Possibility, consistency, connexivity

Luis Estrada-González [1]

Institute for Philosophical Research
Graduate Program in Philosophy of Science
National Autonomous University of Mexico (UNAM)

Abstract

In this paper I explore how to retain Lewis and Langford's characterization of possibility in terms of consistency and Nelson's idea that all propositions are self-consistent. This would amount to having as logical truths all the formulas of the form $\Diamond A$. I show that in using a very simple three-valued connexive logic to evaluate the Lewis and Langford's definition of modalities, one gets some very interesting results connecting possibilism, the thesis according to which everything is possible, with certain styles of connexivism, especially those with room for contradictory theorems.

Keywords: Aristotle's Theses; connexive logic; possibilism; **LP**.

1 Introduction

In their *Symbolic Logic*, [9], C.I. Lewis and C.H. Langford, building upon previous work of the former (see [8]), famously defined implication in terms of possibility, negation and conjunction. Slightly less famously, Lewis and Langford characterized possibility in terms of consistency, and this led to a definition of implication in terms of consistency, which is a very common idea in the field of connexive logic. Then, there are a number of valid arguments with a connexive flavor in *Symbolic Logic*. All of them include instances of Aristotle's Thesis, $\sim(A \rightarrow\sim A)$, in the premises (or as part of the antecedent, in implicational theorems), and in the system are invalid without such instances of instances of Aristotle's Thesis as premises or antecedents.

Unlike Lewis and Langford, and explicitly reacting against some previous work of the former, Everett J. Nelson held in "Intensional relations" ([16]) that all propositions are self-consistent, so he retained the definition of implication in terms of consistency, but rejected the characterization of possibility in terms of consistency. With Aristotle's Thesis as a logical truth, all the valid arguments

[1] This work was written under the support of the PAPIIT project IN403719 "Intensionality all the way down. A new plan for logical relevance". I presented a previous version at the Ninth Conference Non-Classical Logic. Theory and Applications, held in Toruń. I want to thank Hitoshi Omori, Ricardo Arturo Nicolás-Francisco and Heinrich Wansing for extremely helpful comments on previous drafts, as well as the AiML referees for their comments and advice.

with a connexive flavor in *Symbolic Logic* became fully connexive and remain valid even without including Aristotle's Thesis as a premise (or as an antecedent in the implicational theorems).

In this paper, I probe the prospects of having my cake and eating it too. That is, I want to explore how to retain Lewis and Langford's characterization of possibility in terms of consistency and Nelson's idea that all propositions are self-consistent, which amounts in Lewis and Langford's framework to validate all the formulas of the form $\Diamond A$. And here enters a further connexive twist: I will show that in using a very simple three-valued connexive logic introduced in [20] to evaluate Lewis and Langford's definition of modalities, one gets some very interesting results connecting possibilism, the thesis according to which everything is possible, with certain styles of connexivism, especially those with room for contradictory theorems.

Some provisos are in order here. First, my contribution here is not a brand new logic. I am using instead a logic already in circulation to model both Lewis and Langford's characterization of possibility in terms of consistency and Nelson's idea that all propositions are self-consistent, which results in a connexive logic where possibilism is valid. Second, in showing such a model I am not claiming that connexivity and possibilism are equivalent, because they are not. The idea is rather that certain recent brands of connexivity imply possibilism under some characterizations of possibility proposed in the early 20th century, and that the connection has several ramifications for the study of modalities. Third: truth-functional modal logic, of the likes I will analyze here, has been declared a "dead end" many times now, most notably in [5]. I do not think it is, though. True, there are many objections to be overcome, and I will do so in due time in this paper. Lastly: I am not making any strong claims about the truth, correctness or the like of the views presented here. My main claim is about the existence of the connection already mentioned and the worthiness of studying it.

The structure for the remaining of the paper is as follows. In Section 2, I present Lewis and Langford's conceptualization of the notions mentioned in the title. In Section 3, I present Nelson's reaction towards some of the consequences of the Lewis and Langford's proposal and his arguments to prefer a primitive notion of consistency, not definable in terms of possibility. These applications were not considered when such a logic was first presented. In Section 4, I present a limitative result for an attempt to combine both approaches. In Section 5 I show that Omori's **dLP**, which is a connexive logic built upon the $\{\sim, \wedge, \vee\}$-fragment of González-Asenjo/Priest's Logic of Paradox **LP** enriched with a suitable conditional, can support the combination of Lewis and Langford's characterization of possibility in terms of consistency with Nelson's idea that every proposition is self-consistent. By adding more conceptual tools, one can even get a connexive model of Mortensen's possibilism, where not only everything is possible, but nothing is necessary. In Section 6 I address some concerns about this approach. Finally, in Section 7 I present some sets more of modalities allowed in this framework and then I present some conclusions and suggest some paths for future work in this area.

2 Possibility and connexivity

In Lewis and Langford's *Symbolic Logic*, there is a tight connection between the notions of possibility, consistency and (strict) implication. In their formalization ordering, possibility comes first, and implication is defined in the well-known way. However, at the conceptual level, all of them are equally basic. Thus we read, for example: [2]

> The primitive or undefined ideas assumed are the following: (...)
> 4. Self-consistency or possibility: $\Diamond p$. This may be read "p is self-consistent" or "p is possible" or "It is possible that p be true". As will appear later, $\Diamond p$ is equivalent to "It is false that p implies its own negation," (...). The precise logical significance of $\Diamond p$ will be discussed in Section 4. (...)
> The relation of strict implication can be defined in terms of negation, possibility and product:
> *11.02* $(p \rightarrow q) =_{def.} \sim\Diamond(p \wedge \sim q)$

And then in Section 4 there is, as promised, the discussion of the precise logical significance of possibility. Lewis and Langford say:

> When we speak of two propositions as 'consistent,' we mean that it is not possible, with either of them as premise, to deduce the falsity of the other. Thus if $p \rightarrow q$ has the intended meaning "q is deducible from p," then "p is consistent with q" may be defined as follows:
> *17.01* $(p \circ q) =_{def.} \sim(p \rightarrow \sim q)$ [3]

From this, it easily follows that $(p \circ q) \leftrightarrow \Diamond(p \wedge q)$. Thus, possibility or self-consistency $\Diamond p$ would amount to $(p \circ p)$, which in turn would be equivalent to $\sim(p \rightarrow \sim p)$.

The other usual modalities can be defined then as follows:

18.12 $\sim\Diamond p = \sim(p \circ p) = \sim\sim(p \rightarrow \sim p)$
18.13 $\Diamond\sim p = (\sim p \circ \sim p) = \sim(\sim p \rightarrow \sim\sim p)$
18.14 $\sim\Diamond\sim p = \sim(\sim p \circ \sim p) = \sim\sim(\sim p \rightarrow \sim\sim p)$

Now, Lewis and Langford's characterization of possibility, namely

$\Diamond p = (p \circ p) = \sim(p \rightarrow \sim p)$

might look familiar to a logician acquainted with contra-classical logics, as this is a form of *Aristotle's Thesis*, which is one of the characteristic valid schemas

[2] Throughout the paper, the notation of *Symbolic Logic* will be adjusted. Also, Lewis and Langford take classical logic to be basically correct, and that is why they allow certain logical moves that might be in question for other logicians, especially connexivists. Since in this section I am merely presenting their views, I will leave their classical assumptions untouched.

[3] Note that, unlike its Brazilian sibling introduced several years after by da Costa, Nelson's consistency connective is not intended to control Explosion, $A, \sim A \Vdash B$. Note also that this is, at least typographically, the same definition for a connective named variously 'fusion', 'intensional conjunction' or 'multiplicative conjunction' in the relevance logic literature. I will come back to this issue at the end of Section 5.

of *connexive logics.*[4] And this part of *Symbolic Logic* ([9, 154–178]) is indeed full with theorems with a connexive flavor, among them:
17.52 $((p \to q) \wedge (p \to\sim q)) \to\sim (p \circ p)$
17.57 $((p \to q) \wedge \sim (p \circ q)) \to\sim (p \circ p)$
17.58 $((p \circ p) \wedge \sim (p \circ q)) \to\sim (p \to q)$
17.59 $((p \circ p) \wedge (p \to q)) \to\sim (p \to\sim q)$
17.591 $(p \circ p) \to\sim ((p \to q) \wedge (p \to\sim q))$
17.6 $p \to (p \circ p)$

To see this more clearly, consider the contrapositive forms of *17.52* and *17.57*:
$\sim\sim (p \circ p) \to\sim ((p \to q) \wedge (p \to\sim q))$
$\sim\sim (p \circ p) \to\sim ((p \to q) \wedge \sim (p \circ q))$
Substituting all the occurrences of the consistency connective by its definition and employing Double Negation Elimination, one gets that the two above are equivalent to
$\sim (p \to\sim p) \to\sim ((p \to q) \wedge (p \to\sim q))$
which is very close to *17.591*, too. One can read this as expressing that if Aristotle's Thesis holds, Abelard's Principle, $\sim ((p \to q) \wedge (p \to\sim q))$, holds as well. With Residuation, $((p \wedge q) \to r) \leftrightarrow (p \to (q \to r))$, and the definition of the consistency connective, *17.59* becomes
$(p \circ p) \to ((p \to q) \to\sim (p \to\sim q))$
that is, if Aristotle's Thesis holds, Boethius' Thesis, $(p \to q) \to\sim (p \to\sim q)$, holds too. Given the equivalence between $p \circ p$, $\sim (p \to\sim p)$ and $\Diamond p$, these seem to be, although probably unintended, among the earliest appearances of "hypothetical", "default" (in the terminology of [23]) or "humble" (in the terminology of [7]) connexive theses.

3 Consistency and connexivity

Lewis and Langford's theory of consistency, possibility and implication has the following well-known consequences:
19.1 $\sim (p \circ p) \to\sim (p \circ q)$
19.11 $(p \circ q) \to (p \circ p)$
19.74 $\sim \Diamond p \to (p \to q)$
19.75 $\sim \Diamond \sim p \to (q \to p)$

The latter two are the infamous paradoxes of strict implication: an impossible proposition implies every other proposition, and a necessary proposition is implied by every other proposition. From *19.1*, given the interdefinibility of

---

[4] A connexive logic validates
$\sim (A \to\sim A)$ Aristotle's Thesis
$\sim (\sim A \to A)$ Variant of Aristotle's Thesis
$(A \to B) \to\sim (A \to\sim B)$ Boethius' Thesis
$(A \to\sim B) \to\sim (A \to B)$ Variant of Boethius' Thesis
and invalidates
$(A \to B) \to (B \to A)$ Non-symmetry of implication
For good introductions to connexive logics, see [11] or [25].

possibility and consistency, it easily follows that
$\sim\Diamond p \rightarrow\sim(p \circ q)$
whose intuitive phrasing would be "Something impossible is incompatible with everything". From *19.11*, given again the interdefinibility of possibility and consistency, one gets:
$(p \circ q) \rightarrow \Diamond p$
that is, if p is compatible with anything at all, p is possible.

Everett J. Nelson [16] reacted against all these consequences. As the reactions and objections to the paradoxes of strict implication are well-known and have come from different sources, I will focus on Nelson's objections to the unnumbered consequences. His starting point is a notion of consistency different from Lewis and Langford's, because according to Nelson there are pairs of propositions p and q such that each of them is impossible but which are nonetheless mutually consistent, for example "$(2+2) \neq 4$" and "$(3+3) \neq 6$". Nelson not only held that there are impossible propositions that are mutually consistent, but he also held that every proposition is self-consistent, even an impossible one, and that this self-consistency of an impossible does not prevent that it might be inconsistent with some other propositions. For example, "$1 = 0$" is consistent with itself, but it is inconsistent with "$3 \neq 2$". Thus, for Nelson, (in)consistency and (im)possibility come apart, and $(p \circ p)$ is a logical truth, it holds even for "$1 = 0$", but $\Diamond p$ is not, as "$1 = 0$" is not possible. Contradictoriness is a sufficient condition for Nelsonian inconsistency, that is, p and $\sim p$ are inconsistent, but as the example regarding "$(2+2) \neq 4$" and "$(3+3) \neq 6$" shows, $\sim p$ and $\sim q$ might be inconsistent too. Surely Nelson's notion of consistency needs a more precise treatment, and below I will offer a precisification, but this should suffice for now as an exemplification of the differences with his and Lewis and Langford's notions of consistency.

Nelson used the notion of consistency sketched above to give a validity condition for the conditional:
$(p \rightarrow q)$ is true if and only if the antecedent is inconsistent with the negation of the consequent.
Thus, for him $\sim(p \rightarrow\sim p)$ is also a logical truth. [5] Were Nelson right that every proposition is self-consistent, that is, if Aristotle's Thesis was a logical truth, all the theorems with a connexive flavor in *Symbolic Logic* would become fully connexive in the sense that they would be valid even without including Aristotle's Thesis as a premise (or antecedent), because if a logical truth implies a certain proposition p, p itself is a logical truth. Of course, this is not the case with Aristotle's Thesis in Lewis and Langford's theory, but it is in Nelson's (and connexive logics in general). [6]

---

[5] The intuitive argument for it goes as follows. Suppose that every proposition is either true or false, and that a proposition is false if and only if its negation is true. Now consider the conditional $p \rightarrow\sim p$. The negation of the consequent, $\sim\sim p$, is never inconsistent with the antecedent; hence, the conditional is never true. Then, the negation of the conditional is always true.

[6] For a recent detailed study of Nelson's ideas against the background of Lewis' work, see

Now, both ideas —Lewis and Langford's characterization of possibility in terms of consistency and Nelson's self-consistency of every proposition— have certain independent appealing. The problem is that they together entail possibilism, i.e. that every proposition is possible, which is maybe too much to swallow.

4 A limitative result

In fact, Omori proved in [19] that in any logic **L** satisfying

- $A \to A$
- $((A \to A) \to B) \leftrightarrow B$
- $(A \to B) \to ((C \to A) \to (C \to B))$
- $A \leftrightarrow \sim\sim A$
- $\sim (A \to B) \leftrightarrow (A \to \Diamond \sim B)$
- $\sim (A \to B) \leftrightarrow (A \to \sim B)$

plus the rules

- $A, A \to B \vdash_{\mathbf{L}} B$
- $A \to B \vdash_{\mathbf{L}} \Diamond A \to \Diamond B$
- Uniform Substitution

$\Diamond B \leftrightarrow B$ holds as well. It easily follows then that if the possibilist thesis, $\Diamond B$, is added to **L**, it becomes trivial.

For the sake of the argument, one could leave the rules and the first three items out of the discussion as they are valid in positive logic. Double negation can be granted, too. This reduces the room for disagreement to $\sim (A \to B) \leftrightarrow (A \to \Diamond \sim B)$ and $\sim (A \to B) \leftrightarrow (A \to \sim B)$, the Egré-Politzer's and Wansing's theses, respectively. In order to get more sense of what is going in here, let me describe briefly what is behind each of the theses.

Wansing has employed in several contexts a non-standard falsity condition for the conditional. (See for example [24].) More specifically, he has suggested to take the condition of the form "*If* A is true *then* B is false" rather than the condition of the form "A is true *and* B is false" as the falsity condition for a conditional of the form "If A then B", where truth and falsity are not necessarily exclusive. As a byproduct, the resulting logics turn out to be connexive logics which moreover validate the converses of Boethius' Theses, in particular $\sim(A \to B) \to (A \to\sim B)$.

On the other hand, Paul Egré and Guy Politzer [3] carried out an experiment related to the negation of indicative conditionals and considered *weak* conjunctive and conditional formulas of the forms $A \wedge \Diamond \sim B$ and $A \to \Diamond \sim B$, respectively, besides the more familiar *strong* conjunctive and conditional formulas of the forms $A \wedge \sim B$ and $A \to\sim B$, respectively, as formulas equivalent

---

[10].

to $\sim(A \to B)$.

Thus, Omori's result means that, up to non-triviality, it is not possible to have in a single framework, on the one hand, possibilism, and on the other, Wansing's view on negated conditionals and Egré and Politzer's view on negated conditionals. As I have showed, a form of possibilism results from combining Lewis and Langford's view on possibility with Nelson's ideas about consistency. The triviality result means that this mixture cannot be further combined with Wansing's and Egré and Politzer's views on negated conditionals.

The expected casualty is possibilism, i.e. the combination of Lewis and Langford's view on possibility with Nelson's ideas about consistency. Nonetheless, one could also question the assumptions leading to $\Diamond B \leftrightarrow B$. In a sense, this result is more problematic than possibilism itself because, informally, it identifies the possible with the actual. Possibilism may not be that much to swallow, at least in comparison with other options. Fortunately, there are certain formal tools already in circulation that can serve to model this strange mixture. The model is decidedly simple, but it has some notorious features.

In the following section it will be clear that, with good reason, one can blame instead half of the Egré-Politzer Thesis —namely, $(A \to \Diamond\sim B) \to\sim(A \to B)$— and make room for possibilism. In particular, one can show that, according to the model, $\Diamond A$ has to be always false in order to validate the Egré-Politzer Thesis. [7]

5 Connexivity

Consider a language $\mathcal{L}$ with a countable set of propositional variables and with at least the connectives of negation, $\sim$, and conditional, $\to$. Let $V = \{1, 0\}$ be a set of truth values. Consider a family of interpretations of $\mathcal{L}$, that is, relations $\sigma : \mathcal{L} \longrightarrow V$ —excluding, for any $A \in \mathcal{L}$, that both $1 \notin \sigma(A)$ and $0 \notin \sigma(A)$—, with logical validity defined in the following way (where Γ stands for a collection of formulas of $\mathcal{L}$):

$\Gamma \models A$ if and only if, for every σ, if $1 \in \sigma(B)$ for every $B \in \Gamma$, $1 \in \sigma(A)$

Consider now the following evaluation conditions for the conditional:

$1 \in \sigma(A \to B)$ if and only if $1 \notin \sigma(A)$ or $1 \in \sigma(B)$

$0 \in \sigma(A \to B)$ if and only if $1 \notin \sigma(A)$ or $0 \in \sigma(B)$

Such a conditional, together with a relatively uncontroversial evaluation condition for negation, like

$1 \in \sigma(\sim A)$ if and only if $0 \in \sigma(A)$

$0 \in \sigma(\sim A)$ if and only if $1 \in \sigma(A)$

satisfies the core of connexive logics, that is, it validates all of Aristotle's and

[7] Note that one could also decide to save the Egré-Politzer Thesis and blame instead *hyperconnexivity*, i.e. the converse of Boethius' Thesis: $\sim(A \to B) \to (A \to\sim B)$. However, discussing this would require a more complex apparatus; actually, the usual relational semantics for modalities would be more suitable. This is left for the forthcoming second part of this investigation. For a different take on Omori's result, see [17].

Boethius' Theses and their variants, and invalidates the Non-Symmetry of Implication.

What one has got so far is the **LP** negation and the Olkhovikov-Cantwell-Omori (OCO) conditional, first introduced in [18] and then introduced independently in [2] and [20]:

A	B	$\sim A$	$A \to B$
$\{1\}$	$\{1\}$	$\{0\}$	$\{1\}$
$\{1\}$	$\{1,0\}$	$\{0\}$	$\{1,0\}$
$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{1\}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$
$\{1,0\}$	$\{0\}$	$\{1,0\}$	$\{0\}$
$\{0\}$	$\{1\}$	$\{1\}$	$\{1,0\}$
$\{0\}$	$\{1,0\}$	$\{1\}$	$\{1,0\}$
$\{0\}$	$\{0\}$	$\{1\}$	$\{1,0\}$

In what follows, and unless the contrary is stated, the arrow will stand for the OCO conditional.

One nice thing about this choice of connectives is that it allows for double negation elimination, and thus some formulas can be simplified to more manageable and familiar forms, for example
$\sim\sim(A \to\sim A)$ $(\sim\Diamond A)$ can be simplified to $(A \to\sim A)$,
$\sim(\sim A \to\sim\sim A)$ $(\Diamond\sim A)$ can be simplified to $\sim(\sim A \to A)$, and
$\sim\sim(\sim A \to\sim\sim A)$ $(\sim\Diamond\sim A)$ can be simplified to $(\sim A \to A)$.

Below there are the evaluations of the Lewis-Langford modalities according to the OCO conditional and the **LP** negation:

A	$\Diamond_L A$ $\sim(A \to\sim A)$	$\sim\Diamond_L A$ $\sim\sim(A \to\sim A)$	$\Diamond_L\sim A$ $\sim(\sim A \to\sim\sim A)$	$\sim\Diamond_L\sim A$ $\sim\sim(\sim A \to\sim\sim A)$
$\{1\}$	$\{1\}$	$\{0\}$	$\{1,0\}$	$\{1,0\}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$
$\{0\}$	$\{1,0\}$	$\{1,0\}$	$\{1\}$	$\{0\}$

It is easy to check that Wansing's Thesis is valid according to these valuations, but Egré-Politzer's is not. Consider the case when B is true only and A is at least true, and then the right-to-left direction, $(A \to \Diamond \sim B) \to\sim (A \to B)$, is invalid. This seems correct: From the possibility of the consequent's falsity one cannot infer the actual falsity of the whole conditional, this a way too strong falsity condition.

There are several nice things to say about modalities so defined and evaluated with these connectives.

No modal collapse. A, $\Diamond_L A$ and $\sim \Diamond_L \sim A$ are different propositions because they are not equivalent, they do not have the same values under all interpretations, as can be simply checked by looking at the truth tables, and the same goes for $\sim A$, $\Diamond_L\sim A$ and $\sim\Diamond_L A$. True, given the definition of logical consequence as (forwards) truth-preservation in all interpretations, $A \dashv\vdash\sim \Diamond_L \sim A$

holds, which can be seen as a collapse if '$\sim \Diamond_L \sim A$' is understood as "A is necessary". Nonetheless, equivalence and inter-derivability (and co-implication, I would add) are conceptually different, no matter their simultaneous occurrence in several logics, and one has to be careful on what concept is employing to evaluate claims of collapse. I stick to difference in some interpretations as proof of non-equivalence. Nonetheless, $A \dashv\Vdash \sim \Diamond_L \sim A$ is a modal anomaly that has to be explained in due course.

Dualities between modalities and usual modal axioms. For the rest of the paper I will use '$\Box_L A$' as a shorthand for '$\sim \Diamond_L \sim A$'. This is justified, and this is a second nice thing to say about this framework, because the usual dualities between $\Diamond_L A$ and $\Box_L A$ hold [8]:

For all σ,

$\sigma(\Diamond_L A) = \sigma(\sim\Box_L \sim A)$

$\sigma(\sim\Diamond_L A) = \sigma(\Box_L \sim A)$

$\sigma(\Diamond_L \sim A) = \sigma(\sim\Box_L A)$

$\sigma(\sim\Diamond_L \sim A) = \sigma(\Box_L A)$

Also, all the usual modal axioms

(K) $\Box_L(A \to B) \to (\Box_L A \to \Box_L B)$

(T) $\Box_L A \to A$

(4) $\Box_L A \to \Box_L \Box_L A$

hold, as well as the Necessitation Rule

(NEC) From $\Vdash A$ to infer $\Vdash \Box_L A$

So much for the attractive features of the modalities so defined. In the next section, I will discuss some possible objections to this way of combining possibilism and connexivity, but before that, and to make things more interesting, let me add also conjunction and disjunction as evaluated in **LP**, plus the unary consistency connective '$\circ$' defined as

$1 \in \sigma(\circ A)$ if and only if $1 \in \sigma(A)$ and $0 \notin \sigma(A)$ or $0 \in \sigma(A)$ and $1 \notin \sigma(A)$

$0 \in \sigma(\circ A)$ if and only if $1 \in \sigma(A)$ and $0 \in \sigma(A)$

Then one gets the logic **dLP**. [9] Summarizing, (zeroth-order) **dLP** is characterized by the following truth tables:

A	B	$\sim A$	$\circ A$	$A \wedge B$	$A \vee B$	$A \to B$
$\{1\}$	$\{1\}$	$\{0\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\{1\}$	$\{1,0\}$	$\{0\}$	$\{1\}$	$\{1,0\}$	$\{1\}$	$\{1,0\}$
$\{1\}$	$\{0\}$	$\{0\}$	$\{1\}$	$\{0\}$	$\{1\}$	$\{0\}$
$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{0\}$	$\{1,0\}$	$\{1\}$	$\{1\}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$
$\{1,0\}$	$\{0\}$	$\{1,0\}$	$\{0\}$	$\{0\}$	$\{1,0\}$	$\{0\}$
$\{0\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{0\}$	$\{1\}$	$\{1,0\}$
$\{0\}$	$\{1,0\}$	$\{1\}$	$\{1\}$	$\{0\}$	$\{1,0\}$	$\{1,0\}$
$\{0\}$	$\{0\}$	$\{1\}$	$\{1\}$	$\{0\}$	$\{0\}$	$\{1,0\}$

[8] The dualities fail in, for example, Wansing's connexive modal logic **CK**; see [24]. The failure might be a good thing, though; see [26] for an argument to that effect.

[9] First presented, with different primitives though, in [18] and then independently in [20].

which is basically **LP** in the $\{\sim, \wedge, \vee\}$-fragment, augmented with the expressive power allowed by the unary consistency connective and the OCO conditional.

It is easy to check that $\Diamond_L A$ and $\circ A$ are not equivalent, a result that would have pleased Nelson, and this means that $A \circ A$ (self-consistency) and $\circ A$ (consistency simpliciter) are not equivalent, either. This seems in the right track: one thing is that a proposition does not imply its own negation (self-consistency), and another is that a proposition does not imply an arbitrary contradiction (consistency).

When one goes fully to **dLP**, the dualities and the validities are preserved, and even more nice things appear. For example, two "relative consistency" binary connectives can be defined. One of them is more "Lewisian", order-sensitive, non-symmetric, as discussed for example in [21]:
$\sigma(A \circ_L B) = \sigma(B)$ unless $\sigma(A) = \{0\}$, and $\sigma(A \circ_L B) = \{1, 0\}$ in that latter case [10]
which is but the OCO condtional. The other is more "Nelsonian":
$1 \in \sigma(A \circ_N B)$
$0 \in \sigma(A \circ_N B)$ if and only if $0 \in \sigma(A)$ or $0 \in \sigma(B)$

While in general $A \circ_L B$ and $A \circ_N B$ are not equivalent, $A \circ_L A$ and $A \circ_N A$ are for any A, so let me write $A \circ_X A$ to express such indistinctness. Thus, the self-consistency $A \circ_X A$ of any proposition A is a theorem of **dLP**; again, a result that would have pleased Nelson. Nonetheless, the (Nelsonian) relative consistency of any two propositions, $A \circ_N B$, also becomes a theorem in **dLP**, something that definitely would not have pleased Nelson, "because some propositions are inconsistent with others." [16, 443] However, Nelson thought that the inconsistency of two distinct propositions cannot be determined by pure logic alone, and that is reflected both in his truth table and the evaluation conditions of $A \circ_N B$. [11]

6 Some concerns

Combining zeroth-order logic, functionality, finite many-valuedness, modalities and highly non-classical theses seems like a recipe for disaster. Let me address three potential worries here. I do not aim at dispelling all air of doubt, that seems nearly impossible in philosophical issues; I only want to show that some objections usually raised to approaches like the one presented above are far from being knock-down.

---

[10] For simplicity, I use the following convention. Let v_j and v_k be our two truth values. Then '$\sigma(X) = \{v_j\}$' means that $v_j \in \sigma(X)$ and $v_k \notin \sigma(X)$, whereas '$\sigma(X) = \{v_j, v_k\}$' means that $v_j \in \sigma(X)$ and $v_k \in \sigma(X)$.

[11] Finally, recall that
$(A \circ B) =_{def.} \sim(A \to \sim B)$
is the usual definition of fusion (an intensional conjunction) $A \circ B$ in the logic **R**. But $(A \circ_L B)$ can difficultly be regarded as a conjunction, for it is true even when no component is true. This reflects the fact that the OCO conditional is false when both antecedent and consequent are false.

Modal anomalies. It has been a long time since Łukasiewicz offered a many-valued analysis of modalities. Since then, such an approach has lived in discredit because they give rise to *modal anomalies*, that is, highly counterintuitive arguments regarding modalities are validated.

The typical criticism raised against the many-valued approaches to possibility and necessity is of the sort of those found in [5]. The objection is basically that the many-valued notions of possibility and necessity validate several counterintuitive arguments. Dugundji's theorem, that no modal logic between Lewis' **S1** and **S5** can be characterized by a finite many-valued matrix, seems to give more content to the criticisms.

Dugundji's result would be a devastating problem if all and only modalities worth considering lied between **S1** and **S5**, but that is not the case. Consider a modality $\mathcal{J}$ satisfying the following two axiom schemas, where '$\to$' stands again for a generic conditional:
$\mathcal{J}(A \to B) \to (\mathcal{J}A \to \mathcal{J}B)$
$A \to \mathcal{J}A$
When $\mathcal{J}A$ is identified with $\Box A$ and the two axioms schemas above are added to classical logic, the resulting logic is simply $\mathbf{K}$+$(A \to \Box A)$, which is characterized by certain single-element frames. However, $\mathcal{J}A$ cannot be rightly identified with $\Box$ precisely because of $A \to \mathcal{J}A$; on the other hand, it cannot be rightly identified with $\Diamond$ because of $\mathcal{J}(A \to B) \to (\mathcal{J}A \to \mathcal{J}B)$. Moving to a different logic, intuitionistic logic, for example, allows to study interesting models for these axiom schemas. The resulting modality $\mathcal{J}A$ has an hybrid nature, but still closer to possibility, and has appeared in different contexts, interpreted as variedly as "(at some underlying topological space), it is locally the case that" (as in topos theory, where it first appeared) or "under some family of constraints, the hardware device behaves according to" (as in propositional lax logic); see [6, Section 7.6] for an overview of the different standard incarnations of $\mathcal{J}$. This means that in the presence of different logics, some counterintuitive axiom schemas can make sense for certain modalities. And that was Łukasiewicz's reaction to the modal anomalies in his logic: one could try to make sense of the modalities involved as defined within the logic so as to explain away the unintuitiveness of certain axiom schemas. Whether his personal attempt succeeded or not for his logics is a different issue from the correctness of the methodological advice. The case of $\mathcal{J}A$ proves that the attempts are not a priori doomed to fail.

Let me consider explicitly the schemas that worry Font and Hájek, all of them valid in **dLP**:
FH1. $(\Diamond A \land \Diamond B) \to \Diamond(A \land B)$
FH2. $(A \to B) \to (\Box A \to \Box B)$
FH3. $(A \to B) \to (\Diamond A \to \Diamond B)$
FH4. $\Box A \to (B \leftrightarrow \Box B)$
FH5. $\Box A \to (\Diamond B \leftrightarrow \Box B)$

What I have said above on logics not between **S1** and **S5** could serve to partially alleviate the concerns by Font and Hájek. Nonetheless, a more substantial reply can be given. FH1 seemed problematic because contradictions were for a long time the archetype of impossibility. The instance $(\Diamond A \wedge \Diamond \sim A) \rightarrow \Diamond(A \wedge \sim A)$ was regarded as a counterexample to the schema: even if A and $\sim A$ were separately possible, jointly they are not. That they could be true at some states of evaluation in certain semantics —the objection continues— does not make them less impossible: those states are *impossible* states after all. That is a moot point: one could argue that possibility is a property of propositions relative to states, not of states themselves, and that being possible (at a state) just means to be true at some accessible state. Nonetheless, $\Diamond(A \wedge \sim A)$ should be expected in a framework where the idea that everything is possible is taken seriously. More than a drawback of the logic, this should be a welcomed result. However, there is more to be said in favor of it, and it will become evident when discussing the next objection about possibilism.

Of the next couple of schemas, FH2 and FH3, Font and Hájek said that, had Łukasiewicz decided to interpret the arrow in the antecedent as a strict implication and not as a material one, the resulting schemas would have been more acceptable. The schemas also hold in the **dLP** setting. Note that the charge of unacceptability due to the material nature of the arrow in the antecedent does not apply here, as $A \rightarrow_{OCO} B$ is neither equivalent nor interderivable with $\sim(A \wedge \sim B)$ nor $\sim A \vee B$. It is not a strict conditional either, but it still encapsulates a sort of intensionality in not making plainly true a conditional whose antecedent is not true (only). One could object that even if $A \rightarrow_{OCO} B$ comes with some intensionality within it, it is not strong enough as to be counted as a sufficient condition for $(\Box A \rightarrow_{OCO} \Box B)$, so $(A \rightarrow_{OCO} B) \rightarrow_{OCO} (\Box A \rightarrow_{OCO} \Box B)$ could still be regarded as genuinely anomalous. Nonetheless, the intensionality within $A \rightarrow_{OCO} B$ seems sufficient to imply $(\Diamond A \rightarrow_{OCO} \Diamond B)$.

Nevertheless, Font and Hájek say that the validity of the last two schemas "is the main reason for [their] claim that as a logic of possibility and necessity, it [Łukasiewicz four-valued modal logic] is a dead end." Informally, $\Box A \rightarrow (B \leftrightarrow \Box B)$ expresses that if something is necessary, any truth simpliciter is also a necessary truth. But just recall what the modalities mean in this context. '$\Diamond_L A$' means that it is false that A implies its own negation, and '$\Box_L A$' means that A is implied by its own negation. So $B \rightarrow_{OCO} \Box_L B$ becomes $B \rightarrow_{OCO} (\sim B \rightarrow_{OCO} B)$. The validity of this does not seem so abhorrent.

Nonetheless, even if B is true, $(\sim B \rightarrow_{OCO} B)$ contradicts Aristotle's Thesis, so it must be false. And it is. This implies that both $\Box A \rightarrow (B \leftrightarrow_{OCO} \Box_L B)$ and $\sim(\Box A \rightarrow_{OCO} (B \leftrightarrow \Box B))$ are valid in **dLP**, and the same holds for $\Box_L A \rightarrow_{OCO} (\Diamond_L B \leftrightarrow_{OCO} \Box_L B)$ and $\sim(\Box_L A \rightarrow_{OCO} (\Diamond_L B \leftrightarrow_{OCO} \Box_L B))$, as can be easily checked. This means that there is inconsistency surrounding certain combinations of truth, possibility and necessity, and notice that they are the modal anomalies: both $(\Diamond_L A \wedge \Diamond_L \sim A) \rightarrow_{OCO} \Diamond_L(A \wedge \sim A)$ and $\sim((\Diamond_L A \wedge \Diamond_L \sim A) \rightarrow_{OCO} \Diamond_L(A \wedge \sim A))$ hold as well in **dLP**. In fact, among all the axiom schemas highlighted by Font and Hájek, $(A \rightarrow B) \rightarrow (\Diamond A \rightarrow \Diamond B)$ is the only one that does not come with its negation validated too in **dLP**, and

I have already argued that this should not count as a so terribly bad anomaly. (Modal anomalies are not so widespread; $\Diamond_L A \rightarrow_{OCO} A$ and $\Diamond_L A \rightarrow_{OCO} \Box_L A$ simply fail, for example.)

There is another route to alleviate the concerns regarding the modal anomalies, namely that in non-classical contexts not all theorems need to be created equal: one could distinguish between different degrees of satisfiability; in particular, between different degrees of theoremhood. This does not mean that one needs to change the notion of logical validity of **dLP** to get a more refined set of logical truths; one can keep the usual definition of logical validity, and regarding as an extra task selecting among the logically true propositions those that meet additional criteria. Given that theorems are limit cases of logically valid arguments, I can borrow some terminology from the variety of notions of logical validity already available in the literature. [12]

Let me call then 'p-theorems' those formulas that are never antidesignated; 'T-theorems' those formulas that are always designated; 'supertheorems' those formulas that are always designated and at least once (just) true; and 'q-theorems' those formulas that are always (just) true. [13]

Unlike **LP**, **dLP** has q-theorems and they might take the following forms:
(q-t i) $\circ\circ A$,
(q-t ii) $A©B$, for $© \in \{\wedge, \vee, \rightarrow\}$ and where both A and B are themselves q-theorems, or
(q-t iii) $\sim A$, where A has the form $\sim B$ and B is a **dLP** q-theorem.

Then, in **dLP**, for any formula A, both $\Diamond_L A$ and $\Box_L \Diamond_L A$ are **dLP** T-theorems, but in general only $\Diamond_L A$ is a **dLP**-supertheorem and sometimes it can be a q-theorem (when A already is one), while $\Box_L \Diamond_L A$ can only be at most a **dLP** T-theorem. Also, $A \rightarrow \Box_L A$ is always just a **dLP** T-theorem, but cannot even be a **dLP**-supertheorem, let alone a q-theorem, because it is never just true.

With such a distinction between theorems, **dLP** can become even closer to Mortensen's possibilism. If all **dLP**-theorems are treated on equal footing, there are some **dLP**-theorems of the form $\Box_L A$, whereas according to logical possibilism there should be no logical truths of such form. But with the further distinctions just drawn, formulas of the form $\Box_L A$ are at most **dLP** T-theorems, whereas all formulas of the form $\Diamond_L A$ are at the very least **dLP**-supertheorems.

Possibilism. Another concern is about a very special "modal anomaly", namely the commitment to *possibilism*, as per the validity of $\Diamond_L A$. For example, Béziau [1] has objected to evaluations of modalities like the ones presented here on the grounds that they make possibility "trivial", in the sense that they make everything possible, and that is not a good result for a theory of the

[12] For a good introduction to the topic and critical discussion of it, see [27].

[13] And I will stop here. If one gives falsity a treatment independent of truth, as it should be done logically, one could obtain even more shades of theoremhood, but those already introduced suffice for my purposes here.

possible.

However, beyond the incredulous stare towards the validity of $\Diamond_L A$, no reasons to reject possibilism have been put forward. Let me consider two potential objections. The first is that possibilism entails trivialism; another is that possibilism is ruled out by the very definitions of logical notions.

The proof that possibilism entails triviality is as follows:

1.	$\Diamond\Box A \to \Box A$	Axiom (5)
2.	$\Box A \to A$	Axiom (T)
3.	$\Diamond A$	Possibilism
4.	$\Diamond\Box A$	3, Uniform Substitution
5.	$\Box A$	1, 4, Detachment
6.	A	2, 5, Detachment

However, notice that the axiom (5) occurring in the first line of the proof is not validated by the tables in Section 5: make A just false. What is validated is its "contraposed" version, $\Diamond_L A \to \Box_L \Diamond_L A$. They are not equivalent, and that is rightly so because they are conceptually distinct. '$\Diamond_L A \to \Box_L \Diamond_L A$' expresses that one can go, so to speak, from the possibility of something to the necessity of its possibility, which is right according to this version of possibilism. '$\Diamond_L \Box_L A \to \Box_L A$' expresses something different, namely that from the possibility of necessity of something, one can go to its necessity, which is wrong according to possibilism because nothing, except possibilities, is necessary. So, neither the argument from possibilism to triviality is valid here, nor one has anything as strong as **S5** modalities in the current setting.[14]

There is also the concern that possibilism is ruled out by the very definitions of logical notions. By this I mean that logical notions as characterized by, say, evaluation conditions, imply the untruth of possibilism. Let me consider first the very notion of possibility. To minimize the risk of begging the question, let me move to more common ground, the usual relational falsity condition for possibility:

- $\Diamond A$ is false at a state i iff A is false at all state j related to i.

Suppose for the sake of the argument that A is in fact false at all state j related to i. Does this mean that there is no j related to i where A is true? If the answer is affirmative, one should ask whether that conclusion comes from the falsity condition alone or whether it comes from additional considerations, for example, certain ideas about the structure of truth values, that they are exclusive maybe.

Of course, more elaborate anti-possibilist arguments can be given. They might involve the characterization of other logical notions, such as conditionals, quantifiers or even logical consequence itself. I cannot go through all those arguments. What I want to highlight is that, in any case, one must wonder whether possibilism is ruled out by evaluation conditions alone or whether other, logic-specific elements —such as the number and structure of truth val-

[14] Note that the "contraposed" versions of (K), (T), and (4) do hold, though.

ues, or the properties of the accessibility relation— are used as well.[15]

Finally, as we have seen, possibilism does not prevent having a sensible theory of modalities that satisfies their mutual distinguishability, their usual interdefinability through negation and without a plethora of non-standard modal theorems accompanying $\Diamond_L A$.[16]

Dialetheism. Finally, there is the concern that **dLP** is not only dialetheist, that is, that it makes room for sentences of the forms A and $\sim A$ that are simultaneously satisfiable, but is also dialetheic (or contradictory or negation-inconsistent, hence the '**d**' in front of '**LP**'!), since it validates contradictory theorems such as

$(A\wedge \sim A) \rightarrow\sim A$ and $\sim((A\wedge \sim A) \rightarrow\sim A)$;

$(A\wedge \sim A) \rightarrow (B\vee \sim B)$ and $\sim((A\wedge \sim A) \rightarrow (B\vee \sim B))$

Furthermore, one can define in **dLP** a new negation $\neg A$ as $\sim A \wedge \circ A$ (or $\sim\circ(A \rightarrow\sim\circ\circ A)$, if conjunction is not available), and then obtain the following **dLP** theorems:

$(A \wedge \neg A) \rightarrow B$ and $\sim((A \wedge \neg A) \rightarrow B)$.

People way more than talented than I have spent up to 40 years trying to convince others that dialetheism is not outrageous, and they are still struggling; see [22] for a book-length defense of dialetheism. In these paragraphs I can only aspire to push further to the already converted: if they have given dialetheism a chance, maybe they can give a contradictory logic a chance too.

One attempt of reassurance might use the terminology from the discussion about modal anomalies: the contradictory theorems of **dLP** are just **dLP** T-theorems. Furthermore, it must also be noticed that **dLP** exhibits contradictions in already expected and significant places for some connexive logicians, namely around Explosion, certain forms of Simplification —more specifically, simplification of contradictories— and the irrelevant Safety, i.e. $(A\wedge \sim A) \rightarrow (B\vee \sim B)$. This reminds me of the situation in faced by Meyer and Martin in investigating Aristotle's syllogistic. Meyer and Martin wanted to provide a logic for Aristotle's syllogistic, which was not reflexive. In their logic **SI∼I**, see [12], $A \rightarrow A$ was treated as a borderline case, both a fallacy and a validity, hence the validity of both $A \rightarrow A$ and $\sim(A \rightarrow A)$. Perhaps the contradictory theorems in **dLP** can be treated similarly as borderline cases: they should be invalid, as many connexivists have said, but also the validity of such schemas is almost necessitated by a truth-functional, truth-preserving logic, with the standard evluations for negation, conjunction and disjunction.

[15] Chris Mortensen in [14] defends "(logical) possibilism", by which he means the idea that everything is possible (possibilism *stricto sensu*, I would say) and nothing is necessary (non-necessitarianism). See also [15] to complete his picture about possibilism. I have addressed some lacunae and further consequences elsewhere (see [4]).

[16] Mortensen himself found a sort of possibilism around connexivity when in [13] he proved that the logic **E** plus Aristotle's Thesis implies $\Diamond A$ for every A, with $\Diamond A$ defined as $\sim((\sim A \rightarrow\sim A) \rightarrow\sim A)$, which would amount to A in the present context.

7 More modalities

The presence of another, more classical negation in **dLP** allows further definitons of modalities in **dLP**, for example, as follows:

A	$\Diamond A$ $\neg(A \to \neg A)$	$\neg\Diamond A$ $\neg\neg(A \to \neg A)$	$\Diamond\neg A$ $\neg(\neg A \to \neg\neg A)$	$\neg\Diamond\neg A$ $\neg\neg(\neg A \to \neg\neg A)$
$\{1\}$	$\{1\}$	$\{0\}$	$\{0\}$	$\{1\}$
$\{1,0\}$	$\{1\}$	$\{0\}$	$\{0\}$	$\{1\}$
$\{0\}$	$\{0\}$	$\{1\}$	$\{1\}$	$\{0\}$

Let me write '$\Diamond_{L^C}$' for the possibility defined with Aristotle's Thesis written with such a strong negation. In this case, possibilism is lost and $\Diamond_{L^C} A$ and $\neg\Diamond_{L^C}\neg A$, on the one hand, and $\neg\Diamond_{L^C} A$ and $\Diamond_{L^C}\neg A$, on the other, collapse.

But the two negations can interact in interesting ways. For example, $\neg\ (A \ \to \ \neg\ A)$ defines in the three-valued setting what can be called 'the Béziau possibility', a unary connective © such that
$\sigma(©A) = \{0\}$ if and only if $\sigma(A) = \{0\}$, and $\sigma(©A) = \{1\}$ in all other cases, so let me write it as '$\Diamond_B$'. Defining modalities based on the Béziau possibility with $\sim$ instead of $\neg$ —namely, $\sim\Diamond_B A$, $\Diamond_B \sim A$, $\sim\Diamond_B \sim A$— produce modalities different from the L^C modalities as follows:

A	$\Diamond_B A$ $\neg(A \to \neg A)$	$\sim\Diamond_B A$ $\sim\neg(A \to \neg A)$	$\Diamond_B \sim A$ $\neg(\sim A \to \neg \sim A)$	$\sim\Diamond_B \sim A$ $\sim\neg(\sim A \to \neg \sim A)$
$\{1\}$	$\{1\}$	$\{0\}$	$\{0\}$	$\{1\}$
$\{1,0\}$	$\{1\}$	$\{0\}$	$\{1\}$	$\{0\}$
$\{0\}$	$\{0\}$	$\{1\}$	$\{1\}$	$\{0\}$

$\sim\neg(\sim A \to \neg \sim A)$ defines 'Béziau's necessity', a unary conective ©A such that it is true if and only if A is true and is false in every other case.

Dually, defining modalities based on the Lewis-Langford possibility with $\neg$ instead of $\sim$ —namely, $\neg\Diamond_L A$, $\Diamond_L\neg A$, $\neg\Diamond_L\neg A$— produce yet another set of new modalities as follows:

A	$\Diamond_L A$ $\sim(A \to\sim A)$	$\neg\Diamond_L A$ $\neg\sim(A \to\sim A)$	$\Diamond_L\neg A$ $\sim(\neg A \to\sim\neg A)$	$\neg\Diamond_L\neg A$ $\neg\sim(\neg A \to\sim\neg A)$
$\{1\}$	$\{1\}$	$\{0\}$	$\{1,0\}$	$\{0\}$
$\{1,0\}$	$\{1,0\}$	$\{0\}$	$\{1,0\}$	$\{0\}$
$\{0\}$	$\{1,0\}$	$\{0\}$	$\{1\}$	$\{0\}$

Notice that these modalities are even closer to Mortensen's possibilism (possibilism proper and non-necessitarianism) right from the outset, without distinguishing between kinds of theorems: all necessities and impossibilities are just false, and all possibilities are always designated. [17]

[17] Incidentally, rewriting '$\sim((\sim A \to\sim A) \to\sim A)$' —the possibility used in [13] to show that **E** plus Aristotle's Thesis is possibilist— as '$\neg((\neg A \to \neg A) \to \neg A)$' makes it equivalent to $\Diamond_B$, not to A as before.

8 Conclusions

In this paper, I explored how to retain Lewis and Langford's characterization of possibility in terms of consistency and Nelson's idea that all propositions are self-consistent. I started by presenting Lewis and Langford's conceptualization of the notions of consistency and possibility, and how certain connexive notions appear there. Then I quickly reconstructed Nelson's reaction towards some of the consequences of Lewis and Langford's proposal and his arguments to prefer a primitive notion of consistency, not definable in terms of possibility. After that, I showed that Omori's logic **dLP** can support the combination of Lewis and Langford's characterization of possibility in terms of consistency with Nelson's idea that every proposition is self-consistent, with the corresponding outcome that all the formulas of the form $\Diamond A$ are theorems. By adding more conceptual tools, I showed that one can even get a connexive model of Mortensen's possibilism, where not only everything is possible, but nothing is necessary. Finally, I discussed some worries about the project, for example, regarding some modal anomalies or the motivations for a logic with contradictory theorems. Discussing one of those concerns led me to pay closer attention to the two negations available in **dLP** and then consider how the modalities behave in the presence of each negation, to find further conceptual insights and new formulations of modalities even closer to Mortensen's possibilism.

A paper would not be as enjoyable if it did not open at least twice the number of questions it tried to address. Let me indicate then some avenues for further exploration. It is well-known that the expressive power of a logic is inversely proportional to its deductive power: the more you can prove, the less distinctions you can draw. As **dLP** is based on **LP**, the obvious choice for a weaker logic is one based on **FDE**. If one adds the OCO conditional to **FDE** then one gets a connexive dialetheic expansion of **FDE**, already discussed in [25] under the name 'material connexive logic'. [18] An open problem is then, investigating the exact shape of the multi-modal features of both **dLP** and **dFDE**, including further interactions between possibilism and connexivity. Going four-valued could also alter the truth conditions given for the relative consistency connective and this in turn could produce a non-Nelsonian split between self-implication and self-consistency, with tremendous consequences for the theories of modalities, connexivity itself and so on.

Further connections between connexivity and possibilism, that is, how starting with versions of one can lead to versions of the other, using frameworks not necessarily in the vicinity of **dLP** and **dFDE**, would be worth exploring too.

[18] A connexive variant of the more general version of Belnap-Dunn logic, including the negation $\neg$, was studied in [20] under the name '**dBD**'.

References

[1] Jean-Yves Béziau. A new four-valued approach to modal logic. *Logique et Analyse*, 54(213):109–121, 2011.

[2] John Cantwell. The logic of conditional negation. *Notre Dame Journal of Formal Logic*, 49(3):245–260, 2008.

[3] Paul Egré and Guy Politzer. On the negation of indicative conditionals. In Maria Aloni, Michael Franke, and Floris Roelofsen, editors, *Proceedings of the 19th Amsterdam Colloquium, AC 2013*, pages 10–18. 2013.

[4] Luis Estrada-González. Models of possibilism and trivialism. *Logic and Logical Philosophy*, 22(4):175–205, 2012.

[5] Josep María Font and Petr Hájek. On Łukasiewicz's four-valued modal logic. *Studia Logica*, 70(2):157–182, 2002.

[6] Robert Goldblatt. Mathematical modal logic: A view of its evolution. *Journal of Applied Logic*, 1(5-6):309–392, 2003.

[7] Andreas Kapsner. Humble connexivity. *Logical and Logical Philosophy*, 28(3):513–536, 2019.

[8] Clarence Irving Lewis. *A Survey of Symbolic Logic.* University of California Press, Berkeley, 1918.

[9] Clarence Irving Lewis and Cooper Harold Langford. *Symbolic Logic.* Dover Publications, New York, 1932.

[10] Ed Mares and Francesco Paoli. C.I. Lewis, E.J. Nelson, and the modern origins of connexive logic. *Organon F*, 26(3):405–426, 2019.

[11] Storrs McCall. A history of connexivity. In Francis Jeffry Pelletier Dov M. Gabbay and John Woods, editors, *Handbook of the History of Logic. Vol. 11. Logic: A History of its Central Concepts*, pages 415–449. Elsevier, Amsterdam, 2012.

[12] Robert Meyer and Errol Martin. S revisited. *Australasian Journal of Logic*, 16(3):49–67, 2019.

[13] Chris Mortensen. Aristotle's Thesis in consistent and inconsistent logics. *Studia Logica*, 43(1-2):107–116, 1984.

[14] Chris Mortensen. Anything is possible. *Erkenntnis*, 30(3):319–337, 1989.

[15] Chris Mortensen. It isn't so, but could it be? *Logique et Analyse*, 48(189–192):351–360, 2005.

[16] Everett J. Nelson. Intensional relations. *Mind*, 39(156):440–453, 1930.

[17] Ricardo Arturo Nicolás-Francisco. A note on three approaches to connexivity. *Felsefe Arkivi—Archives of Philosophy*, (51):129–138, 2019.

[18] Grigory K. Olkhovikov. On a new three-valued paraconsistent logic. In *Logic of Law and Tolerance*, pages 96–113. Ural State University Press, Yekaterinburg, 2001. In Russian.

[19] Hitoshi Omori. Towards a bridge over two approaches in connexive logic. *Logical and Logical Philosophy*, 28(3):553–566.

[20] Hitoshi Omori. From paraconsistent logic to dialetheic logic. In Holger Andreas and Peter Verdée, editors, *Logical Studies of Paraconsistent Reasoning in Science and Mathematics*, pages 111–134. Springer, Berlin, 2016.

[21] Claudio Pizzi. Cotenability and the logic of consequential implication. *Logic Journal of the IGPL*, 12(6):561–579, 2004.

[22] Graham Priest. *In Contradiction: A Study of the Transconsistent.* Oxford University Press, 2006. Second edition.

[23] Matthias Unterhuber. Beyond system P –Hilbert-style convergence results for conditional logics with a connexive twist. *IFCoLog Journal of Logics and their Applications*, 3(3):377–412, 2016.

[24] Heinrich Wansing. Connexive modal logic. In Renate Schmidt, Ian Pratt-Hartmann, Mark Reynolds, and Heinrich Wansing, editors, *Advances in Modal Logic, Volume 5*, pages 367–383. King's College Publications, London, 2005.

[25] Heinrich Wansing. Connexive logic. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy.* CSLI Stanford, Spring 2020 edition, 2020.

[26] Heinrich Wansing and Sergei Odintsov. Constructive predicate logic and constructive modal logic. formal duality versus semantical duality. In Vincent Hendricks, Fabian Neuhaus, Uwe Scheffler, Stig Andur Pedersen, and Heinrich Wansing, editors, *First-Order Logic Revisited*, pages 269–286. Logos Verlag, Berlin, 2004.

[27] Heinrich Wansing and Yaroslav Shramko. Suszko's Thesis, inferential many-valuedness, and the notion of a logical system. *Studia Logica*, 88(3):405–429, 2008.