generated subframes, we can assume without loss of generality that $\mathfrak{F}$ is finitely generated, i.e., generated by a finite set $U \subseteq W$.

We expand our syntax with a set $\{\, p_A \mid A \in \wp^{\uparrow}(W) \,\}$ (which is possibly uncountably infinite). Similarly to the proof of Theorem 4.2, we can still keep $\mathfrak{F} \Vdash \mathsf{Log}_{\vee\mathsf{A}}(\mathbb{F})$ even if we regard $\mathsf{Log}_{\vee\mathsf{A}}(\mathbb{F})$ as a set of formulas in the expanded language. We define Δ as the set of all the following formulas:

$$p_{A\cap B} \leftrightarrow (p_A \wedge p_B),\; p_{A\cup B} \leftrightarrow (p_A \vee p_B),\; p_{A\Rightarrow B} \leftrightarrow (p_A \to p_B),\; p_{\varnothing} \leftrightarrow \bot,$$

where $A, B \in \wp^{\uparrow}(W)$. Moreover, we put $\Delta_{\mathfrak{F},u} := \{\, p_{\uparrow u} \wedge \varphi \mid \varphi \in \Delta \,\}$ for each $u \in U$. Since $\mathfrak{F}$ is finitely generated by U, $(\Delta_{\mathfrak{F},u})_{u\in U}$ encodes a complete enough description of $\mathfrak{F}$ in terms of the propositional variables $\{\, p_A \mid A \in \wp^{\uparrow}(W) \,\}$.

In what follows, we need to employ a different strategy from our proof of Theorem 4.2. Let us introduce a finite set $\{\, x_u \mid u \in U \,\}$ of mutually disjoint variables and ST_{x_u} be the standard translation from Form to all the formulas in the first-order correspondence model language $\mathcal{L}^1_m$.

We are going to show $\bigcup_{u\in U} \mathrm{ST}_{x_u}[\Delta_{\mathfrak{F},u}]$ is finitely satisfiable in $\mathbb{F}$, where $\mathrm{ST}_{x_u}[\Phi]$ is the direct image of Φ under the standard translation ST_{x_u}, i.e., $\{\, \mathrm{ST}_{x_u}(\varphi) \mid \varphi \in \Phi \,\}$. Let $\Gamma \subseteq \bigcup_{u\in U} \mathrm{ST}_{x_u}[\Delta_{\mathfrak{F},u}]$ be a finite set. Then we may write $\Gamma = \bigcup_{1\leqslant k\leqslant n} \mathrm{ST}_{u_k}[\Gamma_{u_k}]$ for some $u_1, \ldots, u_n \in U$ and some finite $\Gamma_{u_k} \subseteq \Delta_{\mathfrak{F},u_k}$ $(1 \leqslant k \leqslant n)$. Assume for contradiction that Γ is not satisfiable in $\mathbb{F}$, i.e.,

$$\forall\, \mathfrak{G} \in \mathbb{F}\, \forall V\, \forall \overrightarrow{a} \in |\mathfrak{F}|^n\, \exists\, 1 \leqslant k \leqslant n\, \left((\mathfrak{G}, V) \models \neg \mathrm{ST}_{x_{u_k}} \left(\bigwedge \Gamma_{u_k} \right) [\overrightarrow{a}] \right).$$

where $\overrightarrow{a} := (a_1, \ldots, a_n)$. This is also equivalent to:

$$\forall\, \mathfrak{G} \in \mathbb{F}\, \forall V\, \forall \overrightarrow{a} \in |\mathfrak{F}|^n\, \exists\, 1 \leqslant k \leqslant n\, \left((\mathfrak{G}, V) \models \neg \mathrm{ST}_{x_{u_k}} \left(\bigwedge \Gamma_{u_k} \right) [a_k] \right).$$

where we rewrite the assignment for variables. By first-order reasoning (in particular, we use the validity of $\forall x\, \forall y\, (P(x) \vee Q(y)) \to \forall x\, P(x) \vee \forall y\, P(y)$), we get:

$$\forall\, \mathfrak{G} \in \mathbb{F}\, \forall V\, \left((\mathfrak{G}, V) \models \bigvee\nolimits_{1\leqslant k\leqslant n} \forall x_{u_k}\, \neg \mathrm{ST}_{x_{u_k}} \left(\bigwedge \Gamma_{u_k} \right) \right).$$

where "$\neg$" above is the classical negation. By changing our semantics to Kripke semantics, this also implies $\mathbb{F} \Vdash \bigvee_{1\leqslant k\leqslant n} \mathsf{A}\, \neg \bigwedge \Gamma_{u_k}$ by Proposition 2.6, where $\neg$ is the intuitionistic negation. It is noted that $\neg \bigwedge \Gamma_{u_k} \in \mathsf{Form}$, i.e., an intuitionistic formula and so $\bigvee_{1\leqslant k\leqslant n} \mathsf{A}\, \neg \bigwedge \Gamma_{u_k}$ is a disjunctive A-clause. Therefore,

$\bigvee_{1\leqslant k\leqslant n} \mathsf{A}\neg\bigwedge\Gamma_{u_k} \in \mathsf{Log}_{\vee\mathsf{A}}(\mathbb{F})$. Since we have assumed $\mathfrak{F} \Vdash \mathsf{Log}_{\vee\mathsf{A}}(\mathbb{F})$, we obtain $\mathfrak{F} \Vdash \bigvee_{1\leqslant k\leqslant n} \mathsf{A}\neg\bigwedge\Gamma_{u_k}$. This implies that Γ is not satisfiable in $\mathfrak{F}$ in the sense of first-order model theory. But Γ is clearly satisfiable in $\mathfrak{F}$, which implies a desired contradiction. We have shown that $\bigcup_{u\in U} \mathrm{ST}_{x_u}[\Delta_{\mathfrak{F},u}]$ is finitely satisfiable in $\mathbb{F}$.

Since $\mathbb{F}$ is elementary, $\bigcup_{u\in U} \mathrm{ST}_{x_u}[\Delta_{\mathfrak{F},u}]$ is satisfiable in $\mathbb{F}$ by compactness. We can find a frame $\mathfrak{G} \in \mathbb{F}$, a valuation U on $|\mathfrak{G}|$ and a finite sequence $\overrightarrow{w} = (w_u)_{u\in U}$ such that $(\mathfrak{G}, U) \models \bigcup_{u\in U} \mathrm{ST}_{x_u}[\Delta_{\mathfrak{F},u}][\overrightarrow{w}]$. By changing our semantics "$(\models)$" to Kripke semantics ("$\Vdash$"), it follows that $(\mathfrak{G}, U), w_u \Vdash \Delta_{\mathfrak{F},u}$ by Proposition 2.6. Let us put $Z := \{ w_u \mid u \in U \}$. Let $(\mathfrak{G}^*_Z, V^*_Z)$ be an ω-saturated elementary extension of the Z-generated submodel $(\mathfrak{G}_Z, V_Z)$ of $(\mathfrak{G}, V)$. Because $\mathbb{F}$ is elementary and closed under taking generated subframes, we have $\mathfrak{G}_Z \in \mathbb{F}$ hence $\mathfrak{G}^*_Z \in \mathbb{F}$. It is also noted that $(\mathfrak{G}^*_Z, V^*_Z), w^*_u \Vdash \Delta_{\mathfrak{F},u}$ where w^*_u is the corresponding element in $\mathfrak{G}^*_Z$ to w_u in $|\mathfrak{G}_Z|$. Since $(\mathfrak{G}_Z, V_Z) \models \forall x\, \mathrm{ST}_x(\theta)$ for all $\theta \in \Delta$, we also get $(\mathfrak{G}^*_Z, V^*_Z) \models \forall x\, \mathrm{ST}_x(\theta)$ for all $\theta \in \Delta$, which implies $(\mathfrak{G}^*_Z, V^*_Z) \Vdash \Delta$.

Now we claim that $\mathfrak{G}^*_Z \twoheadrightarrow \mathfrak{pe}\,\mathfrak{F}$. By this claim and the closure and reflection properties of $\mathbb{F}$, we can conclude from $\mathfrak{G}^*_Z \in \mathbb{F}$ that $\mathfrak{F} \in \mathbb{F}$, as required. So, let us justify the claim. Define $f : |\mathfrak{G}^*_Z| \to |\mathfrak{pe}\,\mathfrak{F}|$ by: $f(s) := \{ X \subseteq W \mid (\mathfrak{G}^*_Z, V^*_Z), s \Vdash p_X \}$. We prove that f is a surjective bounded morphism. But the proof is almost the same as in the proof of Theorem 4.2, since $(\mathfrak{G}^*_Z, V^*_Z) \Vdash \Delta$. This finishes establishing the goal of $\mathfrak{F} \in \mathbb{F}$. Therefore, we conclude that $\mathsf{Log}_{\vee\mathsf{A}}(\mathbb{F})$ defines $\mathbb{F}$. □

5 Characterizing Elementary Frame Definability by Intuitionistic Inquisitive Logic

5.1 Team Semantics for Intuitionistic Logic

Definition 5.1 Let $\mathfrak{M} = (W, R, V)$ be a model. We say that $t \subseteq W$ is a *team*. Given a model $\mathfrak{M}$, a team $t \subseteq W$ and a formula $\varphi \in \mathsf{Form}$, the satisfaction relation $\mathfrak{M}, t \Vdash \varphi$ is defined inductively as follows:

$$\begin{array}{lll} \mathfrak{M}, t \Vdash p & \text{iff} & t \subseteq V(p) \\ \mathfrak{M}, t \Vdash \bot & \text{iff} & t = \varnothing \\ \mathfrak{M}, t \Vdash \varphi \wedge \psi & \text{iff} & \mathfrak{M}, t \Vdash \varphi \text{ and } \mathfrak{M}, t \Vdash \psi \\ \mathfrak{M}, t \Vdash \varphi \vee \psi & \text{iff} & \exists t_1, t_2 (t = t_1 \cup t_2 \text{ and } \mathfrak{M}, t_1 \Vdash \varphi \text{ and } \mathfrak{M}, t_2 \Vdash \psi) \\ \mathfrak{M}, t \Vdash \varphi \to \psi & \text{iff} & \forall s \subseteq R[t]\, (\mathfrak{M}, s \Vdash \varphi \text{ implies } \mathfrak{M}, s \Vdash \psi), \end{array}$$

where $R[t] := \{ v \in W \mid wRv \text{ for some } w \in t \}$.

For the negation, we can provide the following satisfaction clause:

$$\mathfrak{M}, t \Vdash \neg\varphi \ \text{ iff } \ \forall s \subseteq R[t]\, (s \neq \varnothing \text{ implies } \mathfrak{M}, s \nVdash \varphi).$$

By induction on φ, we can prove the following (see [5, Proposition 3.11]).

Proposition 5.2 *Let $\mathfrak{M}$ be a model. For all formulas $\varphi \in \mathsf{Form}$ and states w, $\mathfrak{M}, \{ w \} \Vdash \varphi$ iff $\mathfrak{M}, w \Vdash \varphi$.*

The following is from [5, Proposition 3.10].

Proposition 5.3 (Flatness) *Let* $\mathfrak{M}$ *be a model. For all formulas* $\varphi \in \mathsf{Form}$ *and teams* $t \subseteq |\mathfrak{M}|$, $\mathfrak{M}, t \Vdash \varphi$ iff $\mathfrak{M}, \{ w \} \Vdash \varphi$ *for all* $w \in t$.

5.2 Intuitionistic Inquisitive Logic

We expand the syntax of intuitionistic logic with *inquisitive disjunction* $\mathbin{\vee\!\!\!\vee}$.

Definition 5.4 The set $\mathsf{Form}(\mathbin{\vee\!\!\!\vee})$ of all formulas for intuitionistic logic is defined inductively as:

$$\mathsf{Form}(\mathbin{\vee\!\!\!\vee}) \ni \varphi ::= p \,|\, \bot \,|\, \varphi \wedge \varphi \,|\, \varphi \vee \varphi \,|\, \varphi \to \varphi \,|\, \varphi \mathbin{\vee\!\!\!\vee} \varphi \quad (p \in \mathsf{Prop}).$$

Given any model $\mathfrak{M} = (W, R, V)$, the satisfaction relation in team semantics for the inquisitive disjunction is defined similarly to Definition 5.1 except:

$$\mathfrak{M}, t \Vdash \varphi \mathbin{\vee\!\!\!\vee} \psi \;\text{ iff }\; \mathfrak{M}, t \Vdash \varphi \textit{ or } \mathfrak{M}, t \Vdash \psi$$

We note that flatness fails for $\mathsf{Form}(\mathbin{\vee\!\!\!\vee})$, but we can still keep the persistency (see [5, Proposition 2.9]).

Proposition 5.5 (Persistency) *If* $M, t \Vdash \varphi$ *and* $s \subseteq R[t]$ *then* $M, s \Vdash \varphi$, *for all* $\varphi \in \mathsf{Form}(\mathbin{\vee\!\!\!\vee})$.

Proposition 5.6 *Given any model* $\mathfrak{M} = (W, R, V)$ *and a formula* $\varphi \in \mathsf{Form}(\mathbin{\vee\!\!\!\vee})$, $\mathfrak{M}, t \Vdash \varphi$ *for all teams* $t \subseteq W$ *iff* $\mathfrak{M}, W \Vdash \varphi$.

Proof. Since $R[W] = W$ (recall that R is reflexive and transitive), the statement follows from Proposition 5.5. □

Definition 5.7 We say that $\varphi \in \mathsf{Form}(\mathbin{\vee\!\!\!\vee})$ is *valid* in a model $\mathfrak{M} = (W, R, V)$ (notation: $\mathfrak{M} \Vdash_T \varphi$, where the subscript "$T$" is used for emphasizing "Team semantics") if $\mathfrak{M}, W \Vdash \varphi$. A set $\Gamma \subseteq \mathsf{Form}(\mathbin{\vee\!\!\!\vee})$ is *valid* in a frame $\mathfrak{M} = (W, R, V)$ (notation: $\mathfrak{F} \Vdash_T \Gamma$) if $(\mathfrak{F}, V) \Vdash_T \varphi$ for all formulas $\varphi \in \Gamma$ and valuations V.

Based on this notion of validity in a frame, we define the notion of frame definability as before. The following proposition is an immediate consequence from [5, Theorem 4.9].

Proposition 5.8 *For every* $\varphi \in \mathsf{Form}(\mathbin{\vee\!\!\!\vee})$, *there are finitely many intuitionistic formulas* $(\psi_i)_{i \in I} \subseteq \mathsf{Form}$ *such that* φ *and* $\mathbin{\vee\!\!\!\vee}_{i \in I} \psi_i$ *are equivalent, i.e.,* $\mathfrak{M}, t \Vdash \varphi$ *iff* $\mathfrak{M}, t \Vdash \mathbin{\vee\!\!\!\vee}_{i \in I} \psi_i$ *for every model* $\mathfrak{M}$ *and team* $t \subseteq |\mathfrak{M}|$.

5.3 Goldblatt-Thomason Theorem for Intuitionistic Inquisitive Logic

Proposition 5.9 *For any finite family* $(\psi_i)_{i \in I} \subseteq \mathsf{Form}$ (*i.e.,* I *is finite*) *and model* $\mathfrak{M} = (W, R, V)$, $\mathfrak{M} \Vdash \bigvee_{i \in I} \mathsf{A}\, \psi_i$ iff $\mathfrak{M} \Vdash_T \mathbin{\vee\!\!\!\vee}_{i \in I} \psi_i$

Proof. The equivalence is verified as follows: $\mathfrak{M} \Vdash \bigvee_{i \in I} \mathsf{A}\, \psi_i$ iff

there exists $i \in I$ such that $[\![\psi_i]\!]_{\mathfrak{M}} = W$
iff there exists $i \in I$ such that $\mathfrak{M}, w \Vdash \psi_i$ for all $w \in W$
iff there exists $i \in I$ such that $\mathfrak{M}, W \Vdash \psi_i$ (by Propositions 5.3 and 5.5)

and the last line is equivalent to $\mathfrak{M}, W \Vdash \mathbin{\vee\!\!\!\vee}_{i \in I} \psi_i$ hence $\mathfrak{M} \Vdash_T \mathbin{\vee\!\!\!\vee}_{i \in I} \psi_i$. □

Proposition 5.10 *For any class $\mathbb{F}$ of frames, the following are equivalent:*

(i) $\mathbb{F}$ *is definable by a set of disjunctive* A*-clauses.*

(ii) $\mathbb{F}$ *is definable by a set of formulas of intuitionistic inquisitive logic.*

Proof. We can establish the direction from (i) to (ii) by Proposition 5.9. The direction from (ii) to (i) follows from Propositions 5.8 and 5.9. □

By Propositions 5.10 and 3.3, we obtain the following frame definability results in the syntax of inquisitive intuitionistic logic.

Proposition 5.11 (i) $p \vee\!\!\!\vee \neg p$ *defines* $\forall\, x, y\, (xRy)$.

(ii) $(p \to q) \vee\!\!\!\vee (q \to p)$ *defines* $\forall\, x, y\, (xRy \text{ or } yRx)$.

(iii) $\neg p \vee\!\!\!\vee \neg\neg p$ *defines* $\forall\, x, y\, \exists\, z\, (xRz \text{ and } yRz)$.

By Proposition 5.10, we can also transfer the undefinability results from Proposition 3.8. For example, all the frame properties listed in Proposition 3.8 are also undefinable in the syntax of intuitionistic inquisitive logic.

Then, we can finally give GT-style characterization to intuitionistic inquisitive logic as follows.

Theorem 5.12 *An elementary frame class $\mathbb{F}$ is definable by a set of formulas of intuitionistic inquisitive logic iff $\mathbb{F}$ is closed under taking bounded morphic images, generated subframes and it reflects finitely generated subframes and prime filter extensions.*

Proof. By Proposition 5.10 and Theorem 4.3. □

6 Further Direction

There are several directions of further research. The first direction is that we may characterize relative frame definability of intuitionistic inquisitive logic within finite frames, as [19] did for modal dependence logic. For intuitionistic formulas, Rodenburg [16] provided a finitary version of Goldblatt-Thomason Theorem. The second direction is on model definability of both intuitionistic logic with the universal modality and intuitionistic inquisitive logic. Goldblatt [10] studied the characterization of intuitionistic definability of modal class. We may extend his result to this context.

As the final direction, we may define the notion of "normal form" of a formula of $\mathsf{Form}(\mathsf{A})$ in the spirit of [9]. Let us define $\mathsf{Form}(\mathsf{A}^+)$ as the set of all formulas φ in $\mathsf{Form}(\mathsf{A})$ such that all occurrences of A in φ are *positive*. For a formula in $\mathsf{Form}(\mathsf{A}^+)$, can we find an equivalent disjunctive A-clause via the normal form? A similar result held for modal logic with the universal modality as in [19]. This is ongoing work with Jonni Virtema.

References

[1] Chang, C. C. and H. J. Keisler, "Model Theory," North-Holland Publishing Company, Amsterdam, 1990, 3 edition.

[2] Ciardelli, I., *Dependency as question entailment*, in: S. Abramsky, J. Kontinen, J. Väänänen and H. Vollmer, editors, *Dependence Logic: Theory and Applications*, Springer, 2016 pp. 129–181.

[3] Ciardelli, I., *Questions as information types*, Synthese **195** (2018), pp. 321–365.

[4] Ciardelli, I., J. Groenendijk and F. Roelofsen, "Inquisitive Semantics," Oxford University Press, 2017.

[5] Ciardelli, I., R. Iemhoff and F. Yang, *Questions and dependency in intuitionistic logic*, Notre Dame Journal of Formal Logic **61** (2020), pp. 75–115.

[6] Ciardelli, I. A. and F. Roelofsen, *Inquisitive dynamic epistemic logic*, Synthese **192** (2015), pp. 1643–1687.

[7] Ebbing, J., L. Hella, A. Meier, J.-S. Müller, J. Virtema and H. Vollmer, *Extended modal dependence logic $\mathcal{EMDL}$*, in: L. Libkin, U. Kohlenbach and R. de Queiroz, editors, *Logic, Language, Information, and Computation. WoLLIC 2013*, Lecture Notes in Computer Science **8071** (2013), pp. 126–137.

[8] Ebbing, J. and P. Lohmann, *Complexity of model checking for modal dependence logic*, in: M. Bieliková, G. Friedrich, G. Gottlob, S. Katzenbeisser and G. Turán, editors, *SOFSEM 2012: Theory and Practice of Computer Science*, Lecture Notes in Computer Science **7147** (2012), pp. 226–237.

[9] Gargov, G. and V. Goranko, *Modal logic with names*, Journal of Philosophical Logic **22** (1993), pp. 607–636.

[10] Goldblatt, R., *Axiomatic classes of intuitionistic models*, Journal of Universal Computer Science **11** (2005), pp. 1945–1962.

[11] Goldblatt, R. I. and S. K. Thomason, *Axiomatic classes in propositional modal logic*, in: J. N. Crossley, editor, *Algebra and Logic*, Springer-Verlag, 1975 pp. 163–173.

[12] Kurz, A. and J. Rosický, *The Goldblatt-Thomason theorem for coalgebras*, in: T. Mossakowski, U. Montanari and M. Haveraaen, editors, *Algebra and Coalgebra in Computer Science. CALCO 2007*, Lecture Notes in Computer Science, **4624** (2007), pp. 342–355.

[13] Punčochář, V., *A generalization of inquisitive semantics*, Journal of Philosophical Logic **45** (2016), pp. 399–428.

[14] Punčochář, V., *Algebras of information states*, Journal of Logic and Computation **27** (2017), pp. 1643–1675.

[15] Punčochář, V., *Substructural inquisitive logics*, The Review of Symbolic Logic **12** (2019), pp. 296–330.

[16] Rodenburg, P. H., "Intuitionistic Correspondence Theory," Ph.D. thesis, Universiteit van Amsterdam (1986).

[17] Sano, K. and M. Ma, *Goldblatt-Thomason-style theorems for graded modal language*, in: L. Beklemishev, V. Goranko and V. Shehtman, editors, *Advances in Modal Logic 2010* (2010), pp. 330–349.

[18] Sano, K. and J. Virtema, *Characterizing relative frame definability in team semantics via the universal modality*, in: J. Väänänen, Åsa Hirvonen and R. de Queiroz, editors, *Logic, Language, Information, and Computation. WoLLIC 2016*, Lecture Notes in Computer Science **9803** (2016), pp. 392–409.

[19] Sano, K. and J. Virtema, *Characterising modal definability of team-based logics via the universal modality*, Annals of Pure and Applied Logic **170** (2019), pp. 1100–1127.

[20] Sano, K. and V. Virtema, *Characterizing frame definability in team semantics via the universal modality*, in: V. de Paiva, R. J. de Queiroz and L. S. Moss, editors, *Logic, Language, Information, and Computation. WoLLIC 2015*, Lecture Notes in Computer Science **9160** (2015), pp. 140–155.

[21] Sindoni, G., K. Sano and J. G. Stell, *Axiomatizing discrete spatial relations*, in: J. Desharnais, W. Guttmann and S. Joosten, editors, *Relational and Algebraic Methods in Computer Science. RAMiCS 2018.*, Lecture Notes in Computer Science **11194**, 2018, pp. 1–18.

[22] ten Cate, B., "Model theory for extended modal languages," Ph.D. thesis, University of Amsterdam, Institute for Logic, Language and Computation (2005).

[23] ten Cate, B., D. Gabelaia and D. Sustretov, *Modal languages for topology: expressivity and definability*, Annals of Pure and Applied Logic **159** (2008), pp. 146–170.

[24] Väänänen, J., *Modal dependence logic*, in: K. R. Apt and R. van Rooij, editors, *New Perspectives on Games and Interaction*, Texts in Logic and Games **4**, Amsterdam University Press, 2008 pp. 237–254.

[25] van Benthem, J., *Modal frame classes revisited*, Fundamenta Informaticae **18** (1993), pp. 303–317.

[26] Yang, F., "On Extensions and Variants of Dependence Logic," Ph.D. thesis, University of Helsinki (2014).