

Non-iterative Modal Logics are Coalgebraic

Jonas Forster and Lutz Schröder

Friedrich-Alexander-Universität Erlangen-Nürnberg

Abstract

A modal logic is *non-iterative* if it can be defined by axioms that do not nest modal operators, and *rank-1* if additionally all propositional variables in axioms are in scope of a modal operator. It is known that every syntactically defined rank-1 modal logic can be equipped with a canonical coalgebraic semantics, ensuring soundness and strong completeness. In the present work, we extend this result to non-iterative modal logics, showing that every non-iterative modal logic can be equipped with a canonical coalgebraic semantics defined in terms of a copointed functor, again ensuring soundness and strong completeness via a canonical model construction. Like in the rank-1 case, the coalgebraic semantics is equivalent to a neighbourhood semantics with suitable frame conditions, so that our main result may be phrased as saying that every non-iterative modal logic is complete over its neighbourhood semantics. As an example application of these results, we establish strong completeness of deontic logics with factual detachment, strengthening previous weak completeness results.

Keywords: Coalgebraic logic, neighbourhood semantics, strong completeness, canonical models, deontic logic

1 Introduction

Modal frame axioms are called *non-iterative* if they do not nest modal operators, and *rank-1* if additionally all occurrences of propositional variables are under modal operators; logics are non-iterative or rank-1, respectively, if they can be axiomatized by axioms of the correspondingly restricted shape. Prominent examples include the K-axiom $\Box(a \rightarrow b) \rightarrow \Box a \rightarrow \Box b$, which is rank-1, and the T-axiom $\Box a \rightarrow a$, which is non-iterative. Lewis [10] shows that every non-iterative modal logic is *weakly* complete over neighbourhood semantics (i.e. every consistent formula is satisfiable). Without restrictions on the axioms, there are modal logics that are weakly complete but not strongly complete (in the sense that every consistent set of formulae is satisfiable) over neighbourhood semantics [18]. Previous work in coalgebraic logic [17] shows that every rank-1 modal logic is even strongly complete over a canonical coalgebraic semantics that coincides with neighbourhood semantics. In the present paper, we extend this result to non-iterative logics: We show that every (syntactically given) non-iterative modal logic is strongly complete over a canonical coalgebraic semantics, and hence over the equivalent neighbourhood semantics.

Generally, the semantic framework of *coalgebraic logic* [2] supports general proof-theoretic, algorithmic, and meta-theoretic results that can be instantiated to the logic of interest, cutting out much of the repetitive labour associated with the iterative process of designing an application-specific logic. The framework is based on casting state-based models of various types (e.g. relational, probabilistic, neighbourhood-based, or game-based) as coalgebras for a functor, the latter to be thought of as encapsulating the structure of the successors of a state.

It has been shown that the modal logic of the class of *all* coalgebras for a given functor can always be axiomatized in rank 1 [15]. Conversely, as indicated above, every rank-1 logic has a coalgebraic semantics [17]; that is, rank-1 axioms can be absorbed into a functor. The coalgebraic treatment of *non-iterative* axioms thus requires a generalization to copointed functors, to be thought of as incorporating the present state as well as its successors. Indeed it turns out that to obtain strong completeness, it is useful to generalize further to *weakly* copointed functors, in which the present state is virtualized as an ultrafilter, and subsequently restrict to *proper* coalgebras of such weakly copointed functors, in which all these virtual points actually materialize. Our main result thus states more precisely that every non-iterative logic is sound and strongly complete for the class of proper coalgebras of a canonical weakly copointed functor we construct; strong completeness w.r.t. a canonical copointed subfunctor then follows. As indicated above, this result translates back into strong completeness w.r.t. neighbourhood semantics. We complement this statement with an (easier) result showing that the modal logic of a copointed functor can always be equipped with a weakly complete non-iterative axiomatization, justifying the slogan that non-iterative logics are precisely the logics of copointed functors.

We illustrate the use of this result on certain deontic logics that on the one hand avoid the deontic explosion problem (ruling out normality, and hence Kripke semantics) and on the other hand allow for *factual detachment*, embodied in properly non-iterative axioms [20]. The only known semantics for such logics is neighbourhood semantics, and it is only known to be weakly complete (by the mentioned results of Lewis [10], alternatively by a concrete proof given in the online appendix of [20]); our results imply that it is in fact strongly complete.

Organization We recall the syntactic notion of non-iterative modal logic [10] in Section 2. In Section 3, we recall the semantic framework of coalgebraic logic, and discuss copointed and weakly copointed functors. Our main technical tool is the 0-1-step logic of a non-iterative coalgebraic logic, introduced in Section 4. We establish the easier direction of the relationship between non-iterative modal logics and coalgebraic modal logic in Section 5, where we show that the modal logic of coalgebras for a copointed functor is always non-iterative (and has the finite model property). Our main result, which states that conversely, every non-iterative modal logic is strongly complete over a canonical coalgebraic semantics that coincides with neighbourhood semantics, is shown in Sections 6 and 7. In Section 8, we present applications to deontic logics.

Proofs that are omitted or only sketched can be found in the full version [3].

2 Non-iterative Modal Logics

A (*modal*) *similarity type* Λ is a set of modal operators with associated finite arity. The set $\mathcal{F}(\Lambda)$ of Λ -formulae is given by the grammar

$$\phi_1, \dots, \phi_n ::= \perp \mid \neg\phi_1 \mid \phi_1 \wedge \phi_2 \mid L(\phi_1, \dots, \phi_n)$$

where $L \in \Lambda$ has arity n . Additional Boolean operators \rightarrow , \leftrightarrow , \vee and \top can then be defined as usual. We denote by $|\phi|$ the *size* of a formula ϕ , measured as the number of subformulae of ϕ . The grammar does not include propositional atoms as a separate syntactic category; however, these can be cast as nullary modalities. We thus distinguish propositional atoms from propositional variables, which are used to formulate axioms and rules.

Definition 2.1 Let $\text{Prop}(V)$ denote the set of propositional formulae ϕ over a given set V (i.e. $\phi ::= \perp \mid a \mid \neg\phi \mid \phi_1 \wedge \phi_2$, with a ranging over V), and put

$$\Lambda(V) = \{L(a_1, \dots, a_n) \mid L \in \Lambda \text{ n-ary}, a_1, \dots, a_n \in V\}.$$

The elements of V are typically thought of as propositional variables. A *one-step formula* or *rank-1 formula* over V is a formula in $\text{Prop}(\Lambda(\text{Prop}(V)))$, and a *0-1-step formula* or a *non-iterative formula* over V is a formula in $\phi \in \text{Prop}(\Lambda(\text{Prop}(V)) \cup V)$. In words, a formula ϕ over V is non-iterative if it does not contain nested modal operators, and a non-iterative formula ϕ is rank-1 if additionally every variable in ϕ lies under a modal operator. We generally refer to maps of the form $\sigma: V \rightarrow Z$ that we use to replace entities of type V with entities of type Z in formulae as *Z-substitutions on V*, and write $\phi\sigma$ for the result of applying σ to a formula ϕ over V .

As indicated previously, the axiom $\Box(a \rightarrow b) \rightarrow \Box a \rightarrow \Box b$ (with a, b propositional variables) is rank-1, and $\Box a \rightarrow a$ is non-iterative. We define modal logics $\mathcal{L} = (\Lambda, \mathcal{A})$ syntactically by a similarity type Λ and a set \mathcal{A} of *axioms* (in the given similarity type), determining the set of derivable formulae via the usual proof system as recalled below. A logic is *non-iterative (rank-1)* if all its axioms are non-iterative (rank-1). Given a logic $\mathcal{L} = (\Lambda, \mathcal{A})$, we say that a Λ -formula ψ is *derivable*, and write $\vdash_{\mathcal{L}} \psi$, if ψ can be derived in finitely many steps via the following rules:

$$(Ax) \frac{}{\psi\sigma} (\psi \in \mathcal{A}, \sigma \text{ an } \mathcal{F}(\Lambda)\text{-substitution})$$

$$(P) \frac{\phi_1 \quad \dots \quad \phi_n}{\psi} (\{\phi_1, \dots, \phi_n\} \vdash_{PL} \psi) \quad (C) \frac{\phi_1 \leftrightarrow \psi_1 \quad \dots \quad \phi_n \leftrightarrow \psi_n}{L(\phi_1, \dots, \phi_n) \leftrightarrow L(\psi_1, \dots, \psi_n)}$$

where by $\{\phi_1, \dots, \phi_n\} \vdash_{PL} \psi$ we indicate that ψ is derivable from assumptions ϕ_1, \dots, ϕ_n by propositional reasoning (e.g. propositional tautologies and modus ponens). The last rule is known as the *congruence rule* or *replacement of*

equivalents. For a set Φ of Λ -formulae, we write $\Phi \vdash_{\mathcal{L}} \psi$ if $\vdash_{\mathcal{L}} (\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \psi$ for some $\phi_1, \dots, \phi_n \in \Phi$. We say that Φ is \mathcal{L} -consistent, or just *consistent*, if $\Phi \not\vdash_{\mathcal{L}} \perp$. A formula ϕ is *consistent* if $\{\phi\}$ is consistent.

Remark 2.2 Non-iterative logics can alternatively be presented in terms of proof rules: A *non-iterative rule* ϕ/ψ over V consists of a *premiss* $\phi \in \mathbf{Prop}(V)$ and a *conclusion* $\psi \in \mathbf{Prop}(\Lambda(V) \cup V)$. There are mutual conversions between the two formats, the conversion from axioms to rules being straightforward, and the conversion from rules to axioms being based on Boolean unification: Given a non-iterative rule ϕ/ψ , pick a *projective unifier* [4] of ϕ , i.e. a substitution σ such that $\phi\sigma$ and $\phi \rightarrow (a \leftrightarrow \sigma(a))$, for all variables a in ψ , are tautologies, and replace ϕ/ψ with the axiom $\psi\sigma$; further details are as in the rank-1 case [15].

Remark 2.3 As the syntax of the logic itself does not include propositional variables, the above system also does not derive formulae with variables. If desired, propositional variables in formulae can be emulated by introducing fresh propositional atoms (treated as nullary modal operators as indicated above). In particular, if substitution is made to apply to these fresh propositional atoms, then the standard substitution rule $\phi/\phi\sigma$ becomes admissible.

3 Coalgebraic Semantics

We next recall basic definitions in universal coalgebra [14] and coalgebraic logic [2], which will form the underlying semantic framework for our main result. We briefly recall requisite categorical definitions; some familiarity with basic category theory will nevertheless be helpful (e.g. [1]).

The underlying principle of (set-based) universal coalgebra is to encapsulate a type of state-based systems as an endofunctor $T : \mathbf{Set} \rightarrow \mathbf{Set}$ (briefly called a *set functor*) where \mathbf{Set} is the category of sets and functions. Thus, T assigns to each set X a set TX , and to each map $f : X \rightarrow Y$ a map $Tf : TX \rightarrow TY$, preserving identities and composition. We think of TX as a type of structured collections over X . A basic example is the (*covariant*) *powerset functor* \mathcal{P} , which assigns to each set X its powerset $\mathcal{P}X$, and to each map $f : X \rightarrow Y$ the map $\mathcal{P}f : \mathcal{P}X \rightarrow \mathcal{P}Y$ that takes direct images, i.e. $\mathcal{P}f(A) = f[A]$ for $A \in \mathcal{P}X$. The most relevant example for our present purposes is the *neighbourhood functor* \mathcal{N} , defined as follows. The *contravariant powerset functor* \mathcal{Q} is a functor of type $\mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}$, i.e. reverses the direction of maps; it maps a set X to its powerset $\mathcal{Q}X = \mathcal{P}X$, and a map $f : X \rightarrow Y$ to the preimage map $\mathcal{Q}f : \mathcal{Q}Y \rightarrow \mathcal{Q}X$, i.e. $\mathcal{Q}f(B) = f^{-1}[B]$ for $B \in \mathcal{Q}Y$. For any functor F , we indicate by F^{op} the functor that acts like F but on the opposite categories, i.e. with arrows reversed in both domain and codomain. Then, we define \mathcal{N} as the composite

$$\mathcal{N} = \mathcal{Q} \circ \mathcal{Q}^{\text{op}} : \mathbf{Set} \rightarrow \mathbf{Set}.$$

We think of elements of $\mathcal{N}X$ as neighbourhood systems over X .

Given a functor T , systems are then abstracted as *T-coalgebras* $C = (X, \xi)$ consisting of a set X of *states* and a *transition function* $\xi : X \rightarrow TX$. We think of ξ as assigning to each state x a structured collection $\xi(x)$ of successors. E.g.

\mathcal{P} -coalgebras are just Kripke frames, assigning as they do to each state a set of successors, and \mathcal{N} -coalgebras are neighbourhood frames, where each state receives a collection of neighbourhoods.

Modal operators are semantically interpreted by *predicate liftings* [12,16]:

Definition 3.1 An n -ary *predicate lifting* for a set functor T is a natural transformation $\lambda : \mathcal{Q}^n \rightarrow \mathcal{Q} \circ T^{op}$, with \mathcal{Q} being the contravariant powerset functor recalled above. So λ is a family of functions λ_X , indexed over all sets X , such that for all $f : X \rightarrow Y$ and $B_i \subseteq Y$, $i = 1, \dots, n$,

$$\lambda_X(f^{-1}[B_1], \dots, f^{-1}[B_n]) = (Tf)^{-1}[\lambda_Y(B_1, \dots, B_n)].$$

A Λ -structure $\mathcal{M} = (T, \llbracket L \rrbracket_{L \in \Lambda})$ for a signature Λ consists of a functor T and an n -ary predicate lifting $\llbracket L \rrbracket$ for every n -ary modal operator $L \in \Lambda$; we say that \mathcal{M} is *based on* T . When there is no danger of confusion, we will occasionally refer to the entire Λ -structure just as T .

Given a Λ -structure \mathcal{M} based on T , we define the satisfaction relation $x \models_C \phi$ between states x in T -coalgebras $C = (X, \xi)$ and Λ -formulae ϕ inductively by

$$\begin{aligned} x \not\models_C \perp & \\ x \models_C \neg\phi & \quad \text{iff } x \not\models_C \phi \\ x \models_C \phi \wedge \psi & \quad \text{iff } x \models_C \phi \text{ and } x \models_C \psi \\ x \models_C L(\phi_1, \dots, \phi_n) & \text{ iff } \xi(x) \in \llbracket L \rrbracket(\llbracket \phi_1 \rrbracket_C, \dots, \llbracket \phi_n \rrbracket_C) \end{aligned}$$

where we write $\llbracket \phi \rrbracket_C$ (or just $\llbracket \phi \rrbracket$) for the *extension* $\{x \in X \mid x \models_C \phi\}$ of ϕ .

Example 3.2 (i) As indicated above, Kripke frames are coalgebras for the powerset functor \mathcal{P} . The standard \Box modality is interpreted over \mathcal{P} via the predicate lifting

$$\llbracket \Box \rrbracket_X(A) = \{B \in \mathcal{P}X \mid B \subseteq A\},$$

which in combination with the above definition of the satisfaction relation induces precisely the usual semantics of \Box .

(ii) *Probabilistic modal logic* [9,6] has unary modal operators L_p indexed over $p \in [0, 1] \cap \mathbb{Q}$, with $L_p\phi$ read ‘ ϕ holds with probability at least p after the next transition step’. It is interpreted over probabilistic transition systems (or Markov chains), which are coalgebras for the *discrete distribution functor* \mathcal{D} , given on sets X by taking $\mathcal{D}X$ to be the set of discrete probability distributions on X . The modal operators are then interpreted using the predicate liftings

$$\llbracket L_p \rrbracket_X(A) = \{\mu \in \mathcal{D}X \mid \mu(A) \geq p\}.$$

(iii) As seen above, *neighbourhood frames* are coalgebras for the neighbourhood functor \mathcal{N} . We capture the usual neighbourhood semantics of the \Box modality by the predicate lifting

$$\llbracket \Box \rrbracket_X(A) = \{N \in \mathcal{N}X \mid A \in N\},$$

that is, a state satisfies $\Box\phi$ iff the extension $\llbracket\phi\rrbracket$ is a neighbourhood of x . More generally, a Λ -neighbourhood frame for a similarity type Λ is a pair $(X, (\nu_L)_{L \in \Lambda})$ consisting of a set X of states and a family of functions $\nu_L: X \rightarrow \mathcal{P}((\mathcal{P}X)^n)$ for $L \in \Lambda$ n -ary. We refer to subsets of $(\mathcal{P}X)^n$ as n -ary neighbourhood systems, and to their elements as n -ary neighbourhoods; if $(A_1, \dots, A_n) \in \nu_L(x)$ for n -ary $L \in \Lambda$, then (A_1, \dots, A_n) is an $(n$ -ary) L -neighbourhood of x . Satisfaction of modalized formulae by states $x \in X$ is then defined by

$$x \models L(\phi_1, \dots, \phi_n) \text{ iff } (\llbracket\phi_1\rrbracket, \dots, \llbracket\phi_n\rrbracket) \in \nu_L(x);$$

in words, $x \models L(\phi_1, \dots, \phi_n)$ iff $(\llbracket\phi_1\rrbracket, \dots, \llbracket\phi_n\rrbracket)$ is an L -neighbourhood of x . Λ -neighbourhood frames are coalgebras for the functor \mathcal{N}_Λ defined by

$$\mathcal{N}_\Lambda = \prod_{L \in \Lambda} \text{\textit{n-ary}} \mathcal{Q} \circ ((\mathcal{Q}^{\text{op}})^n)$$

where product and n -th power $(-)^n$ are pointwise, i.e. $\mathcal{N}_\Lambda X = \prod_{L \in \Lambda} \text{\textit{n-ary}} \mathcal{Q}((\mathcal{Q}X)^n)$. The corresponding predicate liftings are

$$\llbracket L \rrbracket_X(A_1, \dots, A_n) = \{(N_L)_{L \in \Lambda} \in \mathcal{N}_\Lambda X \mid (A_1, \dots, A_n) \in N_L\}.$$

Since we work with classical negation, we can reduce all reasoning problems to satisfiability in the usual manner. Given a Λ -structure based on T , a formula ϕ is *valid* if $x \models_C \phi$ for all states x in T -coalgebras C , and a set Φ of formulae is *satisfiable* if there exists a state x in a T -coalgebra C such that $x \models_C \phi$ for all $\phi \in \Phi$. A formula ϕ is satisfiable if $\{\phi\}$ is satisfiable. A logic $\mathcal{L} = (\Lambda, \mathcal{A})$, or just \mathcal{A} , is *sound* for \mathcal{M} if all L -derivable formulae are valid over \mathcal{M} , *weakly complete* if all consistent formulae are satisfiable (equivalently all valid formulae are derivable), and *strongly complete* if all consistent sets of formulae are satisfiable (which is equivalent to completeness w.r.t. local consequence from possibly infinite sets of assumptions.)

It has been shown that coalgebraic modal logics coincide with rank-1 logics. More precisely, for every Λ -structure \mathcal{M} there exists a rank-1 logic that is weakly complete for \mathcal{M} [15] (strong completeness cannot be expected as coalgebraic modal logics often fail to be compact, e.g. probabilistic modal logic as described in Example 3.2.ii is not compact [15]). Conversely, given a rank-1 logic $\mathcal{L} = (\Lambda, \mathcal{A})$, there is a Λ -structure \mathcal{M} such that \mathcal{L} is sound and strongly complete for \mathcal{M} [17]; this Λ -structure is isomorphic to neighbourhood semantics. Generally speaking, rank-1 axioms can be absorbed into the functor; as a very simple example, the seriality axiom for Kripke frames, $\neg\Box\perp$, can be captured by replacing the powerset functor \mathcal{P} with the non-empty powerset functor \mathcal{P}^* , where $\mathcal{P}^*X = \{A \in \mathcal{P}X \mid A \neq \emptyset\}$.

To cover non-iterative logics, we therefore need additional structure on the functor that additionally caters for base points: A *copointed functor* (T, ε) , or just T when ε is clear from the context, consists of a functor T and a *copoint* ε , i.e. a natural transformation $\varepsilon: T \rightarrow \text{id}$ where id denotes the identity functor. Coalgebras $C = (X, \xi)$ for a copointed functor are by default required to be

proper, i.e. $\varepsilon_X \circ \xi = id_X$. Intuitively, a plain functor encapsulates only the possible (structured collections of) successors that can be assigned to a given present state, while a copointed functor additionally retains the information about the present state itself, accessed via the copoint; the properness condition $\varepsilon_X \circ \xi$ on coalgebras (X, ξ) of a copointed functor effectively demands that this information is accurate, i.e. applying the copoint to $\xi(x)$ actually returns the present state x .

The main purpose of the information about the present state included in T is to allow imposing relationships between the present point and its collection of successors. Indeed, every functor T can be made copointed by passing to the functor $T \times id$ (given on sets X by $(T \times id)X = TX \times X$), with $\varepsilon(t, x) = x$; we refer to copointed functors of this shape as *trivially copointed*, as they impose no relationship between the present state and its collection of successors. Copointed functors can absorb non-iterative axioms; e.g. the modal logic \mathbf{T} is captured by the copointed functor T given by $TX = \{(A, x) \in \mathcal{P}X \times X \mid x \in A\}$ (more details are given in Section 4), which imposes that the present state is among its own successors; that is, proper T -coalgebras are precisely reflexive Kripke frames. This functor T is our first example of a non-trivially copointed functor; note that it is a subfunctor of the trivially copointed functor $\mathcal{P} \times id$. For purposes of our strong completeness result, we make use of a relaxed notion of copointed functor:

Definition 3.3 A *weakly copointed functor* (T, ε) (or just T when ε is clear from the context) consists of a functor T and a *weak copoint* ε , i.e. a natural transformation $\varepsilon : T \rightarrow \mathcal{U}$, where \mathcal{U} denotes the (functor part of) the ultrafilter monad. That is, $\mathcal{U}X$ is the set of ultrafilters on X , and $\mathcal{U}f(\alpha) = \{B \subseteq Y \mid f^{-1}[B] \in \alpha\}$ for $f : X \rightarrow Y$, $\alpha \in \mathcal{U}X$ (so \mathcal{U} is a subfunctor of the neighbourhood functor \mathcal{N} as in Example 3.2.iii). Then, a T -coalgebra structure $\xi : X \rightarrow TX$ is *proper* if $\varepsilon_X \circ \xi = \eta_X$ where $\eta : Id \rightarrow \mathcal{U}$ is the unit of the ultrafilter monad, given by $\eta_X(x) = \dot{x} = \{A \in \mathcal{P}X \mid x \in A\}$. Every functor T induces a *trivially weakly copointed functor* $T \times \mathcal{U}$, with second projection as the weak copoint.

Instead of the identity of the present state, a weakly copointed functor contains only a description of the present state, which in general may fail to be realized as an actual state. However, weakly copointed functors relate tightly to copointed functors in the standard sense:

Lemma and Definition 3.4 Let (T, ε) be a weakly copointed functor. Then

$$T_c X = \{t \in TX \mid \varepsilon(t) \text{ principal}\}$$

defines a copointed subfunctor of T_c , the copointed part of T , with copoint ε_c defined by $\varepsilon_c(t) \in \bigcap \varepsilon(t)$. Moreover, every proper T -coalgebra $C = (X, \xi)$ factors through the inclusion $T_c X \hookrightarrow TX$, inducing a coalgebra C_c for the copointed functor T_c . Given a similarity type Λ with assigned predicate liftings for T , we obtain predicate liftings for T_c by restriction; then, a state $x \in X$ satisfies the same Λ -formulae in C as in C_c .

(Recall that an ultrafilter α is *principal* if $\bigcap \alpha \neq \emptyset$, and then necessarily $|\bigcap \alpha| = 1$.)

Remark 3.5 Indeed, the above lemma implies that weakly copointed functors are not strictly required for our current target results, which are all formulated over proper coalgebras. We nevertheless do involve them in the technical development because of their natural role within coalgebraic logic: The 0-1-step logic, in the sense introduced in the next section, of the canonical Λ -structure, which will be based on a weakly copointed functor, is strongly complete; this would be impossible for any Λ -structure based on a copointed functor. For details, see Remark 6.7.

Remark 3.6 The standard coalgebraic semantics of rank-1 modal logics as recalled above embeds into the copointed setting by converting plain functors T into trivially copointed functors $T \times \text{id}$ or trivially weakly copointed functors $T \times \mathcal{U}$, with modalities interpreted via first projections: Given a Λ -structure based on the functor T , we obtain a Λ -structure based on the trivially copointed functor $T \times \text{id}$ by putting

$$(t, x) \models L(A_1, \dots, A_n) \quad \text{iff} \quad t \models L(A_1, \dots, A_n)$$

for $A_1, \dots, A_n \subseteq X$ and $(t, x) \in TX \times X$, similarly for $T \times \mathcal{U}$. Proper coalgebras for $T \times \text{id}$ and proper coalgebras for $T \times \mathcal{U}$ are both essentially the same as (plain) coalgebras for T , and it is easy to see that the respective modal semantics over T -coalgebras and over proper $T \times \text{id}$ - or $T \times \mathcal{U}$ -coalgebras are equivalent.

Remark 3.7 The categorical concept of a *comonad* extends the notion of copointed functor by additionally assuming an unfolding operation $\delta : T \rightarrow T \circ T$ (the *comultiplication*) satisfying certain equational laws. This amounts to letting T contain information about the entire finite-time future development of the present state: Iterating δ , we can extract evolutions of any depth n , i.e. elements of $T^n X$, from a given element of TX . Comonads can thus be employed to capture iterative frame conditions such as $\Box a \rightarrow \Box \Box a$, with the technical caveat that this requires restricting the branching degree of models to avoid set-theoretic existence problems. Since the meta-theory of iterative frame conditions is in general much less well-behaved than that of non-iterative ones (e.g. recall the cited result on failure of strong completeness over neighbourhood semantics [18]), one should manage expectations regarding the perspective of results in comparable generality as the present one.

4 The 0-1-Step Logic

An important driving principle of coalgebraic logic is to reduce metatheoretic properties of a full-blown modal logic with nested modalities, interpreted over coalgebras, to similar properties of a much simpler *one-step logic* where formulae feature precisely one layer of modalities, and are interpreted over structures that essentially model just one transition step (hence the name). To cover non-iterative logics, we need to extend this principle to cover also the current state

(besides its successors), arriving at the *0-1-step logic* of the given modal logic. For readability, we *restrict the technical development to unary modalities* from now on; covering higher arities requires no more than additional indexing, and we continue to use higher arities in the examples.

Syntax and derivations In formulae of the 0-1-step logic, we intentionally mix syntax and semantics, replacing propositional variables by their values in a powerset Boolean algebra. That is, given a non-iterative logic $\mathcal{L} = (\Lambda, \mathcal{A})$ and a set X , we take $\text{Prop}(\Lambda(\mathcal{P}X) \cup \mathcal{P}X)$ to be the set of *0-1-step formulae over $\mathcal{P}X$* , referring to elements of $\mathcal{P}X$ as (*interpreted*) *propositional atoms*. We denote the evaluation of a $\text{Prop}(\mathcal{P}X)$ -formula ϕ in the Boolean algebra $\mathcal{P}X$ by $\llbracket \phi \rrbracket$, and say that ϕ is *propositionally valid over $\mathcal{P}X$* if $\llbracket \phi \rrbracket = X$. We will identify occurrences of subformulae $\phi \in \text{Prop}(\mathcal{P}X)$ with $\llbracket \phi \rrbracket$ when they lie in scope of a modal operator but not otherwise, i.e. on the uppermost level. This evaluation of inner propositional formulae allows us to omit the modal congruence rule. We thus define *0-1-step derivability* $\vdash_{\mathcal{L}}^{0-1} \psi$ of 0-1-step formulae ψ inductively by the rules

$$\frac{}{\psi\sigma} \quad (\psi \in \mathcal{A}, \sigma \text{ a Prop}(\mathcal{P}X)\text{-substitution})$$

$$\frac{\phi_1, \dots, \phi_n}{\psi} (\{\phi_1, \dots, \phi_n\} \vdash_{PL} \psi) \quad \frac{}{\phi} (\phi \in \text{Prop}(\mathcal{P}X), \llbracket \phi \rrbracket = X).$$

(Non-iterative rules ϕ/ψ as in Remark 2.2, if present, are also applied in substituted form: if $\llbracket \phi\sigma \rrbracket = X$ for a $\text{Prop}(\mathcal{P}X)$ -substitution σ , then derive $\psi\sigma$.) That is, $\vdash_{\mathcal{L}}^{0-1} \psi$ iff ψ is propositionally entailed by

$$\{\psi\sigma \mid \psi \in \mathcal{A}, \sigma \text{ a Prop}(\mathcal{P}X)\text{-substitution}\} \cup \{\phi \mid \phi \in \text{Prop}(\mathcal{P}X), \llbracket \phi \rrbracket = X\}.$$

We write $\Phi \vdash_{\mathcal{L}}^{0-1} \psi$ if $\vdash_{\mathcal{L}}^{0-1} (\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \psi$ for some $\phi_1, \dots, \phi_n \in \Phi$. A set Φ of 0-1-step formulae over $\mathcal{P}X$ is *0-1-step consistent* if $\Phi \not\vdash_{\mathcal{L}}^{0-1} \perp$.

Semantics Fix a weakly copointed functor (T, ε) and a Λ -structure \mathcal{M} based on T . Define the unary predicate lifting ι by $\iota_X(A) = \{t \in TX \mid A \in \varepsilon(t)\}$. The *0-1-step satisfaction* relation $t \models_X^{0-1} \psi$ between functor elements $t \in TX$ and 0-1-step formulae ψ over $\mathcal{P}X$ is inductively defined by

$$\begin{aligned} t &\not\models_X^{0-1} \perp \\ t &\models_X^{0-1} \neg\phi \quad \text{iff } t \not\models_X^{0-1} \phi \\ t &\models_X^{0-1} \phi \wedge \psi \quad \text{iff } t \models_X^{0-1} \phi \text{ and } t \models_X^{0-1} \psi \\ t &\models_X^{0-1} L\phi \quad \text{iff } t \in \llbracket L \rrbracket_X(\phi) \\ t &\models_X^{0-1} B \quad \text{iff } t \in \iota_X(B) \end{aligned}$$

where $B \in \mathcal{P}X$ in the last clause, and $\llbracket \psi \rrbracket_X^{0-1} = \{t \in TX \mid t \models_X^{0-1} \psi\}$. The last clause thus deals with top-level interpreted propositional atoms. Note that in accordance with the above convention, the second to last clause omits interpretation of modal arguments, which are already identified with their interpretation. We say that ψ is *satisfiable* if $\llbracket \psi \rrbracket_X^{0-1} \neq \emptyset$, and we write $TX \models_X^{0-1} \psi$

if $\llbracket \psi \rrbracket_X^{0-1} = TX$. We generally refer to maps $\tau: V \rightarrow \mathcal{P}X$ as $\mathcal{P}X$ -valuations. Given a non-iterative axiom ψ , we write $\psi\tau$ for the 0-1-step formula obtained from ψ by substituting according to τ . Then, ψ is *0-1-step sound* for \mathcal{M} if $TX \models_X^{0-1} \psi\tau$ for every set X and every $\mathcal{P}X$ -valuation τ . Conversely, the logic $\mathcal{L} = (\Lambda, \mathcal{A})$, or just \mathcal{A} , is *0-1-step complete* for \mathcal{M} if every 0-1-step formula ψ over $\mathcal{P}X$ such that $TX \models_X^{0-1} \psi$ is 0-1-step derivable ($\vdash_{\mathcal{L}}^{0-1} \psi$), equivalently if every 0-1-step consistent formula is satisfiable. The same terminology applies to non-iterative rules (Remark 2.2) (specifically, a non-iterative rule ϕ/ψ is 0-1-step sound if $TX \models_X^{0-1} \psi\tau$ whenever $\llbracket \phi\tau \rrbracket = X$).

To enable an appropriate statement of soundness, we extend the semantics of the logic to allow for *frame conditions*: We refer to a pair (C, π) consisting of a T -coalgebra $C = (X, \xi)$ and a valuation $\pi: V \rightarrow \mathcal{P}X$ of the propositional variables as a T -model. We define satisfaction $x \models_{(C, \pi)} \psi$ of 0-1-step formulae $\psi \in \mathbf{Prop}(\Lambda(\mathbf{Prop}(V) \cup V))$ in states x of T -models (C, π) by the same clauses as for \models_C (Section 3), and additionally

$$x \models_{(C, \pi)} a \quad \text{iff} \quad x \in \pi(a)$$

for $a \in V$. We say that C satisfies the *frame condition* ψ if $x \models_{(C, \pi)} \psi$ for all T -models (C, π) . Of course, if C satisfies the frame condition ψ then ψ is *sound* for C , i.e. every state in C satisfies all substitution instances of ψ .

Lemma 4.1 (Soundness) *If a non-iterative axiom ψ over V is 0-1-step sound over a Λ -structure \mathcal{M} based on a weakly copointed functor T , then every proper T -coalgebra satisfies the frame condition ψ ; hence, ψ is sound for the class of all proper T -coalgebras.*

We proceed to discuss in more detail how non-iterative axioms are absorbed into (weakly) copointed functors. Given a (weakly) copointed functor T and a set \mathcal{A}' of additional non-iterative axioms, we can pass to the (weakly) copointed subfunctor $T_{\mathcal{A}'}$ of T given by

$$T_{\mathcal{A}'}X = \{t \in T \mid t \models_X^{0-1} \phi\sigma \text{ for all } \phi \in \mathcal{A}' \text{ and all } \mathcal{P}X\text{-substitutions } \sigma\}$$

and restrict the Λ -structure to $T_{\mathcal{A}'}$ in the evident way. By construction, the axioms in \mathcal{A}' are 0-1-step sound over $T_{\mathcal{A}'}$, and the proper $T_{\mathcal{A}'}$ -coalgebras are precisely those proper T -coalgebras that satisfy the axioms in \mathcal{A}' as frame conditions. Moreover, we have

Lemma 4.2 *In the notation introduced above, suppose that the set \mathcal{A} of non-iterative axioms is 0-1 step sound and 0-1 step complete over T . If \mathcal{A}' mentions only finitely many modalities, then $\mathcal{A} \cup \mathcal{A}'$ is 0-1-step complete over $T_{\mathcal{A}'}$.*

Proof (sketch) Observe that if ψ is a 0-1-step formula over $\mathcal{P}X$ such that $T_{\mathcal{A}'}X \models_X^{0-1} \psi$, with X assumed to be finite w.l.o.g., then $TX \models_X^{0-1} (\bigwedge \Phi) \rightarrow \psi$ where Φ contains representatives up to propositional equivalence of all instances of axioms in \mathcal{A}' under $\mathcal{P}X$ -substitutions; the assumptions guarantee that we can take Φ to be finite. \square

Example 4.3 (i) We have recalled the coalgebraic view on standard Kripke semantics in Example 3.2.i. The usual axioms of the modal logic K ($\Box\top$ and $\Box(a \rightarrow b) \rightarrow \Box a \rightarrow \Box b$) are 0-1-step complete over the trivially copointed functor $\mathcal{P} \times \text{id}$ induced by the functor \mathcal{P} ; this is implied by translating the known *one-step* completeness of these axioms over \mathcal{P} [11] into the copointed setting as indicated in Remark 3.6. It follows by Lemma 4.2 that these axioms, together with the T -axiom $\Box a \rightarrow a$, are 0-1-step complete for the copointed functor T given by

$$TX = \{(B, x) \in \mathcal{P}X \times X \mid (B, x) \models \Box A \rightarrow A \text{ for all } A \in \mathcal{P}X\}.$$

It is easy to see that $TX = \{(B, x) \in \mathcal{P}X \times X \mid x \in B\}$, i.e. T coincides with the copointed functor recalled on p. 235, whose proper coalgebras are the reflexive Kripke frames.

(ii) The assumption that the additional axioms only mention finitely many modalities is really needed; without it, the claim fails even in the rank-1 case. For instance, let \mathcal{S} be the *subdistribution functor*, which assigns to a set X the set $\mathcal{S}X$ of discrete subdistributions on X , where a subdistribution is defined like a distribution except that the weight of the whole set is required to be at most 1 rather than equal to 1. We use modalities L_p ‘with weight at least p ’ with the same semantics as in the probabilistic case (Example 3.2.ii). Take the set

$$\mathcal{A}' = \{\neg L_1\top\} \cup \{L_{1-1/n}\top \mid n \geq 1\}$$

of rank-1 axioms. Then $(\mathcal{S} \times \text{id})_{\mathcal{A}'}X = \emptyset$ for all X , so that $(\mathcal{S} \times \text{id})_{\mathcal{A}'}X \models_X^{0-1} \perp$, but \perp is not derivable under the given axioms (together with any sound axiomatization of \mathcal{S}), as any derivation of \perp could only use a finite subset of \mathcal{A}' , and all such finite subsets are clearly consistent.

A key role in the completeness proof will be played by the following subformula property of the 0-1-step logic, which extends [17, Proposition 24] from rank-1 to non-iterative logics.

Proposition 4.4 *Let ψ be a 0-1-step formula over $\mathcal{P}X$ such that $\vdash_{\mathcal{L}}^{0-1} \psi$. Then ψ is 0-1-step derivable using only $\text{Prop}(\mathfrak{A})$ -instances of axioms and $\text{Prop}(\mathfrak{A})$ -formulae valid over $\mathcal{P}X$, where $\mathfrak{A} \subseteq \mathcal{P}X$ are the sets occurring in ψ .*

The proof requires some facts about propositional logic.

Lemma 4.5 *Let V and W be disjoint finite sets. For an A -valuation τ on V with $A \subseteq \mathcal{P}X$ and a system of Boolean equations $\phi_i\tau = \psi_i\tau$ for $i = 1, \dots, n$ where $\phi_i, \psi_i \in \text{Prop}(V \cup W)$, if there exists an A -valuation κ for W such that $\phi_i\tau\kappa = \psi_i\tau\kappa$ for $i = 1, \dots, n$, then there exists a $\text{Prop}(V)$ -substitution σ on W such that*

- (i) $\phi_i\sigma\tau = \psi_i\sigma\tau$ for $i = 1, \dots, n$
- (ii) $x\kappa \subseteq \llbracket x\sigma\tau \rrbracket$ for $x \in W$ if $|W| = 1$.

(Claim (i) says effectively that if Boolean equations with coefficients in A are solvable in A , then they are solvable by Boolean combinations of the coefficients that actually occur. Claim (ii) is only needed later.)

Proof. (i): This is well-known but we need the construction for Claim (ii). We immediately reduce to a single equation $\phi\tau = \top$ where $\phi = \bigwedge_{i=1}^n (\phi_i \leftrightarrow \psi_i)$. We construct σ by induction over $|W|$, with trivial base $|W| = 0$. In the inductive step, we pick $x \in W$ and obtain, by Boolean expansion,

$$\begin{aligned}\phi &\equiv (x \rightarrow \phi[\top/x]) \wedge (\neg x \rightarrow \phi[\perp/x]) \\ &\equiv (x \rightarrow \phi[\top/x]) \wedge (\neg\phi[\perp/x] \rightarrow x),\end{aligned}$$

which in turn entails $\neg\phi[\perp/x] \rightarrow \phi[\top/x]$, so by assumption the equation $(\neg\phi[\perp/x] \rightarrow \phi[\top/x])\tau = \top$ over $W \setminus \{x\}$ is solved by κ , and hence by induction solvable by some $\text{Prop}(V)$ -substitution σ' . Thus, the substitution

$$\sigma = [\phi[\top/x]/x]\sigma'$$

for W satisfies $\phi\sigma\tau = \top$.

(ii): Let $W = \{x\}$; we then have constructed $\sigma = [\phi[\top/x]/x]$ in (i). We have to show $\kappa(x) \subseteq \llbracket \phi[\top/x]\tau \rrbracket$. Let $y \in \kappa(x)$ and assume w.l.o.g. that ϕ is in CNF, and that x appears in at most one literal in every clause ψ in ϕ . We have to show that $y \in \llbracket \psi[\top/x]\tau \rrbracket$. If the literal x appears in ψ , then this holds trivially. Otherwise, ψ must contain some literal not mentioning x whose interpretation contains y , since $\psi\tau\kappa = \top$ by assumption and $y \notin \llbracket (\neg x)\kappa \rrbracket = \llbracket (\neg x)\tau\kappa \rrbracket$. Therefore $y \in \llbracket \phi[\top/x]\tau \rrbracket$ as required. \square

Lemma 4.6 *Let $\Phi \subseteq \text{Prop}(V)$, let $\psi \in \text{Prop}(V)$, and let σ be a W -substitution on V and τ a U -substitution on V such that $\tau(a) = \tau(b)$ whenever $\sigma(a) = \sigma(b)$ for all $a, b \in V$, and moreover $\Phi\sigma \vdash_{PL} \psi\sigma$. Then $\Phi\tau \vdash_{PL} \psi\tau$.*

Lemma 4.7 *Let $\Phi \subseteq \text{Prop}(V)$, and let $\psi \in \text{Prop}(V)$. Given a U -substitution σ and a W -substitution τ on V , if $\Phi\sigma \vdash_{PL} \psi\sigma$ then $\Phi\tau \cup \Psi \vdash_{PL} \psi\tau$, where $\Psi = \{\tau(a) \leftrightarrow \tau(b) \mid a, b \in V, \sigma(a) = \sigma(b)\}$.*

Lemma 4.8 *Let V and W be disjoint sets, let $W_0 \subseteq W$, let $\Phi \subseteq \text{Prop}(V)$, let $\psi \in \text{Prop}(W_0)$, and let σ and τ be W -substitutions on V such that $\tau(a) = \tau(b)$ whenever $\sigma(a) = \sigma(b)$ and $\tau(a) = c$ whenever $\sigma(a) = c$ for all $a, b \in V$ and $c \in W_0$, and moreover $\Phi\sigma \vdash_{PL} \psi$. Then $\Phi\tau \vdash_{PL} \psi$.*

Proof. Let σ' and τ' be the W -substitutions on $V \cup W_0$ such that $\sigma'(w) = \tau'(w) = w$ for $w \in W_0$ and $\sigma'(v) = \sigma(v)$, $\tau'(v) = \tau(v)$ for $v \in V$. The claim then follows by Lemma 4.6. \square

Proof of Proposition 4.4. Let V be a sufficiently large set of propositional variables. Then there are finite sets Φ_1 of $\text{Prop}(V)$ -instances of axioms and $\Phi_2 \subseteq \text{Prop}(V)$ that we can assume to be instantiated by a single $\mathcal{P}X$ -valuation σ such that the formulae in $\Phi_2\sigma$ are propositionally valid over $\mathcal{P}X$ and $(\Phi_1 \cup \Phi_2)\sigma \vdash_{PL} \psi$. By Lemma 4.8, it suffices to show that there is a $\text{Prop}(\mathfrak{A})$ -substitution τ that solves the following system of equations:

- For all subformulae $L\rho, L\rho'$ in Φ_1 such that $(L\rho)\sigma = (L\rho')\sigma$ in $\Lambda(\mathcal{P}X)$, we have $(L\rho)\tau = (L\rho')\tau$ in $\Lambda(\mathcal{P}X)$. This amounts to an equation $\rho = \rho'$.
- For all subformula LA in ψ and $L\rho$ in Φ_1 such that $LA = (L\rho)\sigma$ in $\Lambda(\mathcal{P}X)$, we have $LA = (L\rho)\tau$ in $\Lambda(\mathcal{P}X)$. This amounts to an equation $A = \rho$.
- For all subformulae ρ, ρ' in $\Phi_1 \cup \Phi_2$ that do not lie beneath a modal operator and are such that $\rho\sigma = \rho'\sigma$, we have $\rho\tau = \rho'\tau$ in $\mathcal{P}X$. This amounts to an equation $\rho = \rho'$.
- For all subformulae A in ψ and ρ in $\Phi_1 \cup \Phi_2$ that do not lie beneath a modal operator and are such that $\rho\sigma = A$ in $\mathcal{P}X$, we have $\rho\tau = A$ in $\mathcal{P}X$. This amounts to an equation $A = \rho$.

By construction, this system of Boolean equations is solvable by σ , and since only sets from \mathfrak{A} appear in the equations, by Lemma 4.5.(i) it is also solvable by a $\text{Prop}(\mathfrak{A})$ -substitution with the required properties. \square

5 Copointed Coalgebraic Logics are Non-Iterative

We next establish that weakly copointed functors are indeed characterized by non-iterative axioms; that is, we *fix for this section a Λ -structure \mathcal{M} based on a weakly copointed functor T* and show that there is a set of non-iterative axioms that is sound and weakly complete over the class of all proper T -coalgebras. (We necessarily restrict to weak completeness, since coalgebraic modal logics in general fail to be compact [15]). In more detail, we show that 0-1-step completeness of a non-iterative axiomatization implies its weak completeness over finite models, and we show that the set of all 0-1-step sound non-iterative axioms is 0-1-step complete. The proofs are fairly straightforward generalizations of the rank-1 case [15]. We begin with the latter step:

Theorem 5.1 *The set of all 0-1-step sound 0-1-step axioms is 0-1-step complete.*

Proof. By Remark 2.2, it suffices to show that the set of all 0-1-step sound non-iterative rules is 0-1-step complete. Let $TX \models_X^{0-1} \psi$ for a 0-1-step formula ψ over $\mathcal{P}X$. Then ψ has the form $\psi = \psi_0\tau$ for $\psi_0 \in \text{Prop}(\Lambda(V_0) \cup V_0)$, with $V_0 \subseteq V$ finite, and a $\mathcal{P}X$ -valuation τ . Let ϕ be the conjunction of all clauses χ over V_0 such that $\llbracket \chi\tau \rrbracket = X$; then $\llbracket \phi\tau \rrbracket = X$. We are thus done once we show that ϕ/ψ_0 is 0-1-step sound. So assume $\llbracket \phi\sigma \rrbracket = Y$ for a $\mathcal{P}Y$ -valuation σ . We have to show $TY \models_Y^{0-1} \psi_0\sigma$. For each $y \in Y$ there is $x \in X$ such that for all $a \in V_0$ we have $x \in \tau(a)$ iff $y \in \sigma(a)$ (otherwise there is a clause χ over V_0 such that $X \models \chi\tau$ but $Y \not\models \chi\sigma$, contradicting $Y \models \phi\sigma$). Therefore there is $f : Y \rightarrow X$ such that $\sigma(a) = f^{-1}[\tau(a)]$ for all $a \in V_0$. By naturality of predicate liftings (including ι) and commutation of preimage with all Boolean operations, we have $\llbracket \psi_0\sigma \rrbracket_Y^{0-1} = Tf^{-1}[\llbracket \psi_0\tau \rrbracket_X^{0-1}]$, and therefore $TY \models_Y^{0-1} \psi_0\sigma$ as required. \square

We will base all our model constructions on the following central notions:

Definition 5.2 A set Σ of formulae is *closed* if it is closed under subformulae and negations of formulae that are not themselves negations. We write C_Σ

for the set of maximally consistent subsets of Σ . For a Λ -formula ϕ , we write $\hat{\phi} = \{\Phi \in C_\Sigma \mid \phi \in \Phi\}$.

Lemma 5.3 [15, Lemma 27] *Let ϕ be a propositional formula over V , σ a Σ -substitution and $\hat{\sigma}$ a $\mathcal{P}(C_\Sigma)$ -valuation with $\hat{\sigma}(a) = \hat{\psi}$ when $\sigma(a) = \psi$. Then $\llbracket \phi \hat{\sigma} \rrbracket = C_\Sigma$ iff $\vdash_{\mathcal{L}} \phi \sigma$.*

Definition 5.4 Let Σ be closed. A coalgebra (C_Σ, ξ) is *coherent* if for all $L\psi \in \Sigma$, $\Phi \in C_\Sigma$,

$$\xi(\Phi) \in \llbracket L \rrbracket_{C_\Sigma}(\hat{\psi}) \quad \text{iff} \quad L\psi \in \Phi.$$

Lemma 5.5 (Truth lemma [15]) *Let Σ be closed, and let $C = (C_\Sigma, \xi)$ be a coherent T -coalgebra and let $\phi \in \Sigma$. For all $\Phi \in C$ we then have $\Phi \models_C \phi$ iff $\phi \in \Phi$.*

Thus, model constructions reduce to showing the existence of coherent coalgebra structures. The latter requires the following lemma, which for later reuse we prove for possibly infinite Σ :

Lemma 5.6 *Let V_Σ denote the set $\{a_\phi \mid \phi \in \Sigma\}$, and let $\Phi \subseteq \text{Prop}(\Lambda(V_\Sigma) \cup V_\Sigma)$. Let σ be the substitution given by $\sigma(a_\phi) = \phi$, and let $\hat{\sigma}$ be the $\mathcal{P}C_\Sigma$ -valuation given by $\hat{\sigma}(a_\phi) = \hat{\phi}$. If $\Phi\sigma$ is consistent, then $\Phi\hat{\sigma}$ is 0-1-step consistent.*

Proof. By contraposition; so assume $\Phi\hat{\sigma} \vdash_{\Sigma}^{0-1} \perp$. By Proposition 4.4, there is a derivation that uses only $\text{Prop}(\mathfrak{A})$ -instances of axioms and $\text{Prop}(\mathfrak{A})$ -formulae valid over $\mathcal{P}C_\Sigma$, for $\mathfrak{A} = \{\hat{\phi} \mid \phi \in \mathcal{F}(\Lambda)\}$. We can write the set of these formulae as $\Theta\hat{\sigma}$ for a set $\Theta \subseteq \text{Prop}(\Lambda(V_\Sigma) \cup V_\Sigma)$. By the definition of 0-1-step derivations, it follows that $(\Phi \cup \Theta)\hat{\sigma} \vdash_{PL} \perp$. Now let Ψ denote the set $\{L\rho \leftrightarrow L\rho' \mid \hat{\rho} = \hat{\rho}'\}$. The formulae in Ψ are derivable in \mathcal{L} by Lemma 5.3 and the congruence rule. Similarly, let $\Gamma = \{\phi \leftrightarrow \phi' \mid \hat{\phi} = \hat{\phi}'\}$; the formulae in Γ are \mathcal{L} -derivable by Lemma 5.3. By Lemma 4.7, it follows that $(\Phi \cup \Theta)\sigma \cup \Psi \cup \Gamma \vdash_{PL} \perp$ and therefore (again using Lemma 5.3) $\Phi\sigma \vdash_{\mathcal{L}} \perp$. \square

Lemma 5.7 (Finite existence lemma) *Let \mathcal{A} be 0-1-step complete, and let Σ be a finite closed set of formulae. Then there exists a coherent proper T -coalgebra structure ξ on C_Σ .*

Proof. Let $\Phi \in C_\Sigma$. We show that the requirements on $\xi(\Phi)$ form a 0-1-step consistent 0-1-step formula, implying existence of $\xi(\Phi)$ by 0-1-step completeness. Take V_Σ , σ and $\hat{\sigma}$ as in Lemma 5.6. Let

$$\chi = \bigwedge_{L\psi \in \Phi} La_\psi \wedge \bigwedge_{\neg L\psi \in \Phi} \neg La_\psi \wedge \bigwedge_{\psi \in \Phi} a_\psi.$$

We need to show that $\chi\hat{\sigma}$ is 0-1-step consistent. By Lemma 5.6, this follows from consistency of $\chi\sigma$, which in turn is implied by consistency of Φ . \square

The announced weak completeness result now follows:

Theorem 5.8 (Weak completeness and bounded model property)

Let \mathcal{A} be 0-1-step complete for the Λ -structure \mathcal{M} . Then \mathcal{A} is weakly complete over finite proper T -coalgebras; specifically, every consistent formula ϕ is satisfiable in a finite proper T -coalgebra of size at most $2^{|\phi|}$.

Proof. Let Σ be the smallest closed set containing ϕ . By the finite existence lemma (Lemma 5.7), there is a proper and coherent T -coalgebra ξ on C_Σ ; note $|C_\Sigma| \leq 2^{|\phi|}$. Since Σ has only finitely many consistent subsets, the consistent set $\{\phi\}$ is contained in some $\Phi \in C_\Sigma$. By the truth lemma, $\Phi \models_{(C_\Sigma, \xi)} \phi$. \square

Remark 5.9 Previous work on the connection between algebraic and coalgebraic semantics [13] has led to results that in particular cover non-iterative frame conditions. The technical setup in the mentioned work features an underlying rank-1 logic, equipped with standard coalgebraic semantics using plain functors, and imposes additional frame conditions as axioms, e.g. non-iterative frame conditions. One of the results obtained [13, Corollary 37] shows that a coalgebraic logic with non-iterative frame conditions is weakly complete over coalgebras satisfying the frame conditions, provided that the frame conditions mention only finitely many modalities. By Remark 3.6 and Lemma 4.2, these assumptions allow combining the given rank-1 logic and the additional frame conditions into a 0-1-step complete logic for the copointed functor defined by the axioms. The weak completeness result therefore follows also from our Theorem 5.8, which moreover applies also to sets of non-iterative frame conditions that mention infinitely many modalities; of course, 0-1-step completeness then needs to be proved without the help of Lemma 4.2. E.g. this will turn out to be possible for the canonical Λ -structure introduced next (Lemma 6.3). All that said, we emphasize again that the results of the present section are obtained by easy extension of previous results in rank-1 coalgebraic logic [15], and included primarily to complete the overall picture; the main technical contribution of the present work lies in the converse implication (to provide a coalgebraic semantics for a given non-iterative logic), tackled next.

6 The Canonical Λ -Structure

We now construct, for a given non-iterative logic $\mathcal{L} = (\Lambda, \mathcal{A})$ that we fix from now on, a canonical Λ -structure $\mathcal{M}_\mathcal{L}$ based on a weakly copointed functor $M_\mathcal{L}$ w.r.t. which we show soundness and strong completeness by means of a canonical model construction. As usual, the state space of the canonical model will be the set of maximally consistent sets, denoted $C_\mathcal{L}$ (so $C_\mathcal{L} = C_{\mathcal{F}(\Lambda)}$ in the notation of Section 5).

We construct the functor $M_\mathcal{L}$ as follows. For a set X , $M_\mathcal{L}X$ is the set of maximally 0-1-step consistent subsets of $\text{Prop}(\Lambda(\mathcal{P}X) \cup \mathcal{P}X)$ (i.e. of the set of 0-1-step formulae over $\mathcal{P}X$). For a function $f: X \rightarrow Y$, we define $M_\mathcal{L}f$ by

$$M_\mathcal{L}f(\Phi) = \{\phi \in \text{Prop}(\Lambda(\mathcal{P}Y) \cup \mathcal{P}Y) \mid \phi\sigma_f \in \Phi\}$$

where σ_f is the $\mathcal{P}X$ -substitution on $\mathcal{P}Y$ given by $\sigma_f(A) = f^{-1}[A]$. We define a weak copoint $\varepsilon: M_\mathcal{L} \rightarrow \mathcal{U}$ by $\varepsilon_X(\Phi) = \Phi \cap \mathcal{P}X$ for $\Phi \in M_\mathcal{L}X$, and interpret $L \in \Lambda$ by

$$[[L]]_X A = \{\Phi \in M_\mathcal{L}X \mid LA \in \Phi\} \quad \text{for } A \subseteq X.$$

Of course, we intend an element of $M_\mathcal{L}X$ to satisfy precisely the 0-1-step formulae that it contains; indeed, we have

Lemma 6.1 (0-1-step truth lemma) *Let ψ be a 0-1-step formula over $\mathcal{P}X$. Then $\Phi \models_X^{0-1} \psi$ iff $\psi \in \Phi$, for $\Phi \in M_{\mathcal{L}}X$.*

Since a maximally consistent set in $M_{\mathcal{L}}X$ must in particular contain all $\mathcal{P}X$ -instances of the axioms in \mathcal{A} , it follows that \mathcal{A} is 0-1-step sound, and hence sound by Lemma 4.1, for $M_{\mathcal{L}}$.

With a view to proving also 0-1-step completeness, we note a 0-1-step version of the well-known Lindenbaum lemma:

Lemma 6.2 (0-1-step Lindenbaum lemma) *Every 0-1-step consistent set of 0-1-step formulae over $\mathcal{P}X$ is contained in a maximal such set.*

From the 0-1-step truth lemma and the 0-1-step Lindenbaum lemma, 0-1-step completeness is immediate:

Lemma 6.3 *The logic \mathcal{L} is 0-1-step complete for $M_{\mathcal{L}}$.*

By Theorem 5.8, this implies weak completeness and the finite (in fact, bounded) model property:

Corollary 6.4 *The logic \mathcal{L} is weakly complete over finite proper $M_{\mathcal{L}}$ -coalgebras.*

Our main result, established in the next section, will show that \mathcal{L} is in fact *strongly* complete over proper $M_{\mathcal{L}}$ -coalgebras (of course, one can then no longer restrict to finite coalgebras). As indicated in the introduction, the canonical Λ -structure is essentially neighbourhood semantics. We proceed to elaborate details.

Recall from Example 3.2.iii that the Λ -neighbourhood functor \mathcal{N}_{Λ} is defined as $\mathcal{N}_{\Lambda} = \prod_{L \in \Lambda} n\text{-ary } \mathcal{Q} \circ ((\mathcal{Q}^{\text{op}})^n)$. Recall that $\mathcal{N}_{\mathcal{L}}$ induces a weakly copointed functor $\mathcal{N}_{\Lambda} \times \mathcal{U}$. Take $\mathcal{N}_{\mathcal{L}}$ to be the weakly copointed subfunctor of $\mathcal{N}_{\Lambda} \times \mathcal{U}$ defined by the the axioms \mathcal{A} , i.e.

$$\mathcal{N}_{\mathcal{L}} = (\mathcal{N}_{\Lambda} \times \mathcal{U})_{\mathcal{A}}$$

in notation introduced in Section 4. It is straightforward to see that the proper $\mathcal{N}_{\mathcal{L}}$ -coalgebras are precisely the Λ -neighbourhood frames satisfying the frame conditions \mathcal{A} . The functors $\mathcal{N}_{\mathcal{L}}$ and $M_{\mathcal{L}}$ are naturally isomorphic via the transformation $\theta: M_{\mathcal{L}} \rightarrow \mathcal{N}_{\mathcal{L}}$ given by

$$\theta_X(\Phi)_L = (\{A \subseteq X \mid LA \in \Phi\}, \{A \subseteq X \mid A \in \Phi\}),$$

which is also compatible with the predicate liftings. We can thus translate Corollary 6.4 into the language of neighbourhood semantics:

Corollary 6.5 *The logic $\mathcal{L} = (\Lambda, \mathcal{A})$ is weakly complete over the class of finite neighbourhood frames that satisfy the axioms in \mathcal{A} as frame conditions.*

That is, one instance of the coalgebraic weak completeness theorem (Theorem 5.8) is weak completeness of non-iterative modal logics over their neighbourhood semantics as originally proved by Lewis [10]

Remark 6.6 The weak completeness result in the above-mentioned previous work on algebraic-coalgebraic semantics [13, Corollary 37] (see Remark 5.9) similarly puts weak neighbourhood completeness of non-iterative logics in a coalgebraic context: Given a rank-1 logic \mathcal{L} , the canonical Λ -structure for the given rank-1 logic satisfies the conditions of [13, Corollary 37], in particular is *one-step complete* (the simpler version of 0-1-step completeness that applies to rank-1 logics) [17], and is isomorphic to the subfunctor of the neighbourhood functor defined by the given rank-1 axioms; [13, Corollary 37] then guarantees that weak completeness is retained in any extension of \mathcal{L} with non-iterative axioms mentioning only finitely many modalities. By comparison, Corollary 6.5 above removes the restriction to finitely many modalities.

Remark 6.7 (Strong 0-1-step completeness) The strong completeness proof for rank-1 canonical structures [17] (which shows that every rank-1 logic is strongly complete over its neighbourhood semantics) can be factored through establishing *strong one-step completeness*, i.e. showing that the one-step logic (the simpler version of the 0-1-step logic that suffices in the rank-1 case) of a canonical structure is strongly complete [17, Remark 55]. Similarly, the 0-1-step logic of the canonical Λ -structure $\mathcal{M}_{\mathcal{L}}$ defined above is strongly complete; that is, for every set X , every consistent set of 0-1-step formulae over $\mathcal{P}X$ is satisfiable over $\mathcal{M}_{\mathcal{L}}$. Indeed, this is immediate from the 0-1-step truth lemma (Lemma 6.1) and the 0-1-step Lindenbaum lemma (Lemma 6.2). On the other hand, the 0-1-step logic of the copointed part of the canonical Λ -structure, or indeed of any copointed functor, clearly fails to be strongly complete: Let α be a non-principal ultrafilter on a set X ; then α can be seen as a set of 0-1-step formulae over $\mathcal{P}X$, and as such is consistent; but α is clearly not satisfiable over any copointed functor. Strong completeness of the 0-1-step logic is the moral reason we include weakly copointed functors in the technical development even though, as indicated in Remark 3.5, we could in principle short-circuit them.

7 Strong Completeness

We proceed to prove our main result, strong completeness of non-iterative modal logics over their canonical structure, and hence over their neighbourhood semantics. The centrepiece of the technical development is an existence lemma; we set out to prepare its proof. As usual, one has

Lemma 7.1 (Lindenbaum Lemma) *Every consistent set of Λ -formulae is contained in a maximally consistent set.*

The existence lemma requires us to show 0-1-step consistency of a set of 0-1-step formulae specifying coherence and properness. We start with the following observation, which is fairly immediate by Lemma 5.6:

Lemma 7.2 *Let $\Phi \in C_{\mathcal{L}}$ be a maximally consistent set. Then the set*

$$\{L\hat{\phi} \mid L\phi \in \Phi\} \cup \{\neg L\hat{\phi} \mid \neg L\phi \in \Phi\} \cup \{\hat{\phi} \mid \phi \in \Phi\}$$

of 0-1-step formulae over $\mathcal{P}C_{\mathcal{L}}$ is 0-1-step consistent.

The key step is then to extend the last component of the union above from expressible subsets of $C_{\mathcal{L}}$ to arbitrary subsets:

Lemma 7.3 *Let $\Phi \in C_{\mathcal{L}}$ be a maximally consistent set. Then the set*

$$\{L\hat{\phi} \mid L\phi \in \Phi\} \cup \{\neg L\hat{\phi} \mid \neg L\phi \in \Phi\} \cup \hat{\Phi}$$

of 0-1-step formulae over $\mathcal{PC}_{\mathcal{L}}$ is 0-1-step consistent.

Recall here that $\hat{\Phi} = \{A \subseteq C_{\mathcal{L}} \mid \Phi \in A\}$ is the principal ultrafilter generated by Φ , and note $\hat{\Phi} \supseteq \{\hat{\phi} \mid \phi \in \Phi\}$. The proof makes central use of Lemma 4.5.(i) and (ii) in a step-wise elimination of atoms in $\hat{\Phi} \setminus \{\hat{\phi} \mid \phi \in \Phi\}$ from 0-1-step derivations. With Lemma 7.3 in place, the existence lemma follows straightforwardly:

Lemma 7.4 (Existence lemma) *There exists a coherent proper $M_{\mathcal{L}}$ -coalgebra on $C_{\mathcal{L}}$.*

Using the Lindenbaum lemma 7.1 and the truth lemma (Lemma 5.5) in the standard fashion, we then obtain our main result:

Theorem 7.5 *The logic \mathcal{L} is strongly complete over proper $M_{\mathcal{L}}$ -coalgebras, and hence over coalgebras for the copointed part (Lemma and Definition 3.4) of $M_{\mathcal{L}}$.*

By the equivalence between the canonical structure and neighbourhood semantics as outlined in Section 6, we thus obtain

Corollary 7.6 *Every non-iterative logic $\mathcal{L} = (\Lambda, \mathcal{A})$ is (sound and) strongly complete over its neighbourhood semantics, i.e. over the class of neighbourhood frames that satisfy the axioms in \mathcal{A} as frame conditions.*

8 Application to Deontic Logic

Deontic logic is concerned with modalities of obligation, such as $O\phi$ ‘ ϕ is obligatory’ and $O(\phi|\psi)$ ‘given ψ , ϕ is obligatory’ (conditional obligation). It is faced with with specific challenges; e.g., conditional obligations are defeasible, and it is therefore nontrivial to come with principles of *factual detachment*, i.e. of deriving actual from conditional obligations, and moreover one needs to avoid the *deontic explosion* that would be caused by unrestricted normality of the obligation modality: If one had an axiom $(Oa \wedge Ob) \rightarrow O(a \wedge b)$, then a single dilemma $(Oa \wedge O\neg a)$ would cause impossible obligations ($O\perp$), making everything obligatory if additionally monotonicity is imposed. Recent developments in deontic logic often are driven mostly axiomatically, so that the only available semantics is neighbourhood semantics.

As an example, we treat axioms for factual detachment proposed by Straßer [20]. The full logical framework uses principles of adaptive logic to govern the actual factual detachment mechanism; here, we concentrate on the underlying deontic logics called the *base logics* of the framework. The logic distinguishes specific types of obligation respectively called *instrumental* and

proper (we refer to [20] for their philosophical definition), and has modalities $O(- | -)$ (binary conditional obligation), O^i (unary instrumental obligation), O^p (unary proper obligation), and $\bullet^i O(- | -)$, $\bullet^p O(- | -)$; the latter two binary modalities serve to block factual detachment of instrumental and proper obligations from conditional obligations, respectively. Corresponding dual permission modalities are denoted by replacing O with P . Various axiomatizations are developed as extensions of Goble's logic **CPDM**, which is aimed at avoiding the deontic explosion and is axiomatized in rank 1 [5]. In the online appendix [19] to [20], it is shown that two such logics **CDPM.2d⁺** and **CDPM.2e⁺** are weakly complete w.r.t. neighbourhood semantics when nesting of modalities is excluded. These logics are non-iterative; they include congruence rules and various rank-1 axioms that we refrain from listing in full, and properly non-iterative axioms

$$\begin{aligned}
(O(a | b) \wedge b \wedge \neg \bullet^p O(a | b)) &\rightarrow O^p a && \text{(FDp)} \\
(O(a | b) \wedge b \wedge \neg \bullet^i O(a | b)) &\rightarrow O^i a && \text{(FDi)} \\
(O(a | b) \wedge \neg a \wedge b) &\rightarrow \bullet^i O(a | b) && \text{(fV)} \\
((P(\neg a | b \wedge c) \vee O(\neg a | b \wedge c)) &&& \text{(Ep)} \\
&\wedge b \wedge c \wedge P(b \wedge c | b) \wedge O(a | b)) && \rightarrow \bullet^p O(a | b) \\
((P(\neg a | b \wedge c) \vee O(\neg a | b \wedge c)) &&& \text{(oV-Ei)} \\
&\wedge b \wedge c \wedge O(a | b)) && \rightarrow \bullet^i O(a | b)
\end{aligned}$$

where we have converted (Ep) and (oV-Ei) from rules to axioms (Remark 2.2). E.g. (FDp) says that we can detach a proper obligation $O^p a$ from a conditional $O(a | b)$ if this is not blocked and b is actually the case, and (fV) say that detaching an instrumental obligation $O^i a$ from a conditional obligation $O(a | b)$ is blocked if the obligation is factually violated ($\neg a \wedge b$). By our Corollary 7.6, the fully modal versions (with nested modalities) of both **CDPM.2d⁺** and **CDPM.2e⁺** are strongly complete w.r.t. neighbourhood semantics.

9 Conclusion and Future Work

We have shown that every non-iterative modal logic is strongly complete over neighbourhood semantics, complementing a classical result by Lewis stating that every such logic is weakly complete and has the finite model property over neighbourhood semantics. Our proof is via coalgebraic semantics, and indeed our main result can be phrased as saying that non-iterative logics are strongly complete over their canonical coalgebraic semantics. A fine point in the coalgebraic semantics is that conceptually, the proof needs to use weakly copointed functors, equipped with a natural transformation into the ultrafilter functor instead of the identity functor like copointed functors, to incorporate non-iterative frame conditions, instead of copointed functors as one would expect. That is, the natural generalization of the construction for the rank-1 case [17], which uses maximally consistent sets in the so-called 0-1-step logic, produces only a weakly copointed functor. Ex post, however, our main result then does

imply completeness w.r.t. a copointed subfunctor. We have applied these results to deontic logics allowing factual detachment, obtaining that these logics are strongly complete over neighbourhood semantics, improving on previous weak completeness results [20]. It will be interesting to connect our results to coalgebraic ultrafilter extensions [7] and the coalgebraic Goldblatt-Thomason theorem [8].

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