Algorithmic properties of first-order modal logics of the natural number line in restricted languages

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Abstract

We study algorithmic properties of first-order predicate monomodal logics of the frames $\langle \mathbb{N}, < \rangle$ and $\langle \mathbb{N}, \leqslant \rangle$ in languages with restrictions on the number of individual variables as well as the number and arity of predicate letters. The languages we consider have no constants, function symbols, or the equality symbol. We show that satisfiability for the logic of $\langle \mathbb{N}, < \rangle$ is Σ^1_1-hard in languages with two individual variables and two monadic predicate letters. We also show that satisfiability for the logic of $\langle \mathbb{N}, \leqslant \rangle$ is Σ^1_1-hard in languages with two individual variables, two monadic, and one 0-ary predicate letter. Thus, these logics are Π^1_1-hard, and therefore not recursively enumerable, in languages with the aforementioned restrictions. Similar results are obtained for the class of first-order predicate monomodal logics of frames $\langle \mathbb{N}, R \rangle$, where R is a binary relation between $<$ and $\leqslant$.

Keywords: first-order modal logic, predicate modal logic, restricted languages, decidability, undecidability, recursive enumerability, validity problem, satisfiability problem, Σ^1_1-hardness, Π^1_1-hardness, classification problem.

1 Introduction

The present paper aims to contribute to the understanding of the algorithmic properties of first-order predicate modal logics in languages with restrictions on

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⋆ This work has been supported by the Russian Foundation for Basic Research, project 18-011-00869.

the number of individual variables, as well as the number and arity of predicate letters.

Interest in the algorithmic properties of non-classical, mostly modal and superintuitionistic (intermediate), predicate logics in restricted languages is a natural outgrowth of the extensive research into the Classical Decision Problem [5]. The study of the Classical Decision Problem aims, in light of undecidability [10] of the classical first-order predicate logic **QCl**, to identify maximal decidable and minimal undecidable fragments of **QCl**, i.e., the decidable fragments that become undecidable when slightly extended and the undecidable fragments that become decidable when slightly restricted. A similar effort has more recently been made to better understand the borderline between the decidable and the undecidable in predicate modal and superintuitionistic logics, mostly by looking at the fragments obtained by limiting the number of individual variables, as well as the number and arity of predicate letters, allowed in the construction of formulas [28], [31], [33], [34], [13], [3], [15], [47], [27], [41], [36].

In the present paper, we attempt to identify the minimal computationally hard fragments of the predicate monomodal logics of the frames $\langle \mathbb{N}, < \rangle$ and $\langle \mathbb{N}, \leqslant \rangle$, i.e., the natural numbers with a natural, respectively, strict and partial order. Interest in these logics is motivated by at least three considerations.

First, these logics are algorithmically quite hard: even thought the exact complexity seems to be unknown, they are, as follows from Lemmas 3.1 and 4.1 below, Π^1_1-hard. Most research into the algorithmic properties of non-classical predicate logics, as can be seen from the references above, has dealt with (un)decidability. While it is natural that (un)decidability is the main concern in the study of the Classical Decision Problem, it is to be expected that predicate modal logics are computationally harder than **QCl**; therefore, research into their algorithmic properties should involve identifying minimal, in the above sense, fragments that are hard in certain classes of the arithmetical, or the analytical, hierarchy. The only study to date, as far as we know, of algorithmic properties of the fragments of not recursively enumerable monomodal predicate logics has been the investigation [36] of the fragments of not recursively enumerable [43], [39, Lemma 3.3] monomodal predicate logics of finite Kripke frames (as discussed in [42], both the logics of finite frames and the logics considered here fall into the category of "awkward" predicate modal logics based on essentially second-order Kripke semantics).

Similar questions have, however, been studied in the context of richer predicate languages containing multiple modal operator—most recently by I. Hodkinson, F. Wolter, and M. Zakharyaschev [24], [47] (see also [14, Chapter 11]; for earlier work, see [2], [45], [46], [1], and [32]). The methods used in this paper are partially inspired by [47, Theorem 2.3], where a Σ^1_1-hard tiling problem is encoded using a predicate language with two modal operators, one corresponding to an atomic accessibility relation and the other to the reflexive transitive closure of that relation. A similar result [24, Theorem 2] has been obtained for the temporal predicate logic of $\langle \mathbb{N}, \leqslant \rangle$, i.e., a predicate logic with two modal operators, one for the "immediate successor" relation on $\mathbb{N}$, the other for its reflexive transitive closure, the partial order $\leqslant$ (of course, both of these can be expressed with a single binary temporal opertator "until"). The novelty of the present work lies, first, in obtaining a similar encoding for languages with a single unary modal operator and, second, similarly to [41] and [36], in further reductions to languages with only two monadic predicate letters—the encodings used in [24, Theorem 2] and [47, Theorem 2.3] require an unlimited supply of monadic predicate letters.

Second, the logics considered here are determined by linear frames, i.e., frames with a restriction on the branching factor in the sense that we cannot freely append to a world of a frame another frame without breaking the structure of the original frame. Modelling, in languages of such logics, of predicate letters with a limited number thereof presents certain difficulties: the methods used in [41] and [36]—which can be traced back to, and inherit the limitations of, the propositional-level techniques used in [20], [7], [38], [37] and [40]—are inapplicable in this setting. On the other hand, the methods used in [4] do not seem to be readily applicable to logics of transitive frames, such as $\langle \mathbb{N}, < \rangle$ and $\langle \mathbb{N}, \leqslant \rangle$. In this respect, the method used here should be of relevance in the study of the algorithmic properties of monomodal logics of various kinds of structures—such as reflexive and irreflexive trees with a limited branching factor—where similar restrictions apply.

Third, the structure $\langle \mathbb{N}, < \rangle$ has long been considered to be a natural model of the flow of time (see, e.g., [19], [17]), and so interest in the algorithmic properties of the predicate modal logics of this structure is partially motivated by applications of first-order temporal logics [9], [8], [17], [24], [25], [23], [14, Chapter 11], [29], [11], [21]. Clearly, the negative results, like those presented here, obtained for languages whose expressive power is weaker than those of predicate temporal logic are directly relevant to that area.

The paper is structured as follows. In Section 2, we introduce the necessary preliminaries on predicate modal logic. In Section 3, we present our results on the logic of $\langle \mathbb{N}, < \rangle$. In Section 4, we present similar results on the logic of $\langle \mathbb{N}, \leqslant \rangle$. We conclude by discussing problems for future research in Section 5.

2 Preliminaries

In this section, we recall the standard definitions related to predicate modal logic, our aim being mainly to fix the terminology and notation used throughout the paper; the reader wishing more background on predicate modal logic may consult [26], [12], [18], [6], and [16].

An unrestricted first-order predicate modal language—as considered in this paper—contains countably many individual variables; countably many predicate letters of every arity, including zero (0-ary predicate letters are propositional variables); the propositional constant $\bot$ (falsity), the binary propositional connective $\to$, the unary modal connective $\Box$, and the quantifier $\forall$. Formulas, as well as the symbols $\neg$, $\vee$, $\wedge$, $\leftrightarrow$, $\exists$, and $\Diamond$, are defined in the usual way. We also use the following abbreviations, where $n \in \mathbb{N}$:

$$\Box^0\varphi = \varphi, \quad \Box^{n+1}\varphi = \Box\Box^n\varphi, \quad \Diamond^n\varphi = \neg\Box^n\neg\varphi,$$
$$\Box^+\varphi = \varphi \wedge \Box\varphi, \quad \Diamond^+\varphi = \varphi \vee \Diamond\varphi.$$

When parentheses are omitted, $\neg$, $\Box$, $\forall$, and $\exists$ are assumed to bind tighter than $\wedge$ and $\vee$, which are assumed to bind tighter than $\to$ and $\leftrightarrow$. We usually write atomic formulas, or atoms, in prefix notation; for some predicate letters, however, we use infix.

A *normal predicate modal logic* is a set of predicate modal formulas containing the validities of the classical predicate logic **QCl**, as well as the formulas of the form $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, and closed under predicate substitution, modus ponens, generalisation, and necessitation. [3]

We use the Kripke semantics to interpret predicate modal formulas.

A *Kripke frame* is a tuple $\mathfrak{F} = \langle W, R\rangle$, where W is a non-empty set of *worlds* and R is a binary *accessibility relation* on W. If wRv, we say that v *is accessible from* w or that w *sees* v. We say that v *is accessible from* w *in* k *steps*, for $k \geqslant 1$, if wR^kv, where R^k is the k-fold composition of R with itself.

A *predicate Kripke frame with expanding domains* is a tuple $\mathfrak{F}_D = \langle W, R, D\rangle$, where $\langle W, R\rangle$ is a Kripke frame and D is a function from W into the set of non-empty subsets of some set, *the domain of* $\mathfrak{F}_D$; the function D is required to satisfy the condition that wRw' implies $D(w) \subseteq D(w')$. We call the set $D(w)$ *the domain of* w. We often write D_w for $D(w)$. We also consider predicate frames satisfying the stronger condition that wRw' implies $D(w) = D(w')$; we call such frames *predicate frames with* (*locally*) *constant domains*. Whenever we say *predicate frame* simpliciter, we mean predicate frame with expanding domains.

A *Kripke model* is a tuple $\mathfrak{M} = \langle W, R, D, I\rangle$, where $\langle W, R, D\rangle$ is a predicate Kripke frame and I, called *the interpretation of predicate letters* with respect to worlds in W, is a function assigning to a world $w \in W$ and an n-ary predicate letter P an n-ary relation $I(w, P)$ on $D(w)$, i.e., $I(w, P) \subseteq D^n(w)$; in particular, if P is 0-ary, $I(w, P) \subseteq D^0(w) = \{\langle\rangle\}$. We often write $P^{I,w}$ for $I(w, P)$. We say that a model $\langle W, R, D, I\rangle$ is *based on* the frame $\langle W, R\rangle$ and is *based on* the predicate frame $\langle W, R, D\rangle$.

An *assignment* in a model is a function g associating with every individual variable x an element $g(x)$ of the domain of the underlying predicate frame. We write $g' \stackrel{x}{=} g$ to mean that assignment g' differs from assignment g in at most the value of x.

The truth of a formula φ at a world w of a model $\mathfrak{M}$ under an assignment g is defined inductively:

- $\mathfrak{M}, w \models^g P(x_1, \dots, x_n)$ if $\langle g(x_1), \dots, g(x_n)\rangle \in P^{I,w}$, where P is an n-ary predicate letter;

[3] The reader wishing a reminder of the definition of these closure conditions may consult [16, Definition 2.6.1]; for a detailed discussion of predicate substitution, see, e.g., [16, §2.3, §2.5].

- $\mathfrak{M}, w \not\models^g \bot$;
- $\mathfrak{M}, w \models^g \varphi_1 \to \varphi_2$ if $\mathfrak{M}, w \models^g \varphi_1$ implies $\mathfrak{M}, w \models^g \varphi_2$;
- $\mathfrak{M}, w \models^g \Box\varphi_1$ if wRw' implies $\mathfrak{M}, w' \models^g \varphi_1$;
- $\mathfrak{M}, w \models^g \forall x\, \varphi_1$ if $\mathfrak{M}, w \models^{g'} \varphi_1$, for every g' such that $g' \stackrel{x}{=} g$ and $g'(x) \in D_w$.

Notice that, given a Kripke model $\mathfrak{M} = \langle W, R, D, I\rangle$ and $w \in W$, the tuple $\mathfrak{M}_w = \langle D_w, I_w\rangle$, where $I_w(P) = I(w, P)$, is a classical predicate model.

Let $\mathfrak{M} = \langle W, R, D, I\rangle$ be a model, $w \in W$, and $a_1, \ldots, a_n \in D_w$; let also $\varphi(x_1, \ldots, x_n)$ be a formula whose free variables are among $x_1, \ldots, x_n$. We write $\mathfrak{M}, w \models \varphi(a_1, \ldots, a_n)$ to mean $\mathfrak{M}, w \models^g \varphi(x_1, \ldots, x_n)$, where $g(x_1) = a_1, \ldots, g(x_n) = a_n$. This notation is unambiguous since the languages we consider lack constants.

We say that a formula φ is *true at a world* w of a model $\mathfrak{M}$ (in symbols, $\mathfrak{M}, w \models \varphi$) if $\mathfrak{M}, w \models^g \varphi$, for every g assigning to the free variables of φ elements of D_w. We say that φ is *true in a model* $\mathfrak{M}$ (in symbols, $\mathfrak{M} \models \varphi$) if $\mathfrak{M}, w \models \varphi$, for every world w of $\mathfrak{M}$. We say that φ is *valid on a predicate frame* $\mathfrak{F}_D$ (in symbols, $\mathfrak{F}_D \models \varphi$) if φ is true in every model based on $\mathfrak{F}_D$. We say that φ is *valid on a frame* $\mathfrak{F}$ (in symbols, $\mathfrak{F} \models \varphi$) if φ is valid on every predicate frame $\langle \mathfrak{F}, D\rangle$. These notions, and the corresponding notation, can be extended to sets of formulas, in a natural way.

We write $w \models \varphi$, rather than $\mathfrak{M}, w \models \varphi$, when $\mathfrak{M}$ is clear from the context.

It is well known that the set of formulas valid on a class of frames is a normal predicate modal logic; this fact is sometimes referred to as soundness of Kripke semantics.

In this paper, we are mostly interested in the logics of frames $\langle \mathbb{N}, <\rangle$ and $\langle \mathbb{N}, \leqslant\rangle$; these logics are denoted, respectively, $L(\mathbb{N}, <)$ and $L(\mathbb{N}, \leqslant)$.

Observe that $L(\mathbb{N}, <) \not\subseteq L(\mathbb{N}, \leqslant)$ since $(\mathbb{N}, <) \models Z$ and $(\mathbb{N}, \leqslant) \not\models Z$, where $Z = \Box(\Box p \to p) \to (\Diamond\Box p \to \Box p)$. Also observe that $L(\mathbb{N}, \leqslant) \not\subseteq L(\mathbb{N}, <)$ since $(\mathbb{N}, \leqslant) \models \Box p \to p$ and $(\mathbb{N}, <) \not\models \Box p \to p$.

3 The first-order logic of $\langle \mathbb{N}, <\rangle$

In this section, we prove that satisfiability for the logic $L(\langle \mathbb{N}, <\rangle)$ is Σ^1_1-hard—hence, $L(\langle \mathbb{N}, <\rangle)$ is Π^1_1-hard, and therefore not recursively enumerable—in languages with two individual variables and two monadic predicate letters.

We do so by encoding the following recurrent tiling problem for $\mathbb{N} \times \mathbb{N}$, known to be Σ^1_1-complete [22].

We are given a set of tiles, a tile t being a 1×1 square, with a fixed orientation, whose edges are "colored" with *left*(t), *right*(t), *up*(t), and *down*(t). A tile type is fully determined by the edge colors. Every tile belongs to one of the finitely many types $T = \{t_0, \ldots, t_n\}$, there being an unlimited supply of tiles of each type. A *tiling* in an arrangement of tiles such that the edge colors of the adjacent tiles match, both horizontally and vertically. We are to determine whether there exists a tiling of an $\mathbb{N} \times \mathbb{N}$ grid, with tiles of the given types, such that a tile of type t_0 occurs infinitely often in the leftmost column. More precisely, we are to determine whether there exists a function $f : \mathbb{N} \times \mathbb{N} \to T$ such that, for every $n, m \in \mathbb{N}$,

(T_1) $\mathit{right}(f(n,m)) = \mathit{left}(f(n+1,m))$;

(T_2) $\mathit{up}(f(n,m)) = \mathit{down}(f(n,m+1))$;

(T_3) the set $\{n \in \mathbb{N} : f(0,n) = t_0\}$ is infinite.

The idea of the encoding we use is based on [14, Theorem 11.1] (see also [44], [30], [47], and [27]). To make the underlying idea clearer, we initially encode the recurrent tiling problem with predicate modal formulas of two individual variables, without regard for the number of predicate letters used; such a concern would complicate the formulas and, possibly, obfuscate their meaning. Subsequently, we eliminate all but two monadic predicate letters in the formulas obtained in the initial encoding.

Let $\triangleleft$ be a binary—while M and P_t, for every $t \in T$, monadic—predicate letters. Let (for brevity, we write l, r, u, and d rather than *left*, *right*, *up*, and *down*)

$$
\begin{aligned}
A_1 &= \forall x \exists y\, (x \triangleleft y);\\
A_2 &= \forall x \forall y\, [(x \triangleleft y \to \Box(x \triangleleft y)) \wedge (\neg(x \triangleleft y) \to \Box\neg(x \triangleleft y))];\\
A_3 &= \exists x\, M(x);\\
A_4 &= \forall x \forall y\, (x \triangleleft y \to \Box^+(M(x) \leftrightarrow \Diamond M(y) \wedge \neg\Diamond^2 M(y)));\\
A_5 &= \Box^+ \forall x\, [\bigvee_{t \in T} P_t(x) \wedge \bigwedge_{t' \neq t} (P_t(x) \to \neg P_{t'}(x))];\\
A_6 &= \forall x \forall y \bigwedge_{t \in T} [\Box^+(x \triangleleft y \wedge P_t(x) \to \bigvee_{r(t)=l(t')} P_{t'}(y))];\\
A_7 &= \forall x \forall y \bigwedge_{t \in T} [\Box^+(M(x) \wedge P_t(y) \to \Box(\exists y\, (x \triangleleft y \wedge M(y)) \to \bigvee_{u(t)=d(t')} P_{t'}(y)))];\\
A_8 &= \forall x\, (M(x) \to \Box\Diamond P_{t_0}(x)),
\end{aligned}
$$

Let A be a conjunction of formulas A_1 through A_8. Notice that A contains only two individual variables.

One may think of the relation represented by $x \triangleleft y$ as an "immediate successor" relation associated with a strict partial order. Then, A_2 says that this "order" is preserved throughout the frame. One may think of an element a of the domain D_w of the world w such that $w \models M(a)$ as "marking" the world w; so, we occasionally say that a is the mark of w. Then, formulas A_3 and A_4 can be understood as saying that every world in a model is "marked" with, as we shall see, a unique element of its domain and that the order of marks of successive worlds agrees with the relation $\triangleleft$. This, as we shall see, gives us an $\mathbb{N} \times \mathbb{N}$ grid whose rows correspond to the worlds of the frame $\langle \mathbb{N}, < \rangle$ and whose columns correspond to the (common) elements of the domains of the worlds. Building on this, formulas A_5 through A_8 describe a sought tiling of thus obtained grid. This is made precise in the following

$$
\begin{array}{rlccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
3 & M(3) & P_{f(0,3)}(0) & P_{f(1,3)}(1) & P_{f(2,3)}(2) & P_{f(3,3)}(3) & \cdots \\
2 & M(2) & P_{f(0,2)}(0) & P_{f(1,2)}(1) & P_{f(2,2)}(2) & P_{f(3,2)}(3) & \cdots \\
1 & M(1) & P_{f(0,1)}(0) & P_{f(1,1)}(1) & P_{f(2,1)}(2) & P_{f(3,1)}(3) & \cdots \\
0 & M(0) & P_{f(0,0)}(0) & P_{f(1,0)}(1) & P_{f(2,0)}(2) & P_{f(3,0)}(3) & \cdots \\
 & & 0 \quad \lhd & 1 \quad \lhd & 2 \quad \lhd & 3 & \cdots
\end{array}
$$

Fig. 1. Model $\mathfrak{M}_0$

Lemma 3.1 *There exists a recurrent tiling of* $\mathbb{N} \times \mathbb{N}$ *if, and only if,* $\langle \mathbb{N}, < \rangle \not\models \neg A$.

Proof. ("if") Suppose that $\mathfrak{M}, m \models A$, for some model $\mathfrak{M} = \langle \mathbb{N}, <, D, I \rangle$ and some $m \in \mathbb{N}$; we may assume without loss of generality that $m = 0$.

Since $0 \models A_3$, there exists $a_0 \in D_0$ such that $0 \models M(a_0)$.

Since $0 \models A_1$, there exists an infinite sequence $a_0, a_1, a_2, \ldots$ of elements of D_0 such that $a_0 \lhd^{I,0} a_1 \lhd^{I,0} a_2 \lhd^{I,0} \ldots$.

Since $0 \models A_2$, clearly, $a_0 \lhd^{I,n} a_1 \lhd^{I,n} a_2 \lhd^{I,n} \ldots$, for every $n \in \mathbb{N}$.

Since $0 \models A_4$, clearly, $n \models M(a_n)$, for every $n \in \mathbb{N}$.

We next show that $a_0, a_1, a_2, \ldots$ are pairwise distinct.

Suppose otherwise, i.e., let $a_i = a_{i+k}$, for some $i, k \in \mathbb{N}$. Then, as we have seen, $i \models M(a_i)$ and $i + k \models M(a_{i+k})$. Since $a_i = a_{i+k}$, we obtain $i + k \models M(a_i)$ and hence, by A_4, $i + k + 1 \models M(a_{i+1})$. Thus, $i \not\models M(a_i) \leftrightarrow \Diamond M(a_{i+1}) \wedge \neg\Diamond^2 M(a_{i+1})$, a contradiction.

Therefore, $w \models M(a_k)$ if, and only if, $w = k$.

Since $0 \models A_5$, for every $m, n \in \mathbb{N}$, there exists a unique $t \in T$ such that $m \models P_t(a_n)$. We can, therefore, define a function $f : \mathbb{N} \times \mathbb{N} \to T$ by

$$f(n, m) = t, \text{ where } t \text{ is such that } m \models P_t(a_n).$$

Since $0 \models A_6 \wedge A_7 \wedge A_8$, the function f satisfies (T_1) through (T_3). Observe that the subformula $\exists y\,(x \lhd y \wedge M(y))$ of A_7 ensures that a vertically matching tile t' is placed right on top of the current tile t.

Therefore, f is a recurrent tiling of $\mathbb{N} \times \mathbb{N}$ with T.

("only if") Suppose that f is a function satisfying (T_1) through (T_3). We define a model $\mathfrak{M}_0$, based on $\langle \mathbb{N}, < \rangle$, satisfying A.

To define $\mathfrak{M}_0$, let $D_n = \mathbb{N}$, for every $n \in \mathbb{N}$, and let I be an interpretation function such that, for every $n \in \mathbb{N}$,

- $n \models k \lhd l \quad \leftrightharpoons \quad l = k + 1;$
- $n \models M(k) \leftrightharpoons k = n;$
- $n \models P_t(k) \leftrightharpoons f(k, n) = t.$

Finally, let $\mathfrak{M}_0 = \langle \mathbb{N}, <, D, I \rangle$ (see Figure 1).

It is straightforward to check that $\mathfrak{M}_0, 0 \models A$. $\Box$

Thus, in the proof of the "if" part of Lemma 3.1, we obtained a grid for the tiling by treating the worlds of the model $\mathfrak{M}$ as rows and elements $a_0, a_1, a_2, \ldots$ of the domain D_0 of world 0 as columns.

We now make, in the following remarks, a few observations about those properties of the model $\mathfrak{M}_0$ defined in the proof of the "only if" part of Lemma 3.1 that we will rely on later on.

Remark 3.2 The model $\mathfrak{M}_0$ defined in the "only if" part of the proof of Lemma 3.1 is based on a predicate frame with a constant domain; even though this domain is $\mathbb{N}$, we denote it by $\mathcal{D}$ when we wish to emphasize that we are talking about the domain, rather than the set of worlds, of $\mathfrak{M}_0$.

Remark 3.3 In the model $\mathfrak{M}_0$ defined in "only if" part of the proof Lemma 3.1, the valuation of the binary predicate letter $\lhd$ is the same at every world.

Remark 3.4 In the model $\mathfrak{M}_0$ defined in "only if" part of the proof of Lemma 3.1, each world is marked with a unique element of the common domain, i.e., for every $w \in \mathbb{N}$, there exists a unique $a \in \mathcal{D}$ such that $w \models M(a)$.

We next eliminate, in a satisfiability-preserving way, all but two monadic predicate letters of the formula A, without increasing the number of individual variables in the resultant formula; we, thus, obtain a reduction of the $\mathbb{N} \times \mathbb{N}$ recurrent tiling problem to satisfiability in $L(\langle \mathbb{N}, < \rangle)$ in languages with two individual variables and two monadic predicate letters.

The elimination of predicate letters is carried out in two steps: first, we model the binary letter $\lhd$ with two monadic ones, obtaining formula A'; then, we model all the monadic letters of A' except M with a single monadic letter, thus obtaining a formula with only two monadic predicate letters and two individual variables.

From now on, we assume, for ease of notation, that A contains monadic predicate letters $P_0, \ldots, P_n$—rather than P_t, for $t \in \{t_0, \ldots, t_n\}$—to refer to the tile types.

First, following ideas of Kripke's [28], we eliminate, in a satisfiability-preserving way, the binary predicate letter $\lhd$ of A, without increasing the number of individual variables in the resultant formula.

Recall that Kripke's construction [28] transforms a model $\mathfrak{M}$ satisfying a formula containing a binary predicate letter, and no modal operators, at a world w in such a way that a sufficiently large number of worlds is added to $\mathfrak{M}$. More precisely, for every pair $\langle a, b \rangle$ of elements of the domain of w, a fresh world is introduced to $\mathfrak{M}$. This construction cannot be applied in a straightforward way in our setting, for two reasons.

Since we are restricted to the frame $\langle \mathbb{N}, < \rangle$, we may not introduce fresh worlds to a model satisfying A; we, rather, have to use the worlds of $\langle \mathbb{N}, < \rangle$ to simulate $\lhd$. Moreover, since $\lhd$ occurs within the scope of the modal operator $\Box$

in A, we need to simulate the valuation of $\lhd$ at every world of the model, not just at the world satisfying A.

We resolve these difficulties by using the fact that A is satisfied in the model $\mathfrak{M}_0$ defined in the "only if" part of the proof of Lemma 3.1 and drawing on the special properties of $\mathfrak{M}_0$—that, as noted in Remark 3.2, it is based on a predicate frame with a constant domain and that, as noted in Remark 3.3, the valuation of $\lhd$ is the same at every world of $\mathfrak{M}_0$.

Let P_{n+1} and P_{n+2} be monadic predicate letters distinct from $M, P_0, \dots, P_n$ and from each other, and let

$$\mu = \exists x\, M(x).$$

Lastly, let A' be the result of substituting $\Diamond(\mu \wedge P_{n+1}(x) \wedge P_{n+2}(y))$ for $x \lhd y$ in A.

Lemma 3.5 *There exists a recurrent tiling of* $\mathbb{N} \times \mathbb{N}$ *if, and only if,* $\langle \mathbb{N}, < \rangle \not\models \neg A'$.

Proof. ("if") This part is argued almost exactly as in the proof of Lemma 3.1, the only difference being that $\Diamond(\mu \wedge P_{n+1}(x) \wedge P_{n+2}(y))$ plays the role of the atom $x \lhd y$.

("only if") Suppose f is a function satisfying conditions (T_1) through (T_3). Let $\mathfrak{M}_0$ be the model defined in "if" part of the proof of Lemma 3.1. As we have seen there, $\mathfrak{M}_0, 0 \models A$. We use $\mathfrak{M}_0$ to define a model $\mathfrak{M}'_0$ satisfying A'.

Let $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a fixed enumeration of the pairs of natural numbers, thought of as elements of the domain $\mathcal{D}$ (i.e., we seek an enumeration of $\mathcal{D} \times \mathcal{D}$). Let α be the infinite sequence of natural numbers

$$0,\ 0,1,\ 0,1,2,\ 0,1,2,3,\ 0,1,2,3,4,\ \dots$$

and let α_k be the kth element of α.

To define $\mathfrak{M}'_0$, we use the predicate frame $\langle \mathbb{N}, <, D \rangle$ underlying the model $\mathfrak{M}_0$, together with the interpretation function I' defined as follows: for $w, a, b, c \in \mathbb{N}$,

$$\mathfrak{M}'_0, w \models P_{n+1}(c) \;\leftrightharpoons\; c = a \text{ and } \mathfrak{M}_0, 0 \models a \lhd b \text{ and } h(\alpha_w) = \langle a, b \rangle;$$
$$\mathfrak{M}'_0, w \models P_{n+2}(c) \;\leftrightharpoons\; c = b \text{ and } \mathfrak{M}_0, 0 \models a \lhd b \text{ and } h(\alpha_w) = \langle a, b \rangle;$$

and

$$\mathfrak{M}'_0, w \models S(c) \leftrightharpoons \mathfrak{M}_0, w \models S(c), \quad \text{for } S \in \{P_0, \dots, P_n, M\}.$$

Finally, let $\mathfrak{M}'_0 = \langle \mathbb{N}, <, D, I' \rangle$.

We prove that $\mathfrak{M}'_0, 0 \models A'$.

Since $\mathfrak{M}_0, 0 \models A$, if suffices to show that $\mathfrak{M}_0, w \models^g x \lhd y$ if, and only if, $\mathfrak{M}'_0, w \models^g \Diamond(\mu \wedge P_{n+1}(x) \wedge P_{n+2}(y))$, for every $w \in \mathbb{N}$ and every g.

Assume $\mathfrak{M}_0, w \models^g x \lhd y$. By definition of $\mathfrak{M}_0$ (see also Remark 3.3), $\mathfrak{M}_0, 0 \models^g x \lhd y$. Let $v \in \mathbb{N}$ be such that $w < v$ and $h(\alpha_v) = \langle g(x), g(y) \rangle$; it is evident from the definition of α that such a v exists. By definition,

$\mathfrak{M}_0', v \models^g P_{n+1}(x) \wedge P_{n+2}(y)$. Since $w < v$ and since, as can be easily checked, $\mathfrak{M}_0', w \models \mu$, we obtain $\mathfrak{M}_0', w \models^g \Diamond(\mu \wedge P_{n+1}(x) \wedge P_{n+2}(y))$.

Conversely, assume $\mathfrak{M}_0', w \models^g \Diamond(\mu \wedge P_{n+1}(x) \wedge P_{n+2}(y))$, and hence $\mathfrak{M}_0', v \models^g P_{n+1}(x) \wedge P_{n+2}(y)$, for some v such that $w < v$. By definition, $\mathfrak{M}_0, 0 \models^g x \lhd y$. Thus, by definition of $\mathfrak{M}_0$ (see also Remark 3.3), $\mathfrak{M}_0, w \models^g x \lhd y$. $\square$

We lastly model, in a satisfiability-preserving way, the occurrences of predicate letters $P_0, \dots, P_{n+2}$ in A' with a single monadic letter P, without increasing the number of individual variables in the resultant formula. We, thus, obtain a reduction of the recurrent tiling problem using formulas with only two individual variables and only two monadic predicate letters, M and P.

Let P be a monadic predicate letter distinct from $P_0, \dots, P_{n+2}, M$, and let, for $k \in \{0, \dots, n+2\}$,

$$\begin{aligned}\beta_k(x) \;=\; \mu \wedge \exists y\, [&\Diamond^{n+4} M(y) \wedge \neg\Diamond^{n+5} M(y) \wedge \\ &\Diamond(\Diamond^{k+1} M(y) \wedge \neg\Diamond^{k+2} M(y) \wedge P(x))];\\ \beta_k(y) \;=\; \mu \wedge \exists x\, [&\Diamond^{n+4} M(x) \wedge \neg\Diamond^{n+5} M(x) \wedge \\ &\Diamond(\Diamond^{k+1} M(x) \wedge \neg\Diamond^{k+2} M(x) \wedge P(y))].\end{aligned}$$

Let $\cdot^*$ be the function replacing $P_k(x)$ with $\beta_k(x)$ and $P_k(y)$ with $\beta_k(y)$, for $k \in \{0, \dots, n+2\}$.

Let A_i^*, where $1 \leqslant i \leqslant 8$ and $i \neq 4$, be the result of applying the function $\cdot^*$ to the formula A_i' and let

$$\begin{aligned}A_4^{\#} \;=\; \forall x \forall y\, (&\Diamond(\beta_{n+1}(x) \wedge \beta_{n+2}(y)) \;\to \\ &\Box(M(x) \leftrightarrow \Diamond^{n+4} M(y) \wedge \neg\Diamond^{n+5} M(y))).\end{aligned}$$

Finally, let

$$A^* = A_1^* \wedge A_2^* \wedge A_3^* \wedge A_4^{\#} \wedge A_5^* \wedge A_6^* \wedge A_7^* \wedge A_8^*.$$

To define a model satisfying A^*, provided a recurrent tiling of $\mathbb{N} \times \mathbb{N}$ exists, we take the model $\mathfrak{M}_0'$ defined in the "only if" part of the proof of Lemma 3.5 and, intuitively, stretch it out to include "additional" worlds whose sole purpose is to simulate the valuation of the predicate letters $P_0, \dots, P_{n+2}$ at worlds of $\mathfrak{M}_0'$: $n+3$ worlds are "inserted" between m and $m+1$ to simulate the valuation of letters $P_0, \dots, P_{n+2}$ at m. The valuation of P_k, where $k \in \{0, \dots, n+2\}$, at m is simulated by the valuation of P at a "newly inserted" intermediate world k steps away from $m+1$ (see Figure 2, where $\beta_{f(a,b)}(x)$ stands for $\beta_k(x)$ such that $f(a,b) = t_k$). This is made precise in the following

Lemma 3.6 *There exists a recurrent tiling of* $\mathbb{N} \times \mathbb{N}$ *if, and only if,* $\langle \mathbb{N}, < \rangle \not\models \neg A^*$.

Proof. ("if") This part is argued as before, the only difference being that $\beta_k(x)$ and $\beta_k(y)$ are used instead of the atoms $P_k(x)$ and $P_k(y)$.

("only if") Suppose f is a function satisfying (T_1) through (T_3). Let $\mathfrak{M}_0'$ be the model defined in the "only if" part of the proof of Lemma 3.5. As we have seen there, $\mathfrak{M}_0', 0 \models A'$. We use $\mathfrak{M}_0'$ to define a model $\mathfrak{M}_0^*$ satisfying A^*.

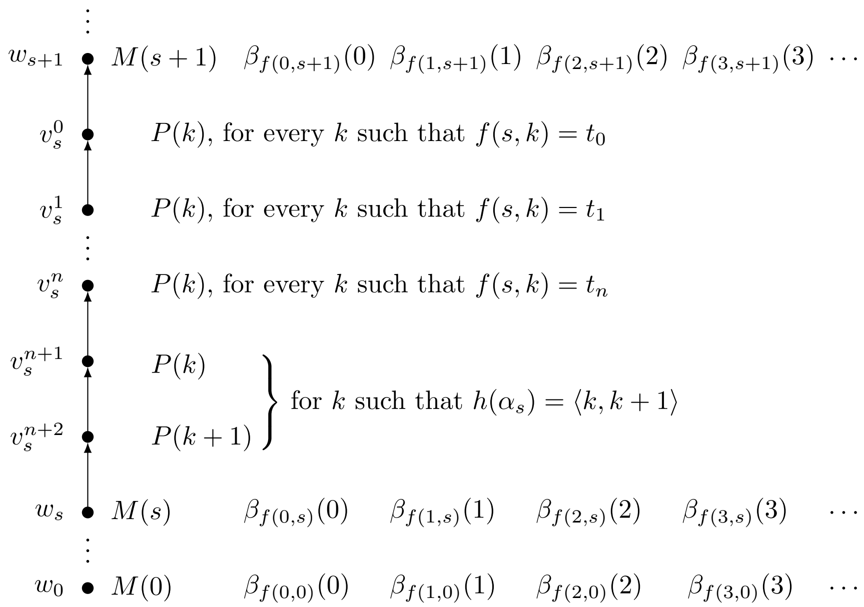

Fig. 2. Model $\mathfrak{M}_0^*$

Think of the worlds of $\mathfrak{M}_0^*$ as being labeled, in the ascending order,

$$w_0, v_0^{n+2}, \ldots, v_0^0, w_1, v_1^{n+2}, \ldots, v_1^0, w_2, v_2^{n+2}, \ldots, v_2^0, w_3, \ldots$$

(i.e., $w_0 = 0$, $v_0^{n+2} = 1$, etc.). Let, as before, $D_w = \mathcal{D} = \mathbb{N}$, for every $w \in \mathbb{N}$. Define the interpretation function I^* on the predicate frame $\langle \mathbb{N}, <, D \rangle$ underlying $\mathfrak{M}_0'$ by

$$\mathfrak{M}_0^*, x \models M(a) \leftrightharpoons x = w_m \text{ and } \mathfrak{M}_0', m \models M(a), \text{ for some } m \in \mathbb{N};$$

$$\mathfrak{M}_0^*, x \models P(a) \leftrightharpoons x = v_m^k \text{ and } \mathfrak{M}_0', m \models P_k(a), \text{ for some } m \in \mathbb{N} \text{ and } k \in \{0, \ldots, n+2\}.$$

We prove that $\mathfrak{M}_0^*, w_0 \models A^*$.

First, we show that, for every $s \in \mathbb{N}$, $k \in \{0, \ldots, n+2\}$, and g,

$$(1)\ \mathfrak{M}_0', s \models^g P_k(x) \iff \mathfrak{M}_0^*, w_s \models^g \beta_k(x);$$

$$(2)\ \mathfrak{M}_0', s \models^g P_k(y) \iff \mathfrak{M}_0^*, w_s \models^g \beta_k(y).$$

Assume $\mathfrak{M}_0', s \models^g P_k(x)$. As we have seen in the proof of Lemma 3.1 (see also Remark 3.4), for every world w in $\mathfrak{M}_0$, there exists a unique $a \in \mathcal{D}$ such that $\mathfrak{M}_0, w \models M(a)$.

Then, $\mathfrak{M}_0', s \models M(a)$, for some unique $a \in \mathcal{D}$; hence, by definition, $\mathfrak{M}_0^*, w_s \models M(a)$. Therefore, $\mathfrak{M}_0^*, w_s \models \mu$.

Similarly, $\mathfrak{M}'_0, s+1 \models M(b)$, for some unique $b \in \mathcal{D}$, and so, by definition, $\mathfrak{M}^*_0, w_{s+1} \models M(b)$. Observe that, due to uniqueness of b for $s+1$, if $t \neq s+1$, then $\mathfrak{M}^*_0, w_t \not\models M(b)$.

By definition, in $\mathfrak{M}^*_0$,

- w_{s+1} is accessible from w_s in $n+4$ steps;
- w_{s+1} is not accessible from w_s in $n+5$ steps;
- $w_s < v^k_s$;
- w_{s+1} is accessible from v^k_s in $k+1$ steps;
- w_{s+1} is not accessible from v^k_s in $k+2$ steps;
- $\mathfrak{M}^*_0, v^k_s \models^g P(x)$.

Therefore, $\mathfrak{M}^*_0, w_s \models^g \beta_k(x)$.

Conversely, assume $\mathfrak{M}^*_0, w_s \models^g \beta_k(x)$. Then, it is immediate from the definition of $\mathfrak{M}^*_0$ that $\mathfrak{M}'_0, s \models^g P_k(x)$.

This proves (1). The argument for (2) is analogous.

From (1) and (2) we immediately obtain $\mathfrak{M}^*_0, w_0 \models A^*_i$, where $1 \leqslant i \leqslant 8$ and $i \neq 4$. It is, moreover, straightforward to check, given (1) and (2), that $\mathfrak{M}^*_0, w_0 \models A^{\#}_4$. Therefore, $\mathfrak{M}^*_0, w_0 \models A^*$. □

We, thus, obtain (the reader wishing a reminder of the basic concepts of computability theory may consult [35])

Theorem 3.7 *Satisfiability for $L(\langle \mathbb{N}, < \rangle)$ is Σ^1_1-hard in languages with two individual variables and two monadic predicate letters.*

Proof. Immediate from Lemma 3.6. □

Thus, $L(\langle \mathbb{N}, < \rangle)$ is not recursively enumerable in such languages:

Theorem 3.8 *The logic $L(\langle \mathbb{N}, < \rangle)$ is Π^1_1-hard in languages with two individual variables and two monadic predicate letters.*

Proof. Immediate from Theorem 3.7. □

4 The first-order logic of $\langle \mathbb{N}, \leqslant \rangle$

We now modify the argument of the preceding section to prove Σ^1_1-hardness of satisfiability for the predicate monomodal logic of $\langle \mathbb{N}, \leqslant \rangle$ in languages with two individual variables, two monadic, and a single 0-ary predicate letter. It follows that the logic of $\langle \mathbb{N}, \leqslant \rangle$ is Π^1_1-hard, and therefore not recursively enumerable, in such languages. In comparison with languages considered in the previous section, we need an additional 0-ary predicate letter to deal with reflexivity.

As before, let $\lhd$ be a binary—while M and P_t, for every $t \in T$, monadic—predicate letters, and let p be a 0-ary predicate letter (i.e., a propositional variable). Given a formula φ in such a language, define

$$\Diamond\!\!\!\Diamond \varphi = \Diamond(\neg p \wedge \Diamond(p \wedge \varphi));$$

$$\Diamond\!\!\!\Diamond^0 \varphi = \varphi, \quad \Diamond\!\!\!\Diamond^{k+1} \varphi = \Diamond\!\!\!\Diamond \Diamond\!\!\!\Diamond^k \varphi.$$

Let

$$A_4^r = \forall x \forall y\, (x \lhd y \to \Box(M(x) \leftrightarrow p \wedge \Diamond\!\!\!\Diamond M(y) \wedge \neg \Diamond\!\!\!\Diamond^2 M(y)),$$

$$A_8^r = \forall x\, (M(x) \to \Box \Diamond\!\!\!\Diamond P_{t_0}(x)),$$

and let $A^r = A_1 \wedge A_2 \wedge A_3 \wedge A_4^r \wedge A_5 \wedge A_6 \wedge A_7 \wedge A_8^r$.

The operator $\Diamond\!\!\!\Diamond$ forces a transition to a different world when valuating a formula $\Diamond\!\!\!\Diamond\varphi$, just as $\Diamond$ does in the absence of reflexivity.

Lemma 4.1 *There exists a recurrent tiling of* $\mathbb{N} \times \mathbb{N}$ *if, and only if,* $\langle \mathbb{N}, \leqslant \rangle \not\models \neg A^r$.

Proof. ("if") Suppose $\mathfrak{M}, m \models A^r$, for some model $\mathfrak{M} = \langle \mathbb{N}, \leqslant, D, I \rangle$ and some $m \in \mathbb{N}$; we may assume without a loss of generality that $m = 0$.

Since $0 \models A_3$, there exists $a_0 \in D_0$ such that $0 \models M(a_0)$.

Since $0 \models A_1$, there exists an infinite sequence $a_0, a_1, a_2, \dots$ of elements of D_0 such that $a_0 \lhd^{I,0} a_1 \lhd^{I,0} a_2 \lhd^{I,0} \dots$.

Since $0 \models A_2$, clearly, $a_0 \lhd^{I,n} a_1 \lhd^{I,n} a_2 \lhd^{I,n} \dots$, for every $n \in \mathbb{N}$.

Since $0 \models A_4^r$, the following holds: $w \models M(a_k)$ if, and only if, $w \models p$ and there exist $w', w'' \in \mathbb{N}$ such that $w \leqslant w' \leqslant w''$ and $w' \not\models p$ and $w'' \models p \wedge M(a_{k+1})$. Observe that the valuation of p guarantees that $w < w' < w''$. Also observe that, if $w'' \leqslant v \leqslant v'$ and $v \not\models p$ and $v' \models p$, then $v' \not\models M(a_{k+1})$. Thus, a mark of the world changes once we have passed through a world refuting p.

For every $k \in \mathbb{N}$, let w_k be, for definiteness' sake, the least world (number) such that $w_k \models M(a_k)$. Observe that, since $0 \models A_5$, for every $m, n \in \mathbb{N}$ there exists a unique $t \in T$ such that $w_m \models P_t(a_n)$. Therefore, we can define a function $f : \mathbb{N} \times \mathbb{N} \to T$ by

$$f(n, m) = t, \text{ where } t \text{ is such that } w_m \models P_t(a_n).$$

Since $0 \models A_6 \wedge A_7 \wedge A_8^r$, conditions (T_1) through (T_3) are satisfied for f. Therefore, f is a recurrent tiling of $\mathbb{N} \times \mathbb{N}$ with T.

("only if") Suppose f is a function satisfying (T_1) through (T_3). We define a model $\mathfrak{M}_0$, based on $\langle \mathbb{N}, \leqslant \rangle$, satisfying A^r.

To define $\mathfrak{M}_0$, let $D_n = \mathbb{N}$, for every $n \in \mathbb{N}$, and let I be an interpretation function such that, for every $n \in \mathbb{N}$,

- $n \models k \lhd l \;\;\leftrightharpoons\;\; l = k + 1$;
- $n \models p \;\;\leftrightharpoons\;\; n = 2m$, for some m;
- $n \models M(k) \;\;\leftrightharpoons\;\; n = 2k$;
- $n \models P_t(k) \;\;\leftrightharpoons\;\; n = 2m$ and $f(k, m) = t$.

Finally, let $\mathfrak{M}_0 = \langle \mathbb{N}, <, D, I \rangle$.

It is straightforward to check that $\mathfrak{M}_0, 0 \models A^r$. $\Box$

Remark 4.2 Observe that Remarks 3.3 and 3.4 apply to those worlds of the model $\mathfrak{M}_0$ defined in the "only if" part of the proof of the Lemma 4.1 where p is true. Also observe that $\mathfrak{M}_0$ is based on a predicate frame with a constant domain.

We next eliminate, in a satisfiability-preserving way, all but two monadic predicate letters of the formula A^r, without increasing the number of individual variables in the resultant formula. As in the preceding section, this is done in two steps. We assume, for convenience, that A^r contains monadic predicate letters $P_0, \ldots, P_n$, rather than P_t, for $t \in \{t_0, \ldots, t_n\}$, to refer to the tile types.

Let P_{n+1} and P_{n+2} be monadic predicate letters distinct form $M, P_0, \ldots, P_n$ and from each other. Let formula μ be defined as before. Lastly, let $(A^r)'$ be the result of substituting $\diamondsuit(\mu \wedge P_{n+1}(x) \wedge P_{n+2}(y))$ for $x \lhd y$ in A^r.

Lemma 4.3 *There exists a recurrent tiling of* $\mathbb{N} \times \mathbb{N}$ *if, and only if,* $\langle \mathbb{N}, \leqslant \rangle \not\models \neg(A^r)'$.

Proof. Similar to the proof of Lemma 3.5. In the proof of the "only if" part, we only simulate the valuation of $\lhd$ at the worlds where p is true—or, equivalently, the worlds w such that $w \models M(a)$, for some a. Therefore, instead of the enumeration h, we use an enumeration of such worlds only. Once this change is made, we proceed as in the proof of Lemma 3.5. □

We, lastly, eliminate all but two monadic predicate letters of $(A^r)'$. Let P be a monadic predicate letter distinct from $P_0, \ldots, P_{n+2}, M$, and let, for $k \in \{0, \ldots, n+2\}$,

$$\begin{aligned} \gamma_k(x) \;&=\; \mu \wedge \exists y\, [\diamondsuit^{n+4} M(y) \wedge \neg\diamondsuit^{n+5} M(y) \wedge \\ &\qquad \diamondsuit(\diamondsuit^{k+1} M(y) \wedge \neg\diamondsuit^{k+2} M(y) \wedge P(x))]; \\ \gamma_k(y) \;&=\; \mu \wedge \exists x\, [\diamondsuit^{n+4} M(x) \wedge \neg\diamondsuit^{n+5} M(x) \wedge \\ &\qquad \diamondsuit(\diamondsuit^{k+1} M(x) \wedge \neg\diamondsuit^{k+2} M(x) \wedge P(y))], \end{aligned}$$

Let $\cdot^*$ be the function replacing $P_k(x)$ with $\gamma_k(x)$ and $P_k(y)$ with $\gamma_k(y)$, for $k \in \{0, \ldots, n+2\}$, in $(A^r)'$.

Let $(A^r_i)^*$, where $1 \leqslant i \leqslant 8$ and $i \neq 4$, be the result of applying the function $\cdot^*$ to $(A^r_i)'$ and let

$$\begin{aligned} (A^r_4)^{\#} \;=\; \forall x \forall y\, (&\diamondsuit(\gamma_{n+1}(x) \wedge \gamma_{n+2}(y)) \;\rightarrow \\ &\Box(M(x) \leftrightarrow \diamondsuit^{n+4}\, M(y) \wedge \neg\diamondsuit^{n+5}\, M(y))). \end{aligned}$$

Finally, let

$$(A^r)^* = (A^r_1)^* \wedge (A^r_2)^* \wedge (A^r_3)^* \wedge (A^r_4)^{\#} \wedge (A^r_5)^* \wedge (A^r_6)^* \wedge (A^r_7)^* \wedge (A^r_8)^*.$$

Lemma 4.4 *There exists a recurrent tiling of* $\mathbb{N} \times \mathbb{N}$ *if, and only if,* $\langle \mathbb{N}, \leqslant \rangle \not\models \neg(A^r)^*$.

Proof. Similar to the proof of Lemma 3.6.

We take the model obtained in the "only if" part of the proof of Lemma 4.3 and, essentially, apply to it the construction used in the proof of Lemma 3.6, the only difference being that we make letter p true at the worlds that we "added" in Lemma 3.6 and "insert" an extra world refuting p in-between every pair of such worlds that are adjacent. □

We, thus, obtain

Theorem 4.5 *Satisfiability for* $L(\langle \mathbb{N}, \leqslant \rangle)$ *is* Σ^1_1*-hard in languages with two individual variables, two monadic predicate letters, and a single* 0*-ary predicate letter.*

Proof. Immediate from Lemma 4.4. □

Thus, $L(\langle \mathbb{N}, \leqslant \rangle)$ is not recursively enumerable in such languages:

Theorem 4.6 *The logic* $L(\langle \mathbb{N}, \leqslant \rangle)$ *is* Π^1_1*-hard in languages with two individual variables, two monadic predicate letters, and a single* 0*-ary predicate letter.*

Proof. Immediate from Theorem 4.5. □

5 Discussion

Observe that the results of Section 4 can be easily extended to predicate monomodal logics of frames $\langle \mathbb{N}, R \rangle$ where R is a binary relation between $<$ and $\leqslant$: given any such logic L, we reduce the recurrent tiling problem to satisfiability for L by applying to the formulas defined in Section 4 the translation replacing occurrences of the modal operator $\Box$ with those of $\Box^+$.

Also observe that we have never relied on the domains of the predicate frames we have been dealing with to be not equal; therefore, all of our results apply to the logics of predicate frames with constant domains.

Lastly, observe that our results apply to the first-order temporal logic of $\langle \mathbb{N}, \leqslant \rangle$, which is essentially the first-order linear time temporal logic **QLTL**.

The results presented here raise the following questions.

The first is whether the results presented here can be strengthened to languages with one fewer predicate letter. Both in [41] and in [36] we have been able to prove undecidability and Σ^0_1-hardness results for languages with a single monadic predicate letter. We conjecture that the logic of $\langle \mathbb{N}, < \rangle$ is Π^1_1-hard in languages with two individual variables and a single monadic predicate letter. If the conjecture is correct, an analogous result for $\langle \mathbb{N}, \leqslant \rangle$, at worst with an additional 0-ary predicate letter, should follow.

The second is whether analogous results can be obtained for the superintuitionistic logic of the frame $\langle \mathbb{N}, \leqslant \rangle$. Given that the accessibility relation in $\langle \mathbb{N}, \leqslant \rangle$ is reflexive and transitive, the only, by not means trivial, hurdle to clear is obtaining a model with a hereditary valuation. Whether this can be done is unclear to us, given the difficulty of modelling the changing values of the tile types on a linear frame with a hereditary valuation.

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