Modal Logic for Induction

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Abstract

We use modal logic to obtain syntactical, proof-theoretic versions of transfinite induction as axioms or rules within an appropriate labelled sequent calculus. While transfinite induction proper, also known as Noetherian induction, can be represented by a rule, the variant in which induction is done up to an arbitrary but fixed level happens to correspond to the Gödel–Löb axiom of provability logic. To verify the practicability of our approach in actual practice, we sketch a fairly universal pattern for proof transformation and test its use in several cases. Among other things, we give a direct and elementary syntactical proof of Segerberg's theorem that the Gödel–Löb axiom characterises precisely the (converse) well-founded and transitive Kripke frames.

Keywords: Induction principles, elementary proofs, modal logic, proof theory, Kripke model, sequent calculus.

1 Introduction

At least since Peano formalised what we all know as mathematical induction, induction as a proof principle has been the main tool for tidily unwrapping the potential infinite as generated by an a priori incomplete process. This is well reflected by the ubiquity of definitions and proofs by induction in today's ever more formal sciences.

Transfinite induction is a generalisation of mathematical induction from the natural numbers to less down-to-earth well-founded orders, such as the ordinal numbers. More precisely, if (and only if) any given order is well-founded, then *induction* holds: in the sense that a predicate holds everywhere in the given order provided that the predicate is progressive, i.e. propagates from all predecessors of a given element to the element itself.

As a rule of thumb, instances of induction are applicable more directly, and are better behaved proof-theoretically, than the corresponding instances of well-foundedness, which come as extremum principles or chain conditions (see, e.g., Proposition 4.2 below). Characteristic examples include Aczel's Set Induction [1–3] versus von Neumann and Zermelo's Axiom of Foundation or Regularity, and Raoult's Open Induction [4,11,27] as opposed to Zorn's Lemma.

Awareness of this phenomenon brought us to carry over to the inductive side some occurrences of well-foundedness in the modal logic of provability. Perhaps Segerberg's theorem [35], which stood right at the beginning of an impressive development [9], is the most prominent case: the Gödel–Löb axiom characterises exactly the (converse) well-founded and transitive Kripke frames. [1] The observation that those occurrences are rather about induction prompted the present investigation.

Inasmuch as instances of induction are about predicates or subsets, they typically go beyond the given logical level, and actually have a somewhat semantic flavour [12, 13]. By modal logic [5, 24, 26] we now obtain syntactical, proof-theoretic variants of induction: they are expressed as axioms or rules within an adequate labelled sequent calculus [21, 23]. While induction proper, for which we say Noetherian induction, can be mirrored by a rule (Lemma 3.3), the variant in which induction is done up to an arbitrary but fixed point of the given order, which we dub Gödel–Löb induction, happens to correspond (Lemma 3.1) to the homonymous axiom of provability logic [6,7,19,36,38]. [2] In fact the usual way to define validity in a Kripke model for the modal operator $\Box$ lends itself naturally to capture universal validity up to a point.

To verify the practicability of our approach in proof practice, we give a fairly universal pattern for proof transformation, from rather algebraic inductive proofs to formal proofs with the required rules, and test this in several cases. Among other things, we prove with the corresponding modal rules that induction necessitates the order under consideration to be irreflexive (Lemma 4.1), and that every meet-closed inductive predicate on a poset propagates from the irreducible elements to any element whatsoever (Example 3.5) [28, 31, 32]. As a by product we gain the curiosity that Noetherian induction is tantamount to the corresponding chain condition plus irreflexivity (Proposition 4.2). [3] Last but not least we give a direct and elementary syntactical proof (Theorem 4.3) of Segerberg's aforementioned theorem that the Gödel–Löb axiom holds exactly in the (converse) well-founded and transitive Kripke frames. All this can also be useful in proof practice: while it might be cumbersome to prove directly that an induction principle holds for a given order, it is often easier to check properties such as irreflexivity and transitivity, or even chain conditions.

[1] See also, for example, Theorem 3.5 of [37], Example 3.9 of [5] and Teorema 7.2 of [24].

[2] This was also called axiom A3 [37], the Löb formula L [5] and axiom G or axiom W [17,23].

[3] Needless to say, this requires some countable choice.

2 Basic modal logic K

Modal logic is obtained from propositional logic by adding the modal operator $\Box$ to the language of propositional logic. A *Kripke model* [18] (X, R, val) is a set X together with an *accessibility relation* R, i.e. a binary relation between elements of X, and a valuation val, i.e. a function assigning one of the truth values 0 or 1 to an element x of X and an atomic formula P. The usual notation is for $\mathrm{val}(x, P) = 1$ is $x \Vdash P$.

We read "xRy" as "y is *accessible* from x" and we read "$x \Vdash P$" as "x *forces* P". Valuations are extended in a unique way to arbitrary formulae by means of inductive clauses:

$$\begin{aligned}
&x \nVdash \bot\\
&x \Vdash A \supset B \text{ if and only if } x \Vdash A \Rightarrow x \Vdash B\\
&x \Vdash A \wedge B \text{ if and only if } x \Vdash A \text{ and } x \Vdash B\\
&x \Vdash A \vee B \text{ if and only if } x \Vdash A \text{ or } x \Vdash B\\
&x \Vdash \Box A \text{ if and only if } \forall y(xRy \Rightarrow y \Vdash A)
\end{aligned}$$

We assume that $x \Vdash P$ is decidable for every $x \in X$ and each atomic formula P, which carries over to arbitrary formulae by the inductive clauses. With the intended applications in mind, in place of R we use the inverse accessibility relation $<$, i.e. we stipulate that $y < x$ if and only if xRy. The pair $(X, <)$ is then dubbed *Kripke frame.*

We adopt the variant $\mathbf{G3K}_<$ (see Table 2) of the calculus **G3K** introduced in [21] for the *basic modal logic* **K** with the additional initial sequents

$$y < x, \Gamma \rightarrow \Delta, y < x \qquad (\sigma_<)$$

$$y = x, \Gamma \rightarrow \Delta, y = x \qquad (\sigma_=)$$

and the rules for equality (see Table 2). With $\neg A$ defined as $A \supset \bot$, the rules $L\neg, R\neg$ are special cases of $L \supset, R \supset$, and we do not give them explicitly.

The basic idea of the calculus is the syntactical internalisation of Kripke semantics: the calculus operates on labelled formulae $x \colon A$, to be read as "x forces A", and on relational formulae $y < x$. For each connective and for the modality $\Box$ the rules are obtained directly from the inductive forcing clauses for compound formulae.

As is common, we denote by $\mathbf{G3K}^*_<$ the extension of $\mathbf{G3K}_<$ with additional rules corresponding to frame properties $*$, The situation is as as laid out in Table 1, in which we use the common abbreviation $\forall y < x\, A$ for $\forall y(y < x \Rightarrow A)$.

Frame property	Rule
Reflexivity $\forall x(x < x)$	$\dfrac{x < x, \Gamma \to \Delta}{\Gamma \to \Delta}\,{\scriptstyle Ref}$
Irreflexivity $\forall x(x \not< x)$	$\dfrac{}{x < x, \Gamma \to \Delta}\,{\scriptstyle Irref}$
Transitivity $\forall x \forall y < x \forall z < y(z < x)$	$\dfrac{x < z, x < y, y < z, \Gamma \to \Delta}{x < y, y < z, \Gamma \to \Delta}\,{\scriptstyle Trans}$

Table 1
Additional rules for $\mathbf{G3K}^*_<$ and the corresponding frame properties

Initial sequents

$x\colon P, \Gamma \to \Delta, x\colon P$

$x\colon \Box A, \Gamma \to \Delta, x\colon \Box A$

$y < x, \Gamma \to \Delta, y < x$

$x = y, \Gamma \to \Delta, x = y$

Propositional rules

$$\frac{x\colon A, x\colon B, \Gamma \to \Delta}{x\colon A \wedge B, \Gamma \to \Delta}\,{\scriptstyle L\wedge} \qquad \frac{\Gamma \to \Delta, x\colon A \quad \Gamma \to \Delta, x\colon B}{\Gamma \to \Delta, x\colon A \wedge B}\,{\scriptstyle R\wedge}$$

$$\frac{x\colon A, \Gamma \to \Delta \quad x\colon B, \Gamma \to \Delta}{x\colon A \vee B, \Gamma \to \Delta}\,{\scriptstyle L\vee} \qquad \frac{\Gamma \to \Delta, x\colon A, x\colon B}{\Gamma \to \Delta, x\colon A \vee B}\,{\scriptstyle R\vee}$$

$$\frac{\Gamma \to \Delta, x\colon A \quad x\colon B, \Gamma \to \Delta}{x\colon A \supset B, \Gamma \to \Delta}\,{\scriptstyle L\supset} \qquad \frac{x\colon A, \Gamma \to \Delta, x\colon B}{\Gamma \to \Delta, x\colon A \supset B}\,{\scriptstyle R\supset}$$

$$\frac{}{x\colon \bot, \Gamma \to \Delta}\,{\scriptstyle L\bot}$$

Modal rules

$$\frac{y\colon A, x\colon \Box A, y < x, \Gamma \to \Delta}{x\colon \Box A, y < x, \Gamma \to \Delta}\,{\scriptstyle L\Box} \qquad \frac{y < x, \Gamma \to \Delta, y\colon A}{\Gamma \to \Delta, x\colon \Box A}\,{\scriptstyle R\Box} \quad (y \text{ fresh})$$

Rules for equality

$$\frac{x = x, \Gamma \to \Delta}{\Gamma \to \Delta}\,{\scriptstyle Eq\text{-}Ref} \qquad \frac{x = z, x = y, y = z, \Gamma \to \Delta}{x = y, y = z, \Gamma \to \Delta}\,{\scriptstyle Eq\text{-}Trans}$$

$$\frac{y < z, x = y, x < z, \Gamma \to \Delta}{x = y, x < z, \Gamma \to \Delta}\,{\scriptstyle Repl_{<_1}} \qquad \frac{x < y, z = y, x < z, \Gamma \to \Delta}{z = y, x < z, \Gamma \to \Delta}\,{\scriptstyle Repl_{<_2}}$$

$$\frac{y\colon P, x = y, x\colon P, \Gamma \to \Delta}{x = y, x\colon P, \Gamma \to \Delta}\,{\scriptstyle Repl_{At}}$$

Table 2
The sequent calculus $\mathbf{G3K}_<$

Derivable sequents

$x\colon A, \Gamma \to \Delta, x\colon A$

$x\colon A \supset B, x\colon A, \Gamma \to \Delta, x\colon B$

$\to x\colon \Box(A \supset B) \supset (\Box A \supset \Box B)$

Admissible rule: Substitution

$$\frac{\Gamma \to \Delta}{\Gamma[y/x] \to \Delta[y/x]}\,{\scriptstyle Subs}$$

Admissible rules: Weakening

$$\frac{\Gamma \to \Delta}{x\colon A, \Gamma \to \Delta}\,{\scriptstyle LW} \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, x\colon A}\,{\scriptstyle RW}$$

$$\frac{\Gamma \to \Delta}{y < x, \Gamma \to \Delta}\,{\scriptstyle LW_<} \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, y < x}\,{\scriptstyle RW_<}$$

Admissible rule: Necessitation

$$\frac{\to x\colon A}{\to x\colon \Box A}\,{\scriptstyle N}$$

Admissible rules: Contraction

$$\frac{x\colon A, x\colon A, \Gamma \to \Delta}{x\colon A, \Gamma \to \Delta}\,{\scriptstyle LC} \qquad \frac{\Gamma \to \Delta, x\colon A, x\colon A}{\Gamma \to \Delta, x\colon A}\,{\scriptstyle RC}$$

$$\frac{y < x, y < x, \Gamma \to \Delta}{y < x, \Gamma \to \Delta}\,{\scriptstyle LC_<} \qquad \frac{\Gamma \to \Delta, y < x, y < x}{\Gamma \to \Delta, y < x}\,{\scriptstyle RC_<}$$

Admissible rule: Replacement

$$\frac{y\colon A, x = y, x\colon A, \Gamma \to \Delta}{x = y, x\colon A, \Gamma \to \Delta}\,{\scriptstyle Repl}$$

Admissible rules: Cut

$$\frac{\Gamma \to \Delta, x\colon A \qquad x\colon A, \Gamma' \to \Delta'}{\Gamma, \Gamma' \to \Delta, \Delta'}\,{\scriptstyle Cut}$$

$$\frac{\Gamma \to \Delta, y < x \qquad y < x, \Gamma' \to \Delta'}{\Gamma, \Gamma' \to \Delta, \Delta'}\,{\scriptstyle Cut_<} \qquad \frac{\Gamma \to \Delta, y = x \qquad y = x, \Gamma' \to \Delta'}{\Gamma, \Gamma' \to \Delta, \Delta'}\,{\scriptstyle Cut_=}$$

Table 3
Structural properties and admissible rules of the sequent calculus $\mathbf{G3K}_<$

The calculus $\mathbf{G3K}_<$ satisfies the following structural properties (for more detail see Table 3, and for a proof see Section 11.4 of [23]):

(i) Sequents of the forms

$$x\colon A, \Gamma \to \Delta, x\colon A$$
$$x\colon A \supset B, x\colon A, \Gamma \to \Delta, x\colon B$$
$$\to x\colon \Box(A \supset B) \supset (\Box A \supset \Box B)$$

are derivable in $\mathbf{G3K}^*_<$ for arbitrary modal formulae A and B.

(ii) The rules of substitution, weakening, contraction and replacement for arbitrary formulae are height-preserving admissible in $\mathbf{G3K}^*_<$.

(iii) The rule of necessitation is admissible in $\mathbf{G3K}^*_<$.

(iv) All the rules of the system $\mathbf{G3K}^*_<$ are height-preserving invertible.

(v) The *Cut* rule is admissible in $\mathbf{G3K}_<$.

Since we add the initial sequents $\sigma_<, \sigma_=$, we also need the following:

Lemma 2.1 *Rules $Cut_<$ and $Cut_=$ are admissible in* $\mathbf{G3K}_<$.

Proof. The proof is induction as the proof of admissibility of *Cut* (see [23], Theorem 11.9), from which we exclude the cases in which the cut formula is principal as no rule has instances of $=, <$ as principal formulae. All the remaining cases are completely analogous to their counterparts in the proof of admissibility of *Cut*. $\Box$

Two important results, to which we will collectively refer as *completeness*, carry over from [22]:

Theorem 2.2 *Let $\Gamma \to \Delta$ be a sequent in the language of $\mathbf{G3K}^*_<$. Then either the sequent is derivable in $\mathbf{G3K}^*_<$ or it has a Kripke countermodel with properties $*$.*

Corollary 2.3 *If a sequent $\Gamma \to \Delta$ is valid in every Kripke model with the frame properties $*$, then it is derivable in the system $\mathbf{G3K}^*_<$.*

2.1 Connective-like rules for propositional variables

In some of the applications below, we will need to add a propositional variable P to the language of $\mathbf{K}$ that will have a "connective-like" behavior. For instance, suppose that we want a variable P to behave at x as $Q(x) \supset R(x)$. In order to avoid self-referential definitions, we ask Q and R not to contain P. We then add the following clause to the definition of val:

$$x \Vdash P \text{ if and only if } Q(x) \Rightarrow R(x)$$

Doing so, we further add to $\mathbf{G3K}^*_<$ a pair of rules that mirror the logical rules:

$$\frac{\Gamma \to \Delta, Q(x) \qquad R(x), \Gamma \to \Delta}{x\colon P, \Gamma \to \Delta}\,{\scriptstyle LP} \qquad \frac{Q(x), \Gamma \to \Delta, R(x)}{\Gamma \to \Delta, x\colon P}\,{\scriptstyle RP}$$

Since they have the same behavior as the logical connectives, all proofs given or referred to in the last section can easily be generalised to extensions of $\mathbf{G3K}_<$ by rules of this kind. In particular, LP and RP are invertible and completeness still holds. We just point out that in the proof of admissibility of *Cut*, we have to be careful when considering the case in which the cut formula is principal in both premisses. For instance when we transform

$$\dfrac{\dfrac{Q(x),\Gamma \to \Delta, R(x)}{\Gamma \to \Delta, x\colon P}\,{\scriptstyle RP} \qquad \dfrac{\Gamma' \to \Delta', Q(x) \quad R(x),\Gamma' \to \Delta'}{x\colon P, \Gamma' \to \Delta'}\,{\scriptstyle LP}}{\Gamma,\Gamma' \to \Delta,\Delta'}\,{\scriptstyle Cut}$$

into

$$\dfrac{\dfrac{\Gamma' \to \Delta', Q(x) \qquad \dfrac{Q(x),\Gamma \to \Delta, R(x) \qquad R(x),\Gamma' \to \Delta'}{Q(x),\Gamma,\Gamma' \to \Delta,\Delta'}\,{\scriptstyle Cut_{(<,=)}}}{\Gamma,\Gamma',\Gamma' \to \Delta,\Delta',\Delta'}\,{\scriptstyle Cut_{(<,=)}}}{\Gamma,\Gamma' \to \Delta,\Delta'}\,{\scriptstyle LC,RC \text{ (multiple times)}}$$

we have to take into consideration that $Q(x), R(x)$ may be instances of $<, =$.

3 Induction principles

Induction principles are typically not expressible within a first-order language. We now present them as ordinary rules of labelled sequent calculus. To start with, we recall *Noetherian Induction* and define *Gödel–Löb Induction*:

$$\forall y(\forall z < y\, Ez \Rightarrow Ey) \Rightarrow \forall y\, Ey \qquad (\textit{Noeth-Ind})$$
$$\forall x(\forall y < x(\forall z < y\, Ez \Rightarrow Ey) \Rightarrow \forall y < x\, Ey) \qquad (\textit{GL-Ind})$$

They prompt us to consider two rules and an axiom on top of $\mathbf{G3K}_<$ (rule $R\Box$-*GLI* is rule $R\Box$-*L* of [23]):

$$\dfrac{y\colon \Box A, \Gamma \to \Delta, y\colon A}{\Gamma \to \Delta, y\colon A}\,{\scriptstyle NI} \qquad \dfrac{y < x, y\colon \Box A, \Gamma \to \Delta, y\colon A}{\Gamma \to \Delta, x\colon \Box A}\,{\scriptstyle R\Box\text{-}GLI}$$

$$\Box(\Box A \supset A) \supset \Box A \qquad (W)$$

Both rules come with the variable condition that y does not appear in Γ, Δ.

Lemma 3.1 *Let a Kripke frame $(X, <)$ be given. The following are equivalent:*

(i) *Axiom W is valid in X for every formula A.*

(ii) *Axiom W is valid in X for every propositional variable A.*

(iii) *Gödel–Löb Induction holds in X, i.e.*

$$\forall x(\forall y < x(\forall z < y\, Ez \Rightarrow Ey) \Rightarrow \forall y < x\, Ey) \qquad (\textit{GL-Ind})$$

for any given predicate $E(x)$ on X.

Proof. (i)⇒(ii). Trivial.
(ii)⇒(iii). Given $E(x)$, pick a propositional variable A and take a valuation such that $x \Vdash A$ if and only if $E(x)$. Then by expanding the definitions we have the following:

$$\begin{aligned}
& x \Vdash \Box(\Box A \supset A) \supset \Box A \\
\Longrightarrow\, & x \Vdash \Box(\Box A \supset A) \Rightarrow x \Vdash \Box A \\
\Longrightarrow\, & \forall y < x\, y \Vdash \Box A \supset A \Rightarrow \forall y < x\, y \Vdash A \\
\Longrightarrow\, & \forall y < x\, (y \Vdash \Box A \Rightarrow y \Vdash A) \Rightarrow \forall y < x\, y \Vdash A \\
\Longrightarrow\, & \forall y < x\, (\forall z < y\, z \Vdash A \Rightarrow y \Vdash A) \Rightarrow \forall y < x\, y \Vdash A \\
\Longrightarrow\, & \forall y < x\, (\forall z < y\, Ez \Rightarrow Ey) \Rightarrow \forall y < x\, Ey
\end{aligned}$$

(iii)⇒(i). Given a formula A, define $E(x)$ as $x \Vdash A$ and read backwards the proof of (ii)⇒(iii). □

Lemma 3.2 *The following are equivalent over* $\mathbf{G3K}_<$ *without* $R\Box$ *(including the structural rules):*

(i) *Rule* $R\Box$*-GLI,*

(ii) *Rule* $R\Box$ *plus axiom W.*

Proof. Claim 1: $R\Box\text{-}GLI \Rightarrow R\Box$.

$$\dfrac{\dfrac{y < x, \Gamma \to \Delta, y\colon A}{y < x, y\colon \Box A, \Gamma \to \Delta, y\colon A}\,{\scriptstyle LW}}{\Gamma \to \Delta, x\colon \Box A}\,{\scriptstyle R\Box\text{-}GLI}$$

Claim 2: $R\Box\text{-}GLI \Rightarrow W$.

$$\dfrac{\dfrac{\dfrac{y < x, y\colon \Box A \supset A, y\colon \Box A, x\colon \Box(\Box A \supset A) \to y\colon A}{y < x, y\colon \Box A, x\colon \Box(\Box A \supset A) \to y\colon A}\,{\scriptstyle L\Box}}{x\colon \Box(\Box A \supset A) \to x\colon \Box A}\,{\scriptstyle R\Box\text{-}GLI}}{\to x\colon \Box(\Box A \supset A) \supset \Box A}\,{\scriptstyle R\supset}$$

Claim 3: $R\Box + W \Rightarrow R\Box\text{-}GLI$.

$$\dfrac{\dfrac{\dfrac{y < x, y\colon \Box A, \Gamma \to \Delta, y\colon A}{y < x, \Gamma \to \Delta, y\colon \Box A \supset A}\,{\scriptstyle R\supset}}{\Gamma \to \Delta, x\colon \Box(\Box A \supset A)}\,{\scriptstyle R\Box} \qquad x\colon \Box(\Box A \supset A) \to x\colon \Box A}{\Gamma \to \Delta, x\colon \Box A}\,{\scriptstyle Cut}$$

where $x\colon \Box(\Box A \supset A) \to x\colon \Box A$ is derivable from W by invertibility of $R\supset$. □

Therefore the sequent calculus obtained by replacing $R\Box$ by $R\Box$-*GLI* is an extension of $\mathbf{G3K}_<$. If we further add the mathematical rules *Trans* and *Irref*, we get the variant $\mathbf{G3KGL}_<$ of the calculus **G3KGL** [21] obtained by adding the initial sequents $\sigma_<, \sigma_=$ and removing the mathematical rules *Trans*, *Irref*.

Lemma 3.3 *Let a Kripke frame $(X, <)$ be given. The following are equivalent:*

(i) *Rule NI is sound in X.*

(ii) *For every propositional variable A, in X we have*

$$\forall y\,(y \Vdash \Box A \Rightarrow y \Vdash A) \Rightarrow \forall y\, y \Vdash A$$

for any given valuation $\Vdash$ on X.

(iii) *Noetherian Induction holds in X, i.e.*

$$\forall y(\forall z < y\, Ez \Rightarrow Ey) \Rightarrow \forall y\, Ey \qquad (\textit{Noeth-Ind})$$

for any given predicate $E(x)$ on X.

Proof. (i)⇒(ii). Suppose that, for all y, $y \Vdash \Box A$ implies $y \Vdash A$. It follows that the sequent $y\colon \Box A \to y\colon A$ is valid, hence, by completeness, derivable. Applying rule NI we can therefore derive $\to y\colon A$

$$\frac{y\colon \Box A \to y\colon A}{\to y\colon A}\,{\scriptstyle NI}$$

and by soundness we obtain that for all y, $y \Vdash A$.
(ii)⇒(iii). Given $E(x)$, pick a propositional variable A and take a valuation such that $x \Vdash A$ if and only if $E(x)$. Then:

$$\begin{aligned} &\forall y\,(y \Vdash \Box A \Rightarrow y \Vdash A) \Rightarrow \forall y\, y \Vdash A \\ \Longrightarrow\, &\forall y\,(\forall z < y\, z \Vdash A \Rightarrow y \Vdash A) \Rightarrow \forall y\, y \Vdash A \\ \Longrightarrow\, &\forall y\,(\forall z < y\, Ez \Rightarrow Ey) \Rightarrow \forall y\, Ey \end{aligned}$$

(iii)⇒(i). Given a formula A, define $E(x)$ as $x \Vdash A$ and read backwards the proof of (ii)⇒(iii). □

The lemmata proved in this section allow us to transform rather algebraic proofs using induction into tree-like derivations in modal logic, following a certain pattern:

Proof transformation pattern Let X be a set endowed with a binary relation $<$. Suppose that we need to show either

(i) a statement of the form $\forall y\, E(y)$ by way of *Noeth-Ind*, or

(ii) a statement of the form $\forall x \forall y < x\, E(y)$ by way of *GL-Ind*.

We consider $(X, <)$ as a Kripke frame, and build a Kripke model as follows. First, we consider a suitable subformula $U(x)$ of $E(x)$ such that it can be encoded in a sequent $Q(x) \to R(x)$, and fix a propositional variable P. We define a valuation such that val: $(x, P) = 1$ if and only if $U(x)$. This is done by adding (variants of) the following rules to the calculus:

$$\frac{\Gamma \to \Delta, Q(x) \qquad R(x), \Gamma \to \Delta}{x\colon P, \Gamma \to \Delta}\,{\scriptstyle LP} \qquad \frac{Q(x), \Gamma \to \Delta, R(x)}{\Gamma \to \Delta, x\colon P}\,{\scriptstyle RP}$$

By means of P, we find a formula A such that $x \Vdash A$ if and only if $E(x)$. We then proceed as follows:

(i) For *Noeth-Ind*: Derive the sequent $y\colon \Box A \to y\colon A$ by using $\mathbf{G3K}_<$ plus *RP* and *LP*, then apply rule *NI*:

$$\dfrac{\begin{array}{c}\vdots\\ y\colon \Box A \to y\colon A\end{array}}{\to y\colon A}\ {\scriptstyle NI}$$

(ii) For *GL-Ind*: Derive the sequent $y < x, y\colon \Box A \to y\colon A$ by using $\mathbf{G3K}_<$ plus *RP* and *LP*, then apply rule *R□-GLI*:

$$\dfrac{\begin{array}{c}\vdots\\ y < x, y\colon \Box A \to y\colon A\end{array}}{\Gamma \to \Delta, x\colon \Box A}\ {\scriptstyle R\Box\text{-}GLI}$$

We point out that this pattern is not fully general, as we do not yet have a universal method to find the subformula $U(x)$ needed to define the valuation.

3.1 Examples

Example 3.4 *GL-Ind* implies that $\forall y < x(y \neq x)$.[4]

Proof. [algebraic] In order to apply *GL-Ind*, we need to show that $\forall y < x(\forall z < y(z \neq x) \Rightarrow y \neq x)$. Fix $y < x$ such that $\forall z < y(z \neq x)$. We need to show that $y \neq x$. Suppose $y = x$. Then $x < x$ and $\forall z < x(z \neq x)$, from which we derive $x \neq x$. Therefore $y \neq x$ and we proved our claim. □

Proof. [modal] Fix x. Pick P such that $y \Vdash P$ if and only if $y = x$. This corresponds to the rules

$$\dfrac{y = x, \Gamma \to \Delta}{y\colon P, \Gamma \to \Delta}\ {\scriptstyle LP} \qquad \dfrac{\Gamma \to \Delta, y = x}{\Gamma \to \Delta, y\colon P}\ {\scriptstyle RP}$$

Then our thesis is equivalent to say that $\to x\colon \Box\neg P$ is derivable in $\mathbf{G3K}_<$ plus *R□-GLI*, *LP* and *RP*:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y = x, y < y, y\colon \Box\neg P \to y\colon \bot, y = x}{y = x, y < y, y\colon \Box\neg P \to y\colon \bot, y\colon P}\ {\scriptstyle RP}}{y\colon \neg P, y = x, y < y, y\colon \Box\neg P \to y\colon \bot}\ {\scriptstyle L\neg}}{y = x, y < y, y\colon \Box\neg P \to y\colon \bot}\ {\scriptstyle L\Box}}{y = x, y < x, y\colon \Box\neg P \to y\colon \bot}\ {\scriptstyle Repl}}{y < x, y\colon \Box\neg P, y\colon P \to y\colon \bot}\ {\scriptstyle LP}}{y < x, y\colon \Box\neg P \to y\colon \neg P}\ {\scriptstyle R\supset}}{\to x\colon \Box\neg P}\ {\scriptstyle R\Box\text{-}GLI}$$

□

[4] If we observe that $\forall y < x(y \neq x)$ is just a variant of irreflexivity $\forall x(x \not< x)$, then this result will be for free once we have proved Lemma 4.1 and Theorem 4.3.

Example 3.5 What follows is a somewhat more general formulation of the fact that by Noetherian induction every meet-closed predicate on a poset propagates from the irreducible elements to any element whatsoever [28, 31, 32].

Consider a ternary predicate $x = y \circ z$. We say that x is $\circ$-*reducible* (for short $R^\circ(x)$) if there are $y < x$ and $z < x$ such that $x = y \circ z$.

Let $E(x)$ be a predicate satisfying

$$\frac{x = y \circ z \quad E(y) \quad E(z)}{E(x)} \tag{$*$}$$

for every y, z. Then *Noeth-Ind* implies $\forall x(R^\circ(x) \vee E(x)) \Rightarrow \forall x\, E(x)$.

Proof. [algebraic] Assume that $\forall x(R^\circ(x) \vee E(x))$. In order to apply induction, we need to show that $\forall x(\forall y < x\, E(y) \Rightarrow E(x))$. Fix x such that $\forall y < x\, E(y)$. It now suffices to show $E(x)$. By assumption, we can distinguish two cases:

- Case $E(x)$: Trivial.
- Case $R^\circ(x)$: Take $y < x$ and $z < x$ such that $x = y \circ z$. By $\forall y < x\, E(y)$ we know that $E(y)$ and $E(z)$. This, by $(*)$ implies $E(x)$.

□

Proof. [modal] Pick a propositional variable P such that $x \Vdash P$ if and only if $E(x)$. The hypothesis $(*)$ can be written as:

$$\frac{x\colon P, y\colon P, z\colon P, x = y \circ z, \Gamma \rightarrow \Delta}{y\colon P, z\colon P, x = y \circ z, \Gamma \rightarrow \Delta}\,{\scriptstyle (*)}$$

The definition of being $\circ$-reducible can be used in the calculus via the rule

$$\frac{x = y \circ z, y < x, z < x, \Gamma \rightarrow \Delta}{R^\circ(x), \Gamma \rightarrow \Delta}\,{\scriptstyle LR^\circ}$$

where y, z are fresh, together with the appropriate RR° rule. The thesis becomes that from the sequent $\rightarrow R^\circ(x), x\colon P$ we can derive $\rightarrow x\colon P$ in $\mathbf{G3K}_<$ using *NI*, $(*)$, LR° and RR°. In fact:

$$\dfrac{\dfrac{\rightarrow R^\circ(x), x\colon P \qquad \dfrac{\dfrac{\dfrac{\dfrac{x = y \circ z, y < x, z < x, x\colon P, z\colon P, y\colon P, x\colon \Box P \rightarrow x\colon P}{x = y \circ z, y < x, z < x, z\colon P, y\colon P, x\colon \Box P \rightarrow x\colon P}\,{\scriptstyle (*)}}{x = y \circ z, y < x, z < x, y\colon P, x\colon \Box P \rightarrow x\colon P}\,{\scriptstyle L\Box}}{x = y \circ z, y < x, z < x, x\colon \Box P \rightarrow x\colon P}\,{\scriptstyle L\Box}}{R^\circ(x), x\colon \Box P \rightarrow x\colon P}\,{\scriptstyle LR^\circ}}{x\colon \Box P \rightarrow x\colon P}\,{\scriptstyle Cut}}{\rightarrow x\colon P}\,{\scriptstyle NI}$$

□

4 Consequences

In this section we apply the tools that we have just developed, in order to revisit certain common properties of the accessibility relation $<$. In particular, this will lead us to useful characterisations of the induction principles that can simplify the task of controlling that they hold in a given structure. We will further shed some more light on the role of transitivity in the calculus.

4.1 Irreflexivity & Noetherianity

The binary relation $<$ on X is said to be *irreflexive* if $\forall x(x \not< x)$, which corresponds to the following rule

$$\frac{}{x < x, \Gamma \to \Delta}\ \mathit{Irref}$$

Lemma 4.1 *Noetherian Induction implies irreflexivity.*[5]

Proof. To show this claim, we use the syntactical proof pattern introduced in Section 3. Pick P such that $x \Vdash P$ if and only if $x < x$, i.e. such that

$$\frac{x < x, \Gamma \to \Delta}{x\colon P, \Gamma \to \Delta}\ LP \qquad\qquad \frac{\Gamma \to \Delta, x < x}{\Gamma \to \Delta, x\colon P}\ RP$$

Then we just need to show $\to x\colon \neg P$ in $\mathbf{G3K}_<$ plus *NI*, *LP* and *RP*:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x\colon \Box\neg P, x < x \to x < x}{x < x, x\colon \Box\neg P \to x\colon P}\ RP}{x\colon \neg P, x < x, x\colon \Box\neg P \to}\ L\neg}{x < x, x\colon \Box\neg P \to}\ L\Box}{x\colon P, x\colon \Box\neg P \to}\ LP}{x\colon \Box\neg P \to x\colon \neg P}\ R\neg}{\to x\colon \neg P}\ NI$$

From this we also get admissibility of the rule version of irreflexivity:

$$\dfrac{\to x\colon \neg P \qquad \dfrac{\dfrac{x < x, \Gamma \to \Delta, x < x}{x < x, \Gamma \to \Delta, x\colon \neg P}\ RP}{x\colon \neg P, x < x, \Gamma \to \Delta}\ L\neg}{x < x, \Gamma \to \Delta}\ Cut$$

□

As in mathematical practice one often talks about ascending chains, we now occasionally switch back to R. So let $y < x$ if and only if xRy: that is, $<$ and R are converse to each other. Notice that $<$ is irreflexive if and only if so is R.

[5] This lemma is a formal direct version of "every well-founded relation is irreflexive", to be compared with "Set Induction implies $\forall x(x \notin x)$" [1–3] as a direct version of "Foundation implies $\forall x(x \notin x)$" in axiomatic set theory.

An *infinite R-sequence* is a sequence $(x_i)_{i\in\mathbb{N}}$ of elements of X such that $x_i R x_{i+1}$ for all $i \in \mathbb{N}$. An infinite R-sequence $(x_i)_{i\in\mathbb{N}}$ is *convergent* if there is $i \in \mathbb{N}$ such that $x_j = x_i$ for all $j > i$. We say that R is *well-founded* if there is no infinite R-sequence; and that R is *Noetherian*—for short, R satisfies *Noeth*—if every infinite R-sequence converges.

While the first and second item of the next lemma are well-known to be equivalent, the occurrence of irreflexivity in the third item is due to the fact that a priori R and $<$ need not possess this feature of an order relation.

Proposition 4.2 *The following are equivalent:*

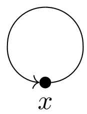
(i) *$<$ satisfies Noetherian Induction.*

(ii) *R is well-founded.*

(iii) *R is irreflexive and Noetherian.*

Proof. The equivalence of the first and the second item is folklore. See Lemma 4.1 for a formal proof that Noetherian Induction implies irreflexivity. If R is well-founded, i.e. there are no infinite R-sequences at all, then R is trivially Noetherian. As for the converse, if R is irreflexive, then no infinite R-sequence converges; whence if, in addition, R is Noetherian, then R is well-founded. □

Notice in this context that if R is Noetherian, it is not always the case that $<$ satisfies *Noeth-Ind*. In fact, the relation R with the following graph



does not satisfy *Noeth-Ind* because it is not irreflexive, but R is Noetherian because the only infinite R-sequence, which is $xRxRxR...$, converges.

4.2 Transitivity & Induction

The binary relation $<$ on X is said to be *transitive* if $\forall x \forall y < x \forall z < y (z < x)$, which corresponds to the following rule

$$\frac{z < x, z < y, y < x, \Gamma \rightarrow \Delta}{z < y, y < x, \Gamma \rightarrow \Delta}\,\textit{Trans}$$

In the light of Proposition 4.2, what we prove next in $\mathbf{G3K}_<$ is a formal version of Segerberg's theorem [35] that the Gödel–Löb axiom describes exactly the (converse) well-founded transitive Kripke frames.

Theorem 4.3 *The following are equivalent:*

(i) *Gödel–Löb Induction,*

(ii) *Noetherian Induction + Transitivity.*

Proof. <u>Claim 1: *GL-Ind* $\Rightarrow$ *Noeth-Ind*</u>. It suffices to show that rule *NI* is admissible in $\mathbf{G3KGL}_{<}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x\colon \Box A,\Gamma \to \Delta, x\colon A}{y\colon \Box A,\Gamma \to \Delta, y\colon A}\,\mathit{Subs}}{y<x, y\colon \Box A,\Gamma \to \Delta, y\colon A}\,\mathit{LW}}{\Gamma \to \Delta, x\colon \Box A}\,R\Box\text{-}\mathit{GLI} \qquad x\colon \Box A,\Gamma \to \Delta, x\colon A}{\Gamma,\Gamma \to \Delta,\Delta, x\colon A}\,\mathit{Cut}}{\Gamma \to \Delta, x\colon A}\,\mathit{LC,RC}\ \text{(multiple times)}$$

<u>Claim 2: *GL-Ind* $\Rightarrow$ *Trans*</u>. To show this claim, we use the syntactical proof pattern introduced in Section 3. Fix x. Pick P such that $y \Vdash P$ if and only if $y < x$, i.e. such that

$$\dfrac{y<x,\Gamma \to \Delta}{y\colon P,\Gamma \to \Delta}\,\mathit{LP} \qquad\qquad \dfrac{\Gamma \to \Delta, y<x}{\Gamma \to \Delta, y\colon P}\,\mathit{RP}$$

It suffices to show that rule *Trans* is admissible in $\mathbf{G3KGL}_{<}$ plus *LP* and *RP*:

$$\dfrac{\to x\colon \Box(\Box P \wedge P) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{z<x, z<y, y<x,\Gamma \to \Delta}{z<x, y\colon \Box P, y\colon P, x\colon \Box(\Box P \wedge P), z<y, y<x,\Gamma \to \Delta}\,\mathit{LW}}{z\colon P, y\colon \Box P, y\colon P, x\colon \Box(\Box P \wedge P), z<y, y<x,\Gamma \to \Delta}\,\mathit{LP}}{y\colon \Box P, y\colon P, x\colon \Box(\Box P \wedge P), z<y, y<x,\Gamma \to \Delta}\,L\Box}{y\colon \Box P \wedge P, x\colon \Box(\Box P \wedge P), z<y, y<x,\Gamma \to \Delta}\,L\wedge}{x\colon \Box(\Box P \wedge P), z<y, y<x,\Gamma \to \Delta}\,L\Box}{z<y, y<x,\Gamma \to \Delta}\,\mathit{Cut}$$

where $\to x\colon \Box(\Box P \wedge P)$ is derived as follows:[6]

$$\dfrac{\dfrac{y<x, y\colon \Box(\Box P \wedge P) \to y\colon \Box P \qquad \dfrac{y<x, y\colon \Box(\Box P \wedge P) \to y<x}{y<x, y\colon \Box(\Box P \wedge P) \to y\colon P}\,\mathit{RP}}{y<x, y\colon \Box(\Box P \wedge P) \to y\colon \Box P \wedge P}\,R\wedge}{\to x\colon \Box(\Box P \wedge P)}\,R\Box\text{-}\mathit{GLI}$$

where $y<x, y\colon \Box(\Box P \wedge P) \to y\colon \Box P$ is derived as follows:

$$\dfrac{\dfrac{z\colon \Box P, z\colon P, z<y, z\colon \Box P, y<x, y\colon \Box(\Box P \wedge P) \to z\colon P}{z\colon \Box P \wedge P, z<y, z\colon \Box P, y<x, y\colon \Box(\Box P \wedge P) \to z\colon P}\,L\wedge}{y<x, y\colon \Box(\Box P \wedge P) \to y\colon \Box P}\,L\Box$$

[6] Notice that the sequent $\to x\colon \Box(\Box P \wedge P)$ corresponds to $\forall x \forall y < x(\forall z < y(z < x)\ \&\ y < x)$, which is a redundant version of transitivity as $y < x$ is repeated both in the premisses and in the conclusions. The reason why we need this version and not the “standard” one (as, for instance, in the case of *Irref* in Lemma 4.1), will become clear in the next subsection.

Claim 3: *Noeth-Ind + Trans* $\Rightarrow$ *GL-Ind*. It suffices to show that Axiom W is derivable in $\mathbf{G3K}_<$ plus *NI* and *Trans*:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y\colon A, y<x, x\colon \Box(\Box(\Box A\supset A)\supset \Box A), x\colon \Box(\Box A\supset A)\to y\colon A \quad \mathcal{D}_1}{y\colon \Box A\supset A, y<x, x\colon \Box(\Box(\Box A\supset A)\supset \Box A), x\colon \Box(\Box A\supset A)\to y\colon A}\,{\scriptstyle L\supset}}{y<x, x\colon \Box(\Box(\Box A\supset A)\supset \Box A), x\colon \Box(\Box A\supset A)\to y\colon A}\,{\scriptstyle L\Box}}{x\colon \Box(\Box(\Box A\supset A)\supset \Box A), x\colon \Box(\Box A\supset A)\to x\colon \Box A}\,{\scriptstyle R\Box}}{x\colon \Box(\Box(\Box A\supset A)\supset \Box A)\to x\colon \Box(\Box A\supset A)\supset \Box A}\,{\scriptstyle R\supset}}{\to x\colon \Box(\Box A\supset A)\supset \Box A}\,{\scriptstyle NI}$$

where $\mathcal{D}_1$ is the following derivation:

$$\dfrac{\dfrac{y\colon \Box A, y<x, x\colon \Box(\Box(\Box A\supset A)\supset \Box A), x\colon \Box(\Box A\supset A)\to y\colon A, y\colon \Box A \qquad \mathcal{D}_2}{y\colon \Box(\Box A\supset A)\supset \Box A, y<x, x\colon \Box(\Box(\Box A\supset A)\supset \Box A), x\colon \Box(\Box A\supset A)\to y\colon A, y\colon \Box A}\,{\scriptstyle L\supset}}{y<x, x\colon \Box(\Box(\Box A\supset A)\supset \Box A), x\colon \Box(\Box A\supset A)\to y\colon A, y\colon \Box A}\,{\scriptstyle L\Box}$$

where $\mathcal{D}_2$ is the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{z\colon \Box A\supset A, z<x, z<y, y<x, x\colon \Box(\Box(\Box A\supset A)\supset \Box A), x\colon \Box(\Box A\supset A)\to y\colon A, y\colon \Box A, z\colon \Box A\supset A}{z<x, z<y, y<x, x\colon \Box(\Box(\Box A\supset A)\supset \Box A), x\colon \Box(\Box A\supset A)\to y\colon A, y\colon \Box A, z\colon \Box A\supset A}\,{\scriptstyle L\Box}}{z<y, y<x, x\colon \Box(\Box(\Box A\supset A)\supset \Box A), x\colon \Box(\Box A\supset A)\to y\colon A, y\colon \Box A, z\colon \Box A\supset A}\,{\scriptstyle Trans}}{y<x, x\colon \Box(\Box(\Box A\supset A)\supset \Box A), x\colon \Box(\Box A\supset A)\to y\colon A, y\colon \Box A, y\colon \Box(\Box A\supset A)}\,{\scriptstyle L\supset}}{y<x, x\colon \Box(\Box(\Box A\supset A)\supset \Box A), x\colon \Box(\Box A\supset A)\to y\colon A, y\colon \Box A, y\colon \Box(\Box A\supset A)}\,{\scriptstyle R\Box}$$

$\Box$

Proposition 4.2 and Theorem 4.2 help to see that *Noeth-Ind* $\not\Rightarrow$ *GL-Ind*. In fact, the structure

$$\begin{tikzcd} z \arrow[r] & y \arrow[r] & x \end{tikzcd}$$

satisfies both *Noeth* and *Irref*, but not *Trans*.

4.3 Transitivity & Cut

The rule *Cut* is known to be admissible in the calculus **G3GL** and thus, by equivalence, in **G3KGL** [23, Theorem 12.20]. As a consequence, *Cut* is also admissible in $\mathbf{G3KGL}_<$ if we add *Trans* and *Irref*. Are these two rules really needed for *Cut* admissibility?

Lemma 4.4 *The following sequents are Cut-free derivable in* $\mathbf{G3KGL}_<$*:*

(i) $x\colon \Box A\to x\colon \Box(A\wedge \Box A)$, [7]

(ii) $x\colon \Box(A\wedge \Box A)\to x\colon \Box\Box A$.

Proof. (i)

$$\dfrac{\dfrac{\dfrac{y\colon A, y<x, y\colon \Box(A\wedge \Box A), x\colon \Box A\to y\colon A}{y<x, y\colon \Box(A\wedge \Box A), x\colon \Box A\to y\colon A}\,{\scriptstyle L\Box} \qquad \mathcal{D}}{y<x, y\colon \Box(A\wedge \Box A), x\colon \Box A\to y\colon A\wedge \Box A}\,{\scriptstyle R\wedge}}{x\colon \Box A\to x\colon \Box(A\wedge \Box A)}\,{\scriptstyle R\Box\text{-}GLI}$$

[7] This is actually the redundant version of transitivity that we had in the proof of Theorem 4.3. Here, the definition of $y\Vdash A$ as $y<x$ is gained by the addition of the premiss $x\colon \Box A$.

where $\mathcal{D}$ is the following derivation:

$$\dfrac{\dfrac{\dfrac{z\colon A, z\colon \Box A, z < y, z\colon \Box A, y < x, y\colon \Box(A \wedge \Box A), x\colon \Box A \to z\colon A}{z\colon A \wedge \Box A, z < y, z\colon \Box A, y < x, y\colon \Box(A \wedge \Box A), x\colon \Box A \to z\colon A}\,{\scriptstyle L\wedge}}{z < y, z\colon \Box A, y < x, y\colon \Box(A \wedge \Box A), x\colon \Box A \to z\colon A}\,{\scriptstyle L\Box}}{y < x, y\colon \Box(A \wedge \Box A), x\colon \Box A \to y\colon \Box A}\,{\scriptstyle R\Box\text{-}GLI}$$

(ii)

$$\dfrac{\dfrac{\dfrac{y\colon A, y\colon \Box A, y < x, y\colon \Box\Box A, x\colon \Box(A \wedge \Box A) \to y\colon \Box A}{y\colon A \wedge \Box A, y < x, y\colon \Box\Box A, x\colon \Box(A \wedge \Box A) \to y\colon \Box A}\,{\scriptstyle L\wedge}}{y < x, y\colon \Box\Box A, x\colon \Box(A \wedge \Box A) \to y\colon \Box A}\,{\scriptstyle L\Box}}{x\colon \Box(A \wedge \Box A) \to x\colon \Box\Box A}\,{\scriptstyle R\Box\text{-}GLI}$$

□

Theorem 4.5 *The Cut rule is not admissible in* **G3KGL**$_<$ *without Trans.*

Proof. If *Cut* were admissible, then by Lemma 4.4 the sequent $x\colon \Box A \to x\colon \Box\Box A$ would be *Cut*-free derivable. [8] Let's try to give a *Cut*-free proof:

$$\dfrac{\dfrac{\dfrac{y\colon A, z < y, z\colon \Box A, y < x, y\colon \Box\Box A, x\colon \Box A \to z\colon A}{y\colon A, z < y, z\colon \Box A, y < x, y\colon \Box\Box A, x\colon \Box A \to z\colon A}\,{\scriptstyle L\Box}}{y < x, y\colon \Box\Box A, x\colon \Box A \to y\colon \Box A}\,{\scriptstyle R\Box\text{-}GLI}}{x\colon \Box A \to x\colon \Box\Box A}\,{\scriptstyle R\Box\text{-}GLI}$$

Observe, however, that the upper-most sequent is not derivable in general. In fact, we have a countermodel:

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \bullet \\ z \Vdash \Box A, z \nVdash A & y \Vdash A, y \Vdash \Box\Box A & x \Vdash \Box A \end{tikzcd}$$

Notice that this is a non-transitive model. □

As a consequence, we get that the assumption of *Trans* is necessary in the aforementioned proof of *Cut*-admissibility in **G3KGL**$_<$. [9]

---

[8] The sequent $x\colon \Box A \to x\colon \Box\Box A$ corresponds to transitivity the same way the sequent $x\colon \Box A \to x\colon \Box(A \wedge \Box A)$ corresponds to redundant transitivity from footnote 6. What we are showing is actually that the "standard" version of transitivity can be deduced from the redundant version by using *Cut* and that *Cut* is necessary in any proof of transitivity. This is why we needed the redundant version in the first place.

[9] This may look a bit counterintuitive: a mathematical principle, transitivity, corresponds to a derivable sequent, but is also equivalent, modulo irreflexivity, to a structural rule. However, this is not really astonishing: *Cut* can be viewed as a form of transitivity, as it is a generalisation of the following:

$$\forall C \forall B (B \supset C \Rightarrow \forall A (A \supset B \Rightarrow A \supset C))$$

which is just transitivity of $\supset$ seen as a relation. This is also the reason for which the *Cut* in literature is sometimes called *Trans*, e.g. when dealing with Scott-style entailment relations (cf [34]; for recent work see, e.g., [10, 15, 16, 29, 30, 33, 39]).

5 Future work

The calculus $\mathbf{G3K}_<$ is classical, but the applications studied up to now have a purely constructive proof in their algebraic counterpart. This makes us confident that we can replace $\mathbf{G3K}_<$ by an intuitionistic modal calculus, such as the one presented in [20].

Furthermore, those applications have not yet suggested a general method to find the subformula $U(x)$ required to define the valuation; whence we will next try to pin down such a general method.

Other principles related to induction are worth a closer look. Apart from the notions of Noetherianity discussed in [13, 25], there is Grzegorczyk induction [14], which is a weaker form of induction compatible with reflexivity. Also the principles of transitivity and irreflexivity deserve further investigation, especially in connection with *Cut*-elimination, as well as the variant GH of the Gödel–Löb axiom [8]. There is already some work in progress on relating this approach with Peano Induction, which will likely lead to similar results in Ordinal Induction.

Acknowledgements

The present study was carried out within the projects "A New Dawn of Intuitionism: Mathematical and Philosophical Advances" (ID 60842) funded by the John Templeton Foundation, and "Reducing complexity in algebra, logic, combinatorics - REDCOM" belonging to the programme "Ricerca Scientifica di Eccellenza 2018" of the Fondazione Cariverona. Additional support for this research was granted by the Academy of Finland (research project no. 1308664) and by the Faculty of Arts of the University of Helsinki. All three authors are members of the "Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni" (GNSAGA) of the Istituto Nazionale di Alta Matematica (INdAM). Last but not least, the authors are grateful to Edi Pavlovic and Daniel Wessel for their interest and suggestions, to Giovanni Sambin for valuable bibliographical indications, and to the anonymous referees for their constructive critique.

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