Indexed Frames and Hybrid Logics

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Abstract

We define and study the notion of 'indexed frames', i.e., tuples (W_1, W_2, R_1, R_2) where each R_i is a binary relation on $W_1 \times W_2$ such that $R_i(w_1, w_2)(v_1, v_2)$ implies $w_i = v_i$. They generalise, among other things, products of Kripke frames. We show that the logic of indexed frames is the fusion logic $\mathsf{K} \oplus \mathsf{K}$. We show the relation between indexed frames and relativised products and we obtain the different logics of indexed frames when we impose certain constraints on the relations R_1 and R_2. Indexed frames were seemingly first used in [8], whithin a proposal for a broader multi-modal framework called Epistemic Logic of Friendship, allowing for both an epistemic accessibility relation and a 'friendship' relation. The set of agents is encoded in the semantics, and these agents are named using nominal variables (a notion borrowed from hybrid logic) with the novelty that these nominals only refer to the elements of one of the sets. [7] provided an axiomatisation for a fragment of the language. We give a simplified proof of this result and we axiomatise an extension of this fragment.

1 Introduction

This paper is concerned with the very interesting (and, to our knowledge, uncharted) mathematical structure that underlies the framework of *Epistemic Logic of Friendship* introduced by Seligman, Liu and Girard in [8]. (Also studied in [9,10]).

It is not in our scope to study the epistemic and social aspects of EFL. Let us nonetheless briefly recall this framework here: we start off with a bimodal language $\mathcal{L}$, defined as:

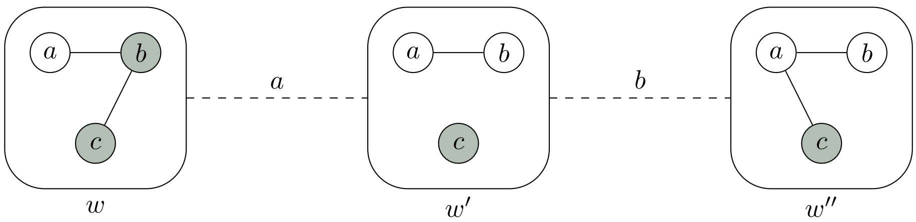
$$\phi ::= p|\bot|\neg\phi|(\phi \wedge \phi)|K\phi|F\phi,$$

where $p \in \mathsf{Prop}$, a countable set of propositional variables. K is meant to be read as an epistemic modality ("I know p"), whereas F is a 'frienship' modality ("all my friends p"). We use $\hat{K}$ and $\hat{F}$ as the duals of these operators. Models are of the form $(W, A, \sim, \asymp, V)$, where W and A are nonempty sets ("states" and "agents", respectively), $\sim = \{\sim_a : a \in A\}$ is a family of binary relations on

W indexed by A ($\sim_a \subseteq W^2$ represents agent a's epistemic accessibility), and $\asymp = \{\asymp_w : w \in W\}$ is a family of binary relations on A indexed by W (each representing which agents are friends at world w). $V : \mathsf{Prop} \to 2^{W \times A}$ is a valuation.

We interpret formulas of $\mathcal{L}$ with respect to pairs $(w, a) \in W \times A$, as follows:
$(w, a) \models K\phi$ iff $(v, a) \models \phi$ for all v such that $w \sim_a v$;
$(w, a) \models F\phi$ iff $(w, b) \models \phi$ for all b such that $a \asymp_w b$.

To illustrate this, see the following diagram. It represents a situation with three agents, Alice, Bob and Charlie, wherein at world w Alice has a friend with the property p (represented by the grey nodes) yet she does not know that:



Indeed, it holds that $w, a \models \hat{F}p \wedge \neg K\hat{F}p$. We could also express more complex things such as "Alice does not know Bob and Charlie are friends". In order to do this, we would need to extend the language, as we shall show later. For now, let us focus on this relational structure.

Indexed frames. We have a multi-relational Kripke frame, whose relations are indexed by a set A, in which each state contains a distinct Kripke frame having A as its underlying set.

We shall call these structures *indexed frames*. In Section 2 we study them and provide the complete axiomatisation of the modal logic they give rise to. Note that indexed frames generalise other ways to combine Kripke frames, such as products: recall that, given two Kripke frames $(W_1, R_1), (W_2, R_2)$, their *product* is the birelational Kripke frame $(W_1 \times W_2, R_1^H, R_2^V)$, where $R_1^H(w_1, w_2)(w_1', w_2')$ iff $w_2 = w_2'$ and $R_1 w_1 w_1'$, and $R_2^V(w_1, w_2)(w_1', w_2')$ iff $w_1 = w_1'$ and $R_2 w_2 w_2'$. R_1^H and R_2^V are referred to as the *horizontal* and *vertical* relations, respectively.

One can easily see that a product of two Kripke frames is simply an indexed frame where $\sim_a = \sim_b$ and $\asymp_w = \asymp_v$ for all a, b, w, v. In Subsection 2.2 we show that any subframe of a product of Kripke frames can be turned in a truth-preserving manner into an indexed frame, which will grant us a bunch of extra completeness results.

In Section 3 we show that every formula that is satisfied in an indexed frame can be satisfied in a finite indexed frame.

Naming the agents. Let us go back to the notion "Alice does not know Bob and Charlie are friends". In order to express this in our language, we need to name the agents. This is done in [8] via the introduction of nominal variables and modality $@_n$, directly imported from hybrid logic: see [1,3,5,6]. The language $\mathcal{L}(@)$ extends $\mathcal{L}$ with the atom n and the operator $@_n\phi$, where n belongs to Nom, a countable set of nominal variables. A model for $\mathcal{L}(@)$ is a tuple $(W, A, \sim, \asymp, V)$, as defined above, with the exception that $V : \mathsf{Prop} \cup \mathsf{Nom} \to 2^{W\times A}$ and, for each $n \in \mathsf{Nom}$, $V(n)$ is of the form $W \times \{a\}$ for some $a \in A$. The nominal n can thus be seen as the name of agent a. We now have: $w, a \models n$ iff $V(n) = W \times \{a\}$, and $w, a \models @_n\phi$ iff $w, b \models \phi$, where b is the agent named by n.

A complete axiomatisation of $\mathcal{L}(@)$ was provided for the first time by Sano in [7]. The proof of completeness works (roughly) as follows: first, a cut-free tree sequent calculus is introduced, which is then shown to be sound and complete. Then Sano shows that a formula which is provable in the Hilbert-style system can be converted into a provable tree sequent and, conversely, that from a provable tree sequent one can obtain a formula which is derivable in the Hilbert-style system.

In the conclusion of [7] it is suggested that finding a proof of this result using canonical models is an interesting area of future research. We present such a proof in Section 4 (Subsection 4.1), along with a proof that the logic possesses the finite model property (Subsection 4.2).

Back to friendship logic. For most of this paper we ignore many of the constraints imposed in [8] upon the models in order to make them a realistic framework for a logic of knowledge and friendship, namely: the set of agents A should be finite, the epistemic relations $\sim_a$ should be equivalence relations, the friendship relations $\asymp_w$ should be symmetric and irreflexive, and, optionally, it should be the case that an agent always knows who her friends are (if $w \sim_a v$ and $a \asymp_w b$, then $a \asymp_v b$). We address these properties in Subsection 4.3 and use all the previous results to provide a logic for the exact class of models proposed in [8]. (It is worth noting that, although in Section 4 we stick to the $\sim$ and $\asymp$ symbols to maintain the notation of [8,7], until this moment the reader should not assume they denote equivalence or symmetric relations.)

Another extension. Another operator from hybrid logic is considered in [8]. The operator $\downarrow x.\phi$ allows to name the current agent x, making it possible to refer to it indexically. The resulting extension of $\mathcal{L}(@)$, let us call it $\mathcal{L}(@\downarrow)$, allows to express things like "I have a friend who knows n is friends with me", $\downarrow x.\hat{F}K@_n\hat{F}x$. In Section 5 we provide a sound and complete axiomatization for $\mathcal{L}(@\downarrow)$.

Some proofs have been moved to the Appendix.

2 Indexed Frames

Definition 2.1 An *indexed frame* is a tuple (W, A, R, S) where W and A are nonempty sets, and $R \subseteq A \times W^2$, $S \subseteq W \times A^2$ are ternary relations. We use $R_a ww'$ and $S_w aa'$ to denote, respectively, $(a, w, w') \in R$ and $(w, a, a') \in S$.

We can see R and S as families of binary relations $\{R_a\}_{a\in A}$ and $\{S_w\}_{w\in W}$.

Alternatively, we can see indexed frames as tuples (W, A, R, S) where R and S are binary relations on $W \times A$ such that $R(w, a)(w', a')$ implies $a = a'$ and $S(w, a)(w', a')$ implies $w = w'$.

Let Prop be a countable set of propositional variables. We will consider a language $\mathcal{L}$ as defined in the introduction. We leave aside the epistemic and social considerations and call our modal boxes $\Box_1$ and $\Box_2$ instead of K and F.

Thus our language $\mathcal{L}$ will be $\phi ::= p|\bot|\neg\phi|(\phi \wedge \phi)|\Box_1\phi|\Box_2\phi$, with $p \in \mathsf{Prop}$. We define the other Boolean connectives as usual, the dual modalities $\Diamond_i\phi := \neg\Box_i\neg\phi$ for $i = 1, 2$, and we adopt the standard rules for omission of the parentheses. Given $\phi \in \mathcal{L}$ we define its set of subformulas $\mathsf{subf}\phi$ in the standard way, and its *modal depth*, $\mathsf{md}(\phi)$, recursively as follows:
$\mathsf{md}(p) = \mathsf{md}(\bot) = 0$, $\mathsf{md}(\neg\phi) = \mathsf{md}(\phi)$, $\mathsf{md}(\phi_1 \wedge \phi_2) = \max_{i=1,2} \mathsf{md}(\phi_i)$, $\mathsf{md}(\Box_i\phi) = 1 + \mathsf{md}(\phi)$.

Definition 2.2 An *indexed model* for $\mathcal{L}$ is a tuple $\mathfrak{M} = (W, A, R, S, V)$ where (W, A, R, S) is an indexed frame and $V : \mathsf{Prop} \to 2^{W \times A}$ is a valuation.

We interpret formulas of $\mathcal{L}$ on indexed models with respect to pairs $(w, a) \in W \times A$ as follows:

$(w, a) \models \Box_1\phi$ iff $w', a \models \phi$ for all $w' \in W$ such that $R_a ww'$;

$(w, a) \models \Box_2\phi$ iff $w, a' \models \phi$ for all $a' \in A$ such that $S_w aa'$.

Global truth of formulas in models and validity of formulas in frames are defined as usual.

2.1 The logic of indexed models

Definition 2.3 Given a unimodal logic L, let $\mathsf{Fr}\, L$ be the class of Kripke frames $\mathcal{F}$ such that $\mathcal{F} \models L$. Given unimodal Kripke-complete logics L_1 and L_2 we define $L_1 \circ L_2$ as the logic of indexed frames (W, A, R, S) such that $(W, R_a) \in \mathsf{Fr}\, L_1$ for all $a \in A$ and $(A, S_w) \in \mathsf{Fr}\, L_2$ for all $w \in W$.

Assuming no constraints on the relations R_a and S_w, the logic of indexed models is the fusion logic $\mathsf{K}\oplus\mathsf{K}$, i.e., the least normal modal logic in $\mathcal{L}$ containing the axioms of the minimal modal logic K for each of the $\Box_i$. To express this in terms of the above definition:

Theorem 2.4 $\mathsf{K} \circ \mathsf{K} = \mathsf{K} \oplus \mathsf{K}$.

This result can be proven using a step-by-step construction. For such a proof, see the Appendix. In the next Subsection we shall prove a more general result, and for this we will employ the notion of *relativized products*, studied in [4].

2.2 Indexed frames and relativized products

The following definitions can be found in [4]:

Definition 2.5 Given two families of frames $\mathcal{K}_1$ and $\mathcal{K}_2$, let $\mathcal{K}_1 \times \mathcal{K}_2$ be the family of products of Kripke frames $\mathcal{F}_1 \times \mathcal{F}_2$ such that $\mathcal{F}_i \in \mathcal{K}_i$. Given Kripke-complete unimodal logics L_1, L_2, we define their *(arbitrary) relativized product* as the logic of arbitrary subframes of products of Kripke frames $\mathcal{F}_1 \times \mathcal{F}_2$ such that $\mathcal{F}_i \in \mathsf{Fr}\, L_i$, i.e.,

$$(L_1 \times L_2)^{SF} = \mathsf{Log}\{\mathcal{G} : \mathcal{G} \subseteq \mathcal{F} \text{ for some } \mathcal{F} \in \mathsf{Fr}\, L_1 \times \mathsf{Fr}\, L_2\}.$$

(We say $\mathcal{G} = (W', R'_1, ..., R'_n)$ is a *subframe* of $\mathcal{F} = (W, R_1, ..., R_n)$, denoted $\mathcal{G} \subseteq \mathcal{F}$, whenever $W' \subseteq W$ and each R'_i is the restriction of R_i to W'.)

A logic L is a *subframe logic* if $\mathcal{F} \in \mathsf{Fr}\, L$ and $\mathcal{G} \subseteq \mathcal{F}$ implies $\mathcal{G} \in \mathsf{Fr}\, L$. (Example: $\mathsf{S4}$, because a subframe of a preorder is a preorder; nonexample: the logic of serial frames $\mathsf{K} + \Diamond\top$, because any finite subframe of $(\mathbb{N}, <)$ is not serial.) The following holds:

Proposition 2.6 ([4, Thm. 9.2]) *If L_1, L_2 are subframe logics, $L_1 \oplus L_2 \subseteq (L_1 \times L_2)^{SF}$.*

Moreover, if $L_1, L_2 \in \{\mathsf{K}, \mathsf{T}, \mathsf{K4}, \mathsf{S4}, \mathsf{S5}, \mathsf{S4.3}\}$, then $L_1 \oplus L_2 = (L_1 \times L_2)^{SF}$.

Let us use these results to give a proof of completeness for the logic of indexed frames. Let $\mathcal{F}_i = (W_i, R'_i)$ for $i = 1, 2$. Let $\mathcal{F} = (W, R_1, R_2)$ be a subframe of $\mathcal{F}_1 \times \mathcal{F}_2$. This means that $W \subseteq W_1 \times W_2$, and R_1 and R_2 are the restrictions to W of the horizontal and vertical relations $(R'_1)^H$ and $(R'_2)^V$ respectively.

Consider the indexed frame $\mathcal{G} = (W_1, W_2, R^H, R^V)$, where, for $w_2 \in W_2$,

$$R^H_{w_2} w_1 w'_1 \text{ iff } \begin{cases} (w_1, w_2) \in W \text{ and } (w'_1, w_2) \in W \text{ and } (w_1, w_2) R_1 (w'_1, w_2); & \text{or} \\ (w_1, w_2) \notin W \text{ and } w'_1 = w_1. \end{cases}$$

and, for $w_1 \in W_1$,

$$R^V_{w_1} w_2 w'_2 \text{ iff } \begin{cases} (w_1, w_2) \in W \text{ and } (w_1, w'_2) \in W \text{ and } (w_1, w_2) R_2 (w_1, w'_2); & \text{or} \\ (w_1, w_2) \notin W \text{ and } w'_2 = w_2. \end{cases}$$

Now, let V be a valuation on $\mathcal{F}$ and set $V'(p) = V(p)$ as a valuation on the indexed frame (W_1, W_2, R^H, R^V). The following holds:

Proposition 2.7 *Let ϕ be a formula in the bimodal language, and let $(w_1, w_2) \in W$. Then $\mathcal{F}, V, (w_1, w_2) \models \phi$ iff $\mathcal{G}, V', (w_1, w_2) \models \phi$.*

Proof. By induction on ϕ. Let us see for instance the case $\phi = \Box_1 \psi$.

If $\mathcal{F}, V, (w_1, w_2) \models \Box_1 \psi$, then let w'_1 such that $R^H_{w_2} w_1 w'_1$. Since $(w_1, w_2) \in W$, by definition we have that $(w'_1, w_2) \in W$ and $(w_1, w_2) R_1 (w'_1, w_2)$ in $\mathcal{F}$, which means that $\mathcal{F}, V, (w'_1, w_2) \models \psi$, and, by induction hypothesis $\mathcal{G}, V', (w'_1, w_2) \models \psi$. But since this is true for all w'_1 such that $R^H_{w_2} w_1 w'_1$, we have that $\mathcal{G}, V', (w_1, w_2) \models \Box_1 \psi$. The converse is analogous, noting that $(w_1, w_2) R_1 (w'_1, w_2)$ implies $R^H_{w_2} w_1 w'_1$. $\Box$

Moreover, we have the following:

Lemma 2.8 *Suppose $\mathcal{F}$ is a subframe of $(W_1, R'_1) \times (W_2, R'_2)$. Suppose R'_1 (respectively R'_2) has one of the following properties: reflexive; transitive; symmetric; connected; Euclidean. Then, for all w_2, $R^H_{w_2}$ (resp. for all w_1, $R^V_{w_1}$) has the same property.*

Proof. Straightforward by construction of R^H and R^V. □

As a consequence:

Theorem 2.9 *If* $L_1, L_2 \in \{\mathsf{K}, \mathsf{T}, \mathsf{K4}, \mathsf{S4}, \mathsf{S5}, \mathsf{S4.3}\}$*, then* $L_1 \circ L_2 = L_1 \oplus L_2$.

Proof. The inclusion $L_1 \circ L_2 \supseteq L_1 \oplus L_2$ holds by definition of $L_1 \circ L_2$. It suffices to see that $L_1 \circ L_2 \subseteq L_1 \oplus L_2$. If $\phi \notin L_1 \oplus L_2$, then by Proposition 2.6 there exist frames $(W_1, R'_1) \in \mathsf{Fr}\, L_1$ and $(W_2, R'_2) \in \mathsf{Fr}\, L_2$, a frame $\mathcal{F} = (W, R_1, R_2) \subseteq (W_1, R'_1) \times (W_2, R'_2)$, a valuation V on $\mathcal{F}$ and a world $(w_1, w_2) \in W$ such that $\mathcal{F}, V, w_1, w_2 \not\models \phi$. But then, the above construction $\mathcal{G} = (W_1, W_2, R^H, R^V)$ satisfies: $(W_1, R^H_{w_2}) \in \mathsf{Fr}\, L_1$ and $(W_2, R^V_{w_1}) \in \mathsf{Fr}\, L_2$ for all w_1, w_2 (Proposition 2.8), and $\mathcal{G}, V', w_1, w_2 \not\models \phi$ (Proposition 2.7); therefore, $\phi \notin L_1 \circ L_2$. □

3 Finite Indexed Model Property

All the logics mentioned so far have the Finite Model Property in the sense that, if a formula is consistent in the logic, there will be a finite model satisfying it [1]. But can we find a finite indexed model satisfying such a formula? The answer is affirmative.

Definition 3.1 A logic L is said to have the *Finite Indexed Model Property (iFMP)* if, given $\phi \notin L$, there exists an indexed model $\mathfrak{M} = (W, A, R, S, V)$ such that W and A are finite, $(W, A, R, S) \models L$, and, for some $(w, a) \in W \times A$, we have $\mathfrak{M}, w, a \not\models \phi$.

Given Kripke-complete unimodal logics L_1 and L_2, let $(L_1 \circ L_2)^f$ be the logic of finite indexed frames of $L_1 \circ L_2$.

Theorem 3.2 $\mathsf{K} \oplus \mathsf{K}$ *has the iFMP, i.e.,* $(\mathsf{K} \circ \mathsf{K})^f = \mathsf{K} \circ \mathsf{K} = \mathsf{K} \oplus \mathsf{K}$.

Proof. This amounts to showing that, if a formula ϕ_0 is satisfied in an indexed model, then there is a finite indexed model that satisfies it. Let $\mathfrak{M} = (W, A, R, S, V)$ and $(w_0, a_0) \in W \times A$ such that $\mathfrak{M}, w_0, a_0 \models \phi_0$.

We define relations $\mathbf{R}$ and $\mathbf{S}$ on $W \times A$ as follows: $(w, a)\mathbf{R}(w', a')$ iff $a = a'$ and $R_a ww'$, and $(w, a)\mathbf{S}(w', a')$ iff $w = w'$ and $S_w aa'$. We will consider chains starting at (w_0, a_0), of the form

$$\alpha = (w_0, a_0)\mathbf{T}_1(w_1, a_1)...\mathbf{T}_k(w_k, a_k),$$

with $k \geq 0$, $\mathbf{T}_i \in \{\mathbf{R}, \mathbf{S}\}$ and $(w_{i-1}, a_{i-1})\mathbf{T}_i(w_i, a_i)$ for $1 \leq i \leq k$. We shall say that such a chain has *length* k (and thus (w_0, a_0) is a chain of length 0). We will call $\mathsf{last}\, \alpha = (w_k, a_k)$.

Fix n to be the modal depth of ϕ_0. We shall construct a finite set of chains of length up to n, in n steps. Let $F_0 = \{(w_0, a_0)\}$. For $0 \leq k \leq n - 1$, suppose F_k is a finite set of chains of length k. Let F_{k+1} be a finite set of minimal cardinality satisfying the following property for all $\alpha \in F_k$ and all $\mathbf{T} \in \{\mathbf{R}, \mathbf{S}\}$:

> for any $(w, a) \in W \times A$, if $(\mathsf{last}\, \alpha)\mathbf{T}(w, a)$, then there exists an element $(w', a') \sim_{\phi_0} (w, a)$ such that $\alpha\mathbf{T}(w', a') \in F_{k+1}$,

[1] Indeed, every logic in the set $\{\mathsf{K}, \mathsf{T}, \mathsf{K4}, \mathsf{S4}, \mathsf{S5}, \mathsf{S4.3}\}$ has the FMP and this property is preserved by fusions: see [11].

where $\sim_{\phi_0}$ is the equivalence relation

$(w, a) \sim_{\phi_0} (w', a')$ iff for all $\psi \in \mathsf{subf}\,\phi_0(\mathfrak{M}, w, a \models \psi$ iff $\mathfrak{M}, w', a' \models \psi)$.

It is not hard to see that there is a set of cardinality at most $2 \cdot |F_k| \cdot 2^{|\,\mathsf{subf}\,\phi_0|}$ satisfying this property. Indeed, for any of the $|F_k|$ choices of α and 2 choices of $\mathbf{T}$, F_{k+1} will contain an element $\alpha\mathbf{T}(w, a)$ for (at most) one representative of each of the (at most) $2^{|\,\mathsf{subf}\,\phi_0|}$ equivalence classes of $\sim_{\phi_0}$.

Let $F' = F_0 \cup ... \cup F_n$. Let F be the closure of F' under the following property:

if $\alpha \in F$, $length(\alpha) < n$, $\mathbf{T} \in \{\mathbf{R}, \mathbf{S}\}$, $w \in W$ and $a \in A$ occur in F, and $(\mathsf{last}\,\alpha)\mathbf{T}(w, a)$, then $\alpha\mathbf{T}(w, a) \in F$.

Obviously, F' is finite, and so is F.

We construct our finite model $\mathfrak{M}^f = (W^f, A^f, R^f, S^f, V^f)$ where W^f and A^f are the restrictions of W and A to those elements occuring in F, i.e,

$$W^f = \{w \in W : w \text{ occurs in } F\};\ A^f = \{a \in A : a \text{ occurs in } F\};$$

and R^f, S^f and V^f are the corresponding restrictions of R, S, and V. The following holds:

Lemma 3.3 *Let $\alpha \in F$ be a chain of length k, i.e,*

$$\alpha = (w_0, a_0)\mathbf{T}_1(w_1, a_1)...\mathbf{T}_k(w_k, a_k),$$

with $\mathbf{T}_i \in \{\mathbf{R}, \mathbf{S}\}$. Let ϕ be a subformula of ϕ_0 such that $\mathsf{md}(\phi) \leq n - k$. Then, $\mathfrak{M}, w_k, a_k \models \phi$ if and only if $\mathfrak{M}^f, w_k, a_k \models \phi$.

This proves our theorem: it suffices to apply the previous Lemma to the chain (w_0, a_0) of length 0 to obtain $\mathfrak{M}^f, w_0, a_0 \models \phi_0$. □

Remark 3.4 The fact that we are taking a submodel of $\mathfrak{M}$ grants us that we can preserve the universal properties of the relations. This means that, if R is reflexive/ transitive/ symmetric/ connected/ Euclidean, so is R^f. Likewise for S and S^f. This fact, paired with Theorem 2.9, gives us the following result immediately:

Theorem 3.5 *If $L_1, L_2 \in \{\mathsf{K}, \mathsf{T}, \mathsf{K4}, \mathsf{S4}, \mathsf{S5}, \mathsf{S4.3}\}$, then $L_1 \oplus L_2$ has the iFMP. In other words, $(L_1 \circ L_2)^f = L_1 \circ L_2 = L_1 \oplus L_2$.*

4 Epistemic Logic of Friendship

We now consider the framework for an 'epistemic logic of friendship' proposed by [8]. For now, this amounts to adding a set $\mathsf{Nom} = \{n, m, ...\}$ of nominal variables to our language, and extending the language to $\mathcal{L}(@)$, defined as:

$$\phi ::= p|n|\bot|\neg\phi|(\phi \wedge \phi)|K\phi|F\phi|@_n\phi,$$

where $p \in \mathsf{Prop}, n \in \mathsf{Nom}$.

Definition 4.1 Models for $\mathcal{L}(@)$ are of the shape $\mathfrak{M} = (W, A, \sim, \asymp, V)$, where $(W, A, \sim, \asymp)$ is an indexed frame and $V : \mathsf{Prop} \cup \mathsf{Nom} \to 2^{W \times A}$ is a valuation function with the property that, for each $n \in \mathsf{Nom}$, $V(n) = W \times \{a\}$ for some $a \in A$. We refer to this unique a as $a = \underline{n}_V$ (or $a = \underline{n}$ if there is no risk of ambiguity).

A model is *named* whenever, for each $a \in A$, there exists $n \in \mathsf{Nom}$ such that $\underline{n} = a$. (Note that, in a named model, A is at most countable.)

We interpret formulas of $\mathcal{L}(@)$ in named models with respect to pairs $(w, a) \in W \times A$ as follows:

$\mathfrak{M}, w, a \models n$ iff $(w, a) \in V(n)$ (iff $\underline{n} = a$);
$\mathfrak{M}, w, a \models @_n\phi$ iff $\mathfrak{M}, w, \underline{n} \models \phi$.

4.1 Axiomatising $\mathcal{L}(@)$ via canonical models

It is proven in [7], via an argument that employs a tree sequent calculus, that the logic of $\mathcal{L}(@)$ is the system EFL, defined in the table below:

(Taut)	all propositional tautologies	(MP)	from ϕ and $\phi \to \psi$, infer ψ
(K_K)	$K(\phi \to \psi) \to (K\phi \to K\psi)$	(Nec_K)	from ϕ, infer $K\phi$
(K_F)	$F(\phi \to \psi) \to (F\phi \to F\psi)$	(Nec_F)	from ϕ, infer $F\phi$
($\mathrm{K}_@$)	$@_n(\phi \to \psi) \to (@_n\phi \to @_n\psi)$	($\mathrm{Nec}_@$)	from ϕ, infer $@_n\phi$
(Ref)	$@_n n$	(Selfdual)	$\neg @_n\phi \leftrightarrow @_n\neg\phi$
(Elim)	$@_n\phi \to (n \to \phi)$	(Agree)	$@_n @_m\phi \to @_m\phi$
(Back)	$@_n\phi \to F@_n\phi$	(DCom)	$@_n K @_n\phi \leftrightarrow @_n K\phi$
($\mathrm{Rigid}_=$)	$@_n m \to K@_n m$	($\mathrm{Rigid}_\neq$)	$\neg @_n m \to K\neg @_n m$
(Name)	From $@_n\phi$ infer ϕ, where n is fresh in ϕ		
(LBG)	From $L(@_n\hat{F}m \to @_m\phi)$ infer $L(@_nF\phi)$, m fresh in $L(@_nF\phi)$.		

In the last line of the above table, the *necessity forms* $L(\#)$ are defined as:

$$L ::= \# | \phi \to L | @_n KL.$$

In this section we present a novel proof of this result using canonical models. To do this, we consider instead the logic EFL^+, obtained by replacing the rule (LBG) in EFL by the following:

(LBG^+) From $L(@_n\hat{F}m \to @_m\phi)$ for all m fresh in $L(@_nF\phi)$, infer $L(@_nF\phi)$.

The following Lemma can be proven by a straightforward induction on derivations.

Lemma 4.2 EFL *and* EFL^+ *prove the same formulas.*

We thus prove completeness of EFL^+. The following validities will be useful:

Proposition 4.3 *The following are derivable in* EFL*:*

(T1) $\vdash @_m @_n \phi \leftrightarrow @_n \phi$;
(T2) $\vdash n \to (@_n \phi \leftrightarrow \phi)$;
(T3) $\vdash @_n m \to (@_n \phi \leftrightarrow @_m \phi)$;
(T4) $\vdash @_n m \leftrightarrow @_m n$;
(T5) $\vdash @_n(\phi \to \psi) \leftrightarrow (@_n \phi \to @_n \psi)$;
(T6) $\vdash @_n m \to (\phi[k/n] \leftrightarrow \phi[k/m])$, *where* $\phi[k/n]$ *is the formula obtained from* ϕ *by replacing each occurrence of* k *by* n.
(T7) $\vdash @_n m \to @_i K @_n m$, *and* $\vdash @_n \neg m \to @_i K @_n \neg m$;
(T8) $\vdash @_n \hat{F} m \wedge @_m \phi \to @_n \hat{F} \phi$;
(T9) $\vdash @_n F \psi \wedge @_n \hat{F} m \to @_m \psi$;
(R1) *if* $\vdash @_n \hat{F} m \wedge @_m \phi \to \psi$, *then* $\vdash @_n \hat{F} \phi \to \psi$, *with* $m \neq n$ *fresh in* ϕ *and* ψ.

We will say that a formula in $\mathcal{L}(@)$ is a *named formula* whenever it is of the form $@_n \phi$. A *BCN formula* is a Boolean combination of named formulas, and we use BCN to denote the set of such formulas. The following is an immediate consequence of (T1), (T5) and (Selfdual):

Corollary 4.4 *If* $\phi \in BCN$, $n \in \mathsf{Nom}$, *then* $\vdash @_n \phi \leftrightarrow \phi$.

A formula ϕ is *consistent* if $\neg\phi$ is not derivable. The following lemma will be useful later.

Lemma 4.5 *If* n *does not occur in* ϕ, *then* ϕ *is consistent if and only if* $@_n \phi$ *is consistent.*

Proof. If ϕ is inconsistent we have $\vdash \neg\phi$ and thus by ($\text{Nec}_@$), $\vdash @_n \neg\phi$, which by (Selfdual) gives that $\vdash \neg @_n \phi$. If $@_n \phi$ is inconsistent then $\vdash \neg @_n \phi$ which by (Selfdual) means $\vdash @_n \neg \phi$ and thus, by (Name), $\vdash \neg\phi$. □

Now we can start our completeness proof. The two above results allow us to focus only on BCN formulas. A *theory* is a set of BCN formulas T such that:

i. $\mathsf{EFL}^+ \cap BCN \subseteq T$;
ii. T is closed under Modus Ponens;
iii. If $L(@_n \hat{F} m \to @_m \phi) \in T$ for all $m \neq n$ not occurring in L or in ϕ, then $L(@_n F \phi) \in T$.

A theory is *consistent* whenever $@_n \bot \notin T$ (for any/all n). It is easy to see that $\mathsf{EFL}^+ \cap BCN$ is the least consistent theory. A consistent theory is *maximal* if no proper superset of it is a consistent theory.

Lemma 4.6 *Given a theory* T, *the set*

$$T_{K_n} = \{\psi \in BCN : \ \vdash \psi \leftrightarrow @_n \phi \text{ for some } @_n K \phi \in T\}$$

is a theory.

Proof. Note the following: for any $\phi \in BCN$, we have that $\phi \in T_{K_n}$ iff $@_n K \phi \in T$. Indeed,if $\phi \in T_{K_n}$, then $\vdash \phi \leftrightarrow @_n \psi$ for some $@_n K \psi \in T$. But then, using (Nec_K), ($\text{Nec}_@$) and (DCom) in that order we obtain $\vdash @_n K \phi \leftrightarrow$

$@_n K\psi$, and thus $@_n K\phi \in T$. The other direction is trivial and uses that $\vdash @_n\phi \leftrightarrow \phi$. With this:

Rule i. If $\phi \in \mathsf{EFL}^+ \cap BCN$, $m \in \mathsf{Nom}$, $@_n K\phi \in \mathsf{EFL}^+ \cap BCN$ (by applying two Nec rules) and thus $@_n K\phi \in T$, so $\phi \in T_{K_n}$.

Rule ii. If ϕ and $\phi \to \psi \in T_{K_n}$, then $@_n K\phi$, $@_n K(\phi \to \psi) \in T$ and, by applying the K axioms and modus ponens, $@_n K\psi \in T$, and thus $\psi \in T_{K_n}$.

Rule iii. If $L(@_k \hat{F} m \to @_m \phi) \in T_{K_n}$ for all fresh m, then $@_n KL(@_k \hat{F} m \to @_m \phi) \in T$ for all fresh m, and thus, since $@_n KL$ is an admissible form, $@_n KL(@_k F\phi) \in T$, whence $L(@_k F\phi) \in T_{K_n}$.

Lemma 4.7 *Given a theory T and a formula $\phi \in BCN$, the set $T_\phi = \{\psi \in BCN : \phi \to \psi \in T\}$ is a theory containing T and including the formula ϕ, and it is consistent whenever T is consistent and $\neg\phi \notin T$.*

Proof. Rule i. If $\psi \in \mathsf{EFL}^+ \cap BCN$, then $\phi \to \psi \in \mathsf{EFL}^+ \cap BCN$, thus $\psi \in T_\phi$.

Rule ii. Follows from classical propositional logic.

Rule iii. Follows from the fact that, if L is an admissible form, so is $\phi \to L$.

The fact that $\phi \in T_\phi \supseteq T$ is because $\vdash \phi \to \phi$ and $\vdash \psi \to (\phi \to \psi)$. If $\neg\phi \notin T$, then $@_n\neg\phi \notin T$, thus $@_n(\phi \to \bot) \notin T$. Using the K axiom and $\vdash \phi \leftrightarrow @_n\phi$, we obtain $\phi \to @_n\bot \notin T$, and thus $@_n\bot \notin T_\phi$.

Now,

Lemma 4.8 (Lindenbaum's lemma) *A consistent theory can be extended to a maximal consistent theory.*

Proof. Let T_0 be a consistent theory and $(\phi_k)_{k\in\omega}$ be an enumeration of BCN where each formula occurs infinitely many times.

Given a consistent theory T_k, we define a consistent theory T_{k+1} (which extends T_k) as follows:

- If $\neg\phi_k \notin T_k$, then $T_{k+1} = (T_k)_{\phi_k}$.
- If $\neg\phi_k \in T_k$, then:
  - · If $\neg\phi_k$ is of the form $\neg L(@_n F\phi)$, then for some fresh m it must be the case that $L(@_n \hat{F} m \to @_m \phi) \notin T_k$, for otherwise we would have by rule iii. that $L(@_n F\phi) \in T_k$, contradicting its consistency. Then we set $T_{k+1} = (T_k)_{\neg L(@_n \hat{F} m \to @_m \phi)}$.
  - · Otherwise, $T_{k+1} = T_k$.

Let $T = \bigcup_{k\in\omega} T_k$. Then T is a maximal consistent theory. Consistency is obvious, for each T_k is consistent. Maximality comes from the fact that, for every formula ϕ_k, either $\neg\phi_k$ was already in T_k, or ϕ_k was added to T_{k+1}, therefore it cannot have consistent supersets closed under modus ponens. To see that it is a theory, it suffices to check that Rule iii. is satisfied. And indeed, if $L(@_n F\phi) \notin T$, then $\neg L(@_n F\phi) \in T_k$ for some k. Consider some $k' > k$ such that $\phi_{k'} = \neg L(@_n F\phi)$. Then, by construction, $T_{k'+1}$ must contain $\neg L(@_n \hat{F} m \to @_m \phi)$ for some fresh m, and therefore it is not the case that $L(@_n \hat{F} m \to @_m \phi) \in T$ for all fresh m.

Let MCT denote the set of maximal consistent theories. Given $T, S \in$

MCT, and $n \in \mathsf{Nom}$, we define: $T \sim_n S$ iff $T_{K_n} \subseteq S$.

Lemma 4.9 (Diamond Lemma) *Let $T \in MCT$. We have:*

i. If $@_n\hat{K}\phi \in T$, then there exists $S \in MCT$ such that $T \sim_n S \ni @_n\phi$.

ii. If $@_n\hat{F}\phi \in T$, then there is some $m \neq n$ fresh in ϕ such that $@_n\hat{F}m \wedge @_m\phi \in T$.

Proof. i. Take the consistent theory $(T_{K_n})_{@_n\phi}$ and extend it to the desired successor using Lindenbaum's lemma. Note that T_{K_n} is consistent, for if not, $@_n\bot \in T_{K_n}$, and thus $@_nK@_n\bot \in T$. But, since $@_n\bot$ is equivalent to $\bot$, this means that $@_nK\bot \in T$, contradicting $@_n\hat{K}\phi \in T$. Note moreover that $\neg@_n\phi \notin T_{K_n}$, for if that was the case, $@_nK\neg@_n\phi \in T$, which is equivalent to $\neg@_n\hat{K}\phi \in T$: contradiction. Thus $(T_{K_n})_{@_n\phi}$ is consistent.

ii. If $@_n\hat{F}m \wedge @_m\phi \notin T$ for all fresh m, then $\neg(@_n\hat{F}m) \vee \neg(@_m\phi) \in T$ for all fresh m, and thus, by logical equivalence, $@_n\hat{F}m \to @_m\neg\phi \in T$ for all fresh m, which entails $@_nF\neg\phi \in T$, and therefore $\neg@_n\hat{F}\phi \in T$.

Lemma 4.10 *Let $i \in \mathsf{Nom}$. If $\Gamma \sim_i \Delta$ then, for any $n, m \in \mathsf{Nom}$, we have: $@_nm \in \Gamma$ if and only if $@_nm \in \Delta$.*

Proof. By (T7) of Prop. 4.3: if $@_nm \in \Gamma$, then $@_iK@_nm \in \Gamma$, which entails $@_i@_nm \in \Delta$, and therefore, by the (Agree) axiom, $@_nm \in \Delta$. If $@_nm \notin \Gamma$, by maximal consistency and the (Selfdual) axiom we have that $@_n\neg m \in \Gamma$ and we can proceed similarly to obtain that $@_n\neg m \in \Delta$ and thus $@_nm \notin \Delta$. □

Let ϕ_0 be a consistent formula and let us build a model satisfying it. Take a nominal n_0 not occurring in ϕ_0 and note that $@_{n_0}\phi_0$ is a consistent BCN formula (by Lemma 4.5) and thus the consistent theory $(BCN \cap \mathsf{EFL}^+)_{@_{n_0}\phi_0}$ can be extended (by Lindembaum's lemma) to $\Gamma_0 \in MCT$.

Let W be the set of elements reachable from Γ_0 by the $\sim_n$ relations, i.e.

$$W = \{\Delta \in MCT : \Gamma_0 = \Delta_0 \sim_{n_1} \Delta_1 \sim_{n_2} ... \sim_{n_k} \Delta_k = \Delta \\ \text{for some } n_1, ..., n_k \in \mathsf{Nom}, \Delta_0, ..., \Delta_k \in MCT\}.$$

Note that this construction guarantees (by Lemma 4.10) that for any $\Gamma \in W$, $@_nm \in \Gamma$ iff $@_nm \in \Gamma_0$. Note moreover that the theorems

$$\vdash @_nn \text{ (Ref)}; \vdash @_nm \leftrightarrow @_mn \text{ (T4)}; \vdash @_nm \wedge @_mi \to @_ni \text{ (conseq. of T3)}$$

guarantee that the binary relation on Nom defined as $n \equiv m$ iff $@_nm \in \Gamma_0$ is an equivalence relation. Let $[n]$ denote the equivalence class of $n \in \mathsf{Nom}$ and let $A = \{[n] :\in \mathsf{Nom}\}$.

For $[n] \in A$, we define $\sim_{[n]} = \sim_n$. Let us see that this is well-defined, which amounts to showing that $\sim_n = \sim_m$ whenever $n \equiv m$. But given $\Gamma, \Delta \in W$, and $n \equiv m$, the fact that $@_nm \in \Gamma \cap \Delta$ paired with (T3) give us that $@_nK\phi \in \Gamma$ iff $@_mK\phi$ in Γ, and $@_n\phi \in \Delta$ iff $@_m\phi \in \Delta$, which entails $\Gamma \sim_n \Delta$ iff $\Gamma \sim_m \Delta$.

For $\Gamma \in W$ we define

$$[n] \asymp_\Gamma [m] \text{ iff } @_n\hat{F}m \in \Gamma.$$

Let us see that this definition does not depend on the choice of representative for the equivalence classes: suppose $@_n \hat{F} m \in \Gamma$ and take $n' \in [n], m' \in [m]$. We have that $@_{n'} \hat{F} m \in \Gamma$, by (T3), and therefore, by (T6), $@_{n'} \hat{F} m' \in \Gamma$.

Finally we define a valuation by setting

$$\begin{aligned} V(p) &= \{(\Gamma, [n]) \in W \times A : @_n p \in \Gamma\}, & p &\in \mathsf{Prop}; \\ V(n) &= \{(\Gamma, [n]) : \Gamma \in W\}, & n &\in \mathsf{Nom}. \end{aligned}$$

Note that we have defined V so that $\underline{n} = [n]$. We have that

$$\mathfrak{M}^C = (W, A, \sim_{[n] \in A}, \asymp_{\Gamma \in W}, V)$$

is a named model and, moreover:

Lemma 4.11 (Truth Lemma) *For any formula $\phi \in \mathcal{L}(@)$, it is the case that $\mathfrak{M}^C, \Gamma, [n] \models \phi$ if and only if $@_n \phi \in \Gamma$.*

Proof. By induction on ϕ. For the case $\phi = m \in \mathsf{Nom}$ we recall that $\underline{m} = [m]$. For the case $\phi = K\psi$, we use the Diamond Lemma. For the case $\phi = F\psi$, we use the Diamond Lemma for one direction and (T9) for the other. □

With this:

Theorem 4.12 EFL^+ *(and therefore* EFL*) is complete with respect to the class of (not necessarily finite) named indexed models.*

Proof. If ϕ_0 is consistent, so is $@_{n_0} \phi_0$ for n_0 not occurring in ϕ_0, and thus we can construct $\mathfrak{M}^C$ as above and we have that $\mathfrak{M}^C, \Gamma_0, [n_0] \models \phi_0$. □

4.2 Finite models

The following also holds:

Theorem 4.13 EFL *is complete with respect to the class of finite named indexed models.*

This is a consequence of a result very similar to Theorem 3.2: if a formula is satisfied in a model $(W, A, \sim, \asymp, V)$, then there is a finite submodel which satisfies it.

The proof of this result has minimal changes with respect to the proof of Thm. 3.2, and it is sketched in the Appendix.

4.3 Extensions of EFL

In [8] some assumptions are made about the epistemic and social relations in the models. The epistemic relations $\sim_a$ are equivalence relations, whereas the friendship relation $\asymp_w$ is irreflexive and symmetric.

One would expect, for instance, that if the relations $\sim_a$ that give the semantics of the knowledge modality K are reflexive, transitive and symmetric, then this modality should follow the axioms of $\mathsf{S5}$, namely:

$$\vdash K\phi \to \phi; \quad \vdash K\phi \to KK\phi; \quad \vdash \phi \to K\neg K\neg\phi.$$

Similarly, if $\sim_a$ is a preorder, the extra axioms of S4 (i.e. the first two above), should be included to the logic. Let EFL+S5$_K$ denote the logic resulting from adding these three axioms to EFL, and let EFL+S4$_K$ be the logic resulting from adding the first two. And indeed:

Theorem 4.14 ([7]) EFL + S5$_K$ *is sound and complete with respect to the class of models where the* $\sim_a$ *are equivalence relations. Moreover,* EFL + S4$_K$ *is sound and complete with respect to the class of models where each* $\sim_a$ *is a preorder.*

The proof of this result in [7] consists in adding corresponding rules to the tree sequent calculus and showing that a provable formula in the Hilbert-style system can be transformed into a provable sequent and vice versa. With the canonical models presented in this text this proof becomes quite straightforward. First, note that thanks to (T5) the following are easily provable in EFL + S5$_K$ (and the first two in EFL + S4$_K$):

$$\vdash @_n K\phi \to @_n\phi; \quad \vdash @_n K\phi \to @_n KK\phi; \quad \vdash @_n\phi \to @_n K\neg K\neg\phi.$$

With this, the proof of the following lemma is straightforward:

Lemma 4.15 *If the axioms of* S5 *for* K *(resp.* S4*) are present in the logic, each relation* $\sim_n$ *in the canonical model is an equivalence relation (resp. a preorder).*

Remark 4.16 Given that $@_n$ distributes over $\to, \wedge, \vee, \neg$, one can see that there are many examples of formulas ϕ defining a certain frame property from which it is trivial to compute a formula $@_n\psi$ defining the same property in the $\sim_n$ relations of indexed frames. Some obvious questions arise: is this true of any Sahlqvist formula? Can we adapt the notion of Sahlqvist formula to this setting and prove an analogue of the Sahlqvist Completeness Theorem ([2, Thm. 4.42])? We conjecture the answer is affirmative.

Similarly, as pointed out by [7] the following axioms encode irreflexivity and symmetry of the friendship relation $\asymp_w$:

$$\text{(irr)} \quad \neg @_n \hat{F} n \qquad \text{(sym)} \quad @_n \hat{F} m \to @_m \hat{F} n$$

The proof of this lemma is also straightforward:

Lemma 4.17 *If (irr) and (sym) are present in the logic, each relation* $\asymp_\Gamma$ *in the canonical model is irreflexive and symmetric.*

Therefore, and since the rest of the completeness proof proceeds as before, we have a complete axiomatisation of the models proposed by [8]:

Theorem 4.18 EFL + S5$_K$ + (*irr*) + (*sym*) *is the logic of finite indexed frames* $(W, A, \sim, \asymp)$ *where each* $\sim_a$ *is an equivalence relation and each* $\asymp_w$ *is irreflexive and symmetric.*

Finally, an optional further constraint is that an agent should not doubt who her own friends are. For this one would consider frames with the property:

if $w \sim_a v$, then $a \asymp_w b$ implies $a \asymp_v b$. We will call these *KYF frames* (for "know your friends"). It is again very easy to check that, by adding to the logic the axiom

$$\text{(kyf)} \quad \hat{F}m \to K\hat{F}m,$$

the resulting canonical model is a KYF frame.

5 Axiomatisation of $\mathcal{L}(@\downarrow)$

In [8] another operator is borrowed from hybrid logic, namely $\downarrow x.\phi$, which names the current agent 'x', allowing to refer to her indexically. We now have, on top of Prop and Nom, a countable set $\mathsf{SVar} = \{x, y, ...\}$ of *state variables*. $\mathcal{L}(@\downarrow)$ is simply $\mathcal{L}(@)$ extended with x and $\downarrow x.\phi$, where $x \in \mathsf{SVar}$. Formulas are read on named indexed models with respect to triples (g, w, a), where $g : \mathsf{SVar} \to A$ is an assignment function, as follows:

$\mathfrak{M}, g, w, a \models x$ iff $g(x) = a$;
$\mathfrak{M}, g, w, a \models \downarrow x.\phi$ iff $\mathfrak{M}, g^x_a, w, a \models \phi$,
where $g^x_a(y) = g(y)$ for $y \neq x$ and $g^x_a(x) = a$.

Given a formula ϕ and a nominal n, we define $\phi[x/n]$ to be the formula resulting from replacing each *free* occurence of x in ϕ by n. Formally:

Definition 5.1 Given $x \in \mathsf{SVar}$, $n \in \mathsf{Nom}$ and $\phi \in \mathcal{L}(@\downarrow)$:
$\phi[x/n] = \phi$ if $\phi = p \in \mathsf{Prop}, \bot, m \in \mathsf{Nom}$ or $y \in \mathsf{SVar}\backslash\{x\}$; $\quad x[x/n] = n$;
$(\phi \wedge \psi)[x/n] = \phi[x/n] \wedge \psi[x/n]$; $\quad (\downarrow x.\phi)[x/n] = \downarrow x.\phi$;
$(B\phi)[x/n] = B(\phi[x/n])$ if $B = \neg, K, F, @_m$, or $\downarrow y$ $(y \neq x)$;

With this, we can define the logic of the fragment $\mathcal{L}(@\downarrow)$:

Definition 5.2 $\mathsf{EFL}_\downarrow$ is the logic containing the axioms and rules of EFL plus the following axiom and rule:
(DA) $@_n(\downarrow x.\phi \leftrightarrow \phi[x/n])$.
(FV) from $\phi[x/n]$ (with n fresh in ϕ), infer ϕ.

The fact that (DA) is sound can be checked by just unpacking the semantics. The soundness of the (FV) rule is a consequence of the following Lemma, whose proof is an easy induction on ϕ:

Lemma 5.3 *Let $\phi \in \mathcal{L}(@\downarrow)$ and n be fresh in ϕ. Let $\mathfrak{M} = (W, A, \sim, R, V)$ be a model and g an assignment. We define a new valuation in $\mathfrak{M}$ by: $V'(n) = W \times \{g(x)\}$, $V'(m) = V(m)$ for $n \neq m$, $V'(p) = V(p)$ for $p \in \mathsf{Prop}$. Let $\mathfrak{M}' = (W, A, \sim, R, V')$. Then $\mathfrak{M}, w, a, g \models \phi$ iff $\mathfrak{M}', w, a, g \models \phi[x/n]$.*

For completeness we shall use these two lemmas; respectively an application of the (FV) rule, and a straightforward induction on ϕ:

Lemma 5.4 *If ϕ is consistent and n_1,...,n_k are fresh, then $\phi[x_1/n_1]...[x_k/n_k]$ is consistent.*

Lemma 5.5 *Let $\mathfrak{M}$ be a model, ϕ be a formula, g an assignment and $x_1, ..., x_k \in \mathsf{SVar}$. Let $n_1, ..., n_k \in \mathsf{Nom}$ such that $\underline{n}_i = g(x_i)$. Then*

$$\mathfrak{M}, w, a, g \models \phi \textit{ iff } \mathfrak{M}, w, a, g \models \phi[x_1/n_1]...[x_k/n_k].$$

Now, we construct our canonical model exactly like before with one caveat: our sets MCT will only contain *BCN formulas without free variables* (i.e. *BCN sentences*). We prove the following variant of the Truth Lemma:

Proposition 5.6 *Let g be an assignment and ϕ a formula whose free variables are $x_1, ..., x_k$. Let $[n_i] = g(x_i)$. Then*

$$\mathfrak{M}, \Gamma, [n], g \models \phi \textit{ iff } @_n \phi[x_1/n_1]...[x_n/n_k] \in \Gamma.$$

With this we can prove completeness:

Theorem 5.7 $\mathsf{EFL}_{\downarrow}$ *is complete with respect to indexed models.*

Proof. Suppose ϕ_0 is a consistent formula. Let $x_1, ..., x_k$ be the free variables of ϕ_0 and $n_0, n_1, ..., n_k$ fresh. Then $\phi_0[x_1/n_1]...[x_k/n_k]$ is a consistent sentence (by Lemma 5.4) and so is

$$@_{n_0} \phi_0[x_1/n_1]...[x_k/n_k]$$

(by Lemma 4.5). We extend this to $\Gamma_0 \in MCT$, we construct the corresponding canonical model and we let g be any assignment such that $g(x_i) = [n_i]$. Then we have by Prop. 5.6 that $\mathfrak{M}, \Gamma_0, [n_0], g \models \phi_0$. □

6 Conclusion

In this paper we have studied several aspects of indexed frames, introduced for the first time (as far as we know) in [8]. We have as well provided axiomatisations for the fragments $\mathcal{L}$ (with several constraints in the relations) and $\mathcal{L}(@\downarrow)$, on top of a novel proof of completeness of EFL for the fragment $\mathcal{L}(@)$.

Some interesting directions for future work include studying the decidability of $\mathcal{L}(@\downarrow)$, resolving the conjecture in Remark 4.16, or otherwise providing a more general version of Thm. 2.9.

But perhaps the most fruitful direction to go from here would be the application of indexed frames to different modal logics wherein some interdependence between the modalities exists. Just as an example, we could think of an epistemic temporal logic where each possible world is a timeline and the set of epistemically accessible worlds changes at every time, modelled using indexed frames.

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Appendix

Proof of Theorem 2.4. First we introduce a notion of *indexed pseudo-model*.

Definition .1 An *indexed pseudo-model* is a tuple (W, A, R, S, σ) where (W, A, R, S) is an indexed frame and σ is a function which assigns to every pair $(w, a) \in W \times A$ a $\mathsf{K} \oplus \mathsf{K}$-maximal consistent set, with the following properties:

(C1) If $\Box_1 \phi \in \sigma(w, a)$ and $R_a w w'$, then $\phi \in \sigma(w', s)$;

(C2) If $\Box_2 \phi \in \sigma(w, a)$ and $S_w a a'$, then $\phi \in \sigma(w', s')$.

The right-to-left direction of C1 and C2 need not hold for certain formulas ϕ and pairs (w, a). We call these situations *defects*. Formally:

Definition .2 A *1-defect* is a tuple (ϕ, w, a) such that $\neg\Box_1\phi \in \sigma(w, a)$ and, for all $w' \in W$ such that $R_a ww'$, $\phi \in \sigma(w', a)$. A *2-defect* is a tuple (ϕ, w, a) such that $\neg\Box_2\phi \in \sigma(w, a)$ and, for all $a' \in A$ such that $S_w aa'$, $\phi \in \sigma(w, a')$.

Given a 1-defect (ϕ, w, a) we can update our pseudo-model into a new pseudo-model without this defect by simply adding a point, as we detail below.

Let $\mathfrak{M} = (W, A, R, S, \sigma)$ be an indexed pseudo-model and (ϕ, w, a) be a 1-defect. That means that $\neg\Box_1\phi \in \sigma(w, a)$ yet $\phi \in \sigma(w', a)$ for all w' such that $R_a ww'$. Note that the set $\{\neg\phi\} \cup \{\psi : \Box_1\psi \in \sigma(w, a)\}$ is consistent,therefore it can be extended by Lindenbaum's lemma to a maximal consistent set Δ. Let $w_0 \notin W$. We define a new pseudo-model in which the defect is not present by $\mathfrak{M}'_1 = (W', A', R', S', \sigma')$, where:

- $W' = W \cup \{w_0\}$; $A' = A$;
- $R' = R \cup \{(a, w, w_0)\}$; $S' = S$;
- for all $a' \in A$, $\sigma'(w_0, a') = \Delta$ and $\sigma'(w', a') = \sigma(w', a')$ for $w' \neq w_0$.

$\mathfrak{M}'_1$ is an indexed pseudo-model. Indeed, suppose $\Box_1\psi \in \sigma'(w', a')$ and $R_{a'} w'w''$. If $w'' \neq w_0$, then $\sigma'(w'', a') = \sigma(w'', a') \ni \psi$. Otherwise, if $w'' = w_0$, then by construction we have that $w' = w$ and $a' = a$. Therefore, since $\Box_1\psi \in \sigma'(w', a') = \sigma(w, a)$ we have by construction that $\psi \in \Delta = \sigma'(w'', a')$. Moreover, we have built $\mathfrak{M}'_1$ such that (ϕ, w, a) is no longer a 1-defect.

In a completely analogous manner, given a 2-defect (ϕ, w, a) we can add an extra point a_0 to A to build a pseudo-model which does not present this defect: $\mathfrak{M}'_2 = (W', A', R', S', \sigma')$ with $W = W'$, $A' = A \cup \{a_0\}$, $R' = R$, $S' = S \cup \{(w, a, a_0)\}$, and $\sigma'(w', a) = \Delta$, for some maximal consistent set Δ containing $\{\psi : \Box_2\psi \in \sigma(w, a)\} \cup \{\neg\phi\}$.

Definition .3 Given an indexed pseudo-model $\mathfrak{M} = (W, A, R, S, \sigma)$ and a 1-defect (resp. a 2-defect) (ϕ, w, a), the $(1, \phi, w, a)$*-update* (resp. $(2, \phi, w, a)$*-update*) of $\mathfrak{M}$ is $\mathfrak{M}'_1$ (resp. $\mathfrak{M}'_2$) as constructed above.

We can now prove that $\mathsf{K} \oplus \mathsf{K}$ is the logic of indexed frames.

Fix a maximal consistent set Σ_0. Let us construct a chain of indexed pseudo-models

$$(\mathfrak{M}^k)_{k\in\omega} = (W^k, A^k, R^k, S^k, \sigma^k)_{k\in\omega}$$

such that, for all k,

i. Σ_0 is in the image of σ^k;
ii. $W^k \subseteq W^{k+1} \subseteq \mathbb{Q}$ and $A^k \subseteq A^{k+1} \subseteq \mathbb{Q}$;
iii. $R^k \subseteq R^{k+1}$ and $S^k \subseteq S^{k+1}$;
iv. $\sigma^{k+1}(w, a) = \sigma^k(w, a)$ if $(w, a) \in W^k \times A^k$.

Initial step: Take $w_0, a_0 \in \mathbb{Q}$ and set $W^0 = \{w_0\}$, $A^0 = \{a_0\}$, $R^0 = S^0 = \varnothing$, and $\sigma^0(w_0, a_0) = \Sigma_0$.

Recursive step. Let $(i_n, \psi_n, w_n, a_n)_{n\in\omega}$ be an enumeration of the set $\{1, 2\} \times \mathcal{L} \times \mathbb{Q} \times \mathbb{Q}$ in which every element appears infinitely many times. Suppose we have constructed $\mathfrak{M}^k = (W^k, A^k, R^k, S^k, \sigma^k)$. Then:

- If $i_k = 1$ and $(w_k, a_k) \in W^k \times A^k$ and (ψ_k, w_k, a_k) is a 1-defect of $\mathfrak{M}^k$, then $\mathfrak{M}^{k+1}$ is the $(1, \psi_k, w_k, a_k)$-update of $\mathfrak{M}^k$;

- If $i_k = 2$ and $(w_k, a_k) \in W^k \times A^k$ and (ψ_k, w_k, a_k) is a 2-defect of $\mathfrak{M}^k$, then $\mathfrak{M}^{k+1}$ is the $(2, \psi_k, w_k, a_k)$-update of $\mathfrak{M}^k$;
- Otherwise, $\mathfrak{M}^{k+1} = \mathfrak{M}^k$.

Finally, let $\mathfrak{M}^\omega = (W^\omega, A^\omega, R^\omega, S^\omega, \sigma^\omega)$, where:

- $W^\omega = \bigcup_{k\in\omega} W^k$; $A^\omega = \bigcup_{k\in\omega} A^k$;
- $R^\omega = \bigcup_{k\in\omega} R^k$; $S^\omega = \bigcup_{k\in\omega} S^k$;
- σ^ω is the unique function such that $\sigma^\omega|_{W^k\times A^k} = \sigma^k$ for all k.

We have:

Lemma .4 $\mathfrak{M}^\omega$ *is an indexed pseudo-model with no defects.*

Proof. The fact that $\mathfrak{M}^\omega$ is an indexed pseudo-model is rather straightforward. Suppose $\Box_1\phi \in \sigma(w, a)$ and $R^\omega_a ww'$ for some $\phi \in \mathcal{L}$, $w, w' \in W^\omega$ and $a \in A^\omega$. Let $k \in \omega$ be the least natural number such that $w, w' \in W^k$ and $a \in A^k$. Then we have that $\Box_1\phi \in \sigma^k(w, a)$ and $R^k_a ww'$, and thus $\phi \in \sigma^k(w', a) = \sigma(w', a)$. Therefore, (C1) is satisfied (and (C2) too via an analogous reasoning).

Let us now see there are no 1-defects (the proof that there are no 2-defects is completely analogous). Suppose that (ϕ, w, a) is a 1-defect of $\mathfrak{M}^\omega$, i.e., $\neg\Box_1\phi \in \sigma^\omega(w, a)$ yet $\phi \in \sigma^\omega(w', a)$ whenever $R^\omega_a ww'$.

Let us consider the least $k \in \omega$ such that $(w, a) \in W^k\times A^k$ and the least $n \geq k$ such that $(1, \phi, w, a) = (i_n, \psi_n, w_n, a_n)$ in the aforementioned enumeration. Then we have that (ϕ, w, a) is a 1-defect in $\mathfrak{M}^n$, and therefore it gets "fixed" in the update $\mathfrak{M}^{n+1}$, i.e., there exists some $w' \in W^{n+1} \setminus W^n$ such that $R^{n+1}_a ww'$ and $\neg\phi \in \sigma^{n+1}(w', a)$. But this means that $R^\omega_a ww'$ and $\neg\phi \in \sigma^\omega(w', a)$: a contradiction. □

Now,

Lemma .5 (Truth lemma.) *Define a valuation V on $\mathfrak{M}^\omega$ by:*

$$V(p) = \{(w, a) \in W^\omega \times A^\omega : p \in \sigma^\omega(w, a)\}.$$

Then for all $w \in W^\omega$, $a \in A^\omega$ and $\phi \in \mathcal{L}$, $\mathfrak{M}^\omega, w, a \models \phi$ if and only if $\phi \in \sigma^\omega(w, a)$.

Proof. By induction on the structure of ϕ. If $\phi = p$, then the definition of V gives us the result trivially. The induction steps corresponding to $\neg\phi$ and $\phi_1 \wedge \phi_2$ are straightforward.

Now let $\phi = \Box_1\psi$. If $w, a \models \Box_1\psi$, this means that $(w', a) \models \psi$ for every $w' \in W^\omega$ such that $R^\omega_a ww'$. But then by induction hypothesis $\psi \in \sigma^\omega(w', a)$ whenever $R_a ww'$. So, if $\Box_1\psi \notin \sigma^\omega(w, a)$, then $\neg\Box_1\psi \in \sigma^\omega(w, a)$ and thus (ψ, w, a) is a 1-defect, in contradiction with Lemma .4. Thus $\Box_1\psi \in \sigma^\omega(w, a)$. Conversely, suppose $\Box_1\psi \in \sigma^\omega(w, a)$ and $R^\omega_a ww'$. By (C1), this means that $\psi \in \sigma^\omega(w', a)$ which entails, by induction hypothesis, that $(w', a) \models \psi$. Since this is true for all w' with $R_a ww'$, we have that $w, a \models \Box_1\psi$.

The case $\phi = \Box_2\psi$ is analogous. □

With all this, we can prove the following theorem, from which Thm. 2.4 immediately follows:

Theorem .6 *The fusion logic* $\mathsf{K} \oplus \mathsf{K}$ *is complete with respect to indexed models.*

Proof. Given a consistent formula ϕ, extend it to a maximal consistent set Σ_0 and construct $\mathfrak{M}^\omega$ by the procedure described above, making sure that Σ_0 is in the image of σ^0. Then we have that there exist $w_0, a_0 \in W^\omega\times A^\omega$ such that $\sigma^\omega(w_0, a_0) = \Sigma_0 \ni \phi$, and therefore by the Truth Lemma $w_0, a_0 \models \phi$. □

Remark .7 *It is not hard to tweak this proof to show, for instance, that the fusion logic* $\mathsf{S4}_{\Box_1} \oplus \mathsf{K}_{\Box_2}$ *is the logic of indexed models* (W, A, R, S) *where* R_a *is a preorder for all* $a \in A$*, or that* $\mathsf{K}_{\Box_1} \oplus \mathsf{S5}_{\Box_2}$ *is the logic of indexed models wherein the* S_w *are equivalence relations. More generally, this procedure can easily be tweaked in order to provide a proof for every individual instance of Thm. 2.9. However, this proof can help us to go beyond that Theorem and allows us to show, for instance, that the result is true of the logic of serial frames, i.e.,* $(\mathsf{K}+\Diamond\top)\circ(\mathsf{K}+\Diamond\top) = (\mathsf{K}+\Diamond\top)\oplus(\mathsf{K}+\Diamond\top)$.

Proof of Lemma 3.3. By induction on ϕ. The cases for $\phi = p$ and $\phi = \top$ are trivial, and so is the inductive step for $\phi = \neg\psi$.

Case $\phi = \psi_1 \wedge \psi_2$. If $\mathfrak{M}, w_k, a_k \models \psi_1 \wedge \psi_2$, then $\mathfrak{M}, w_k, a_k \models \psi_i$ for $i = 1$ and 2. But then, since $\mathsf{md}\,\psi_i \leq \mathsf{md}\,\psi \leq n-k$, we have by induction hypothesis that $\mathfrak{M}^f, w_k, a_k \models \psi_i$ and thus $\mathfrak{M}^f, w_k, a_k \models \phi$. The converse is analogous.

Case $\phi = \Box_1\psi$. Suppose that $\mathfrak{M}^f, w_k, a_k \models \Box_1\psi$ and take w such that $R_{a_k} w_k w$. Note that $k < n$ because $n - k \geq \mathsf{md}\,\Box_1\psi > 0$, and thus F_{k+1} is defined and contains an element $\alpha\mathbf{R}(w_{k+1}, a_{k+1})$ such that $a_{k+1} = a_k$, $R_{a_k} w_k w_{k+1}$ (and therefore $R^f_{a_k} w_k w_{k+1}$) and $(w_{k+1}, a_{k+1}) \sim_{\phi_0} (w, a_k)$. We have that $\mathfrak{M}^f, w_{k+1}, a_{k+1} \models \psi$ and, since $n-(k+1) = n-k-1 \geq \mathsf{md}(\Box_1\psi) - 1 = \mathsf{md}\,\psi$, induction hypothesis gives us that $\mathfrak{M}, w_{k+1}, a_{k+1} \models \psi$. By the $\sim_{\phi_0}$ relation, this means that $\mathfrak{M}, w, a_k \models \psi$, and we have thus proven that $\mathfrak{M}, w_k, a_k \models \Box_1\psi$.

Conversely, suppose $\mathfrak{M}, w_k, a_k \models \Box_1\psi$ and $R^f_{a_k} w_k w$. We have that $R_{a_k} w_k w$ and thus $\mathfrak{M}, w, a_k \models \psi$. Since $\alpha\mathbf{R}(w, a_k) \in F$ and its length is $k+1$, and since $n-(k+1) \geq \mathsf{md}\,\psi$, induction hypothesis applies and we have that $\mathfrak{M}^f, w, a_k \models \psi$. This entails $\mathfrak{M}^f, w_k, a_k \models \Box_1\psi$.

The case $\phi = \Box_2\psi$ is completely analogous. □

Proof of Prop. 4.3.

(T1) to (T6) are proven in Prop. 3 of [7] and Lemma 2 of [3].

(T7)	$\vdash @_n m \to @_i K @_n m$.	
	$\vdash @_n m \to K @_n m$	(Rigid$_=$)
	$\vdash @_i @_n m \to @_i K @_n m$	(K$_@$+Nec$_@$)
	$\vdash @_n m \to @_i K @_n m$	(T1)

The derivation of $\vdash @_n \neg m \to @_i K @_n \neg m$ is identical but using (Rigid$_{\neq}$ + Selfdual) in the first step.

(T8)	$\vdash @_n \hat{F} m \wedge @_m \phi \to @_n \hat{F} \phi$.	
	$\vdash @_n \hat{F} m \wedge @_m \phi \to @_n \hat{F} m \wedge F @_m \phi$	(Back)
	$\vdash @_n \hat{F} m \wedge @_m \phi \to @_n \hat{F} m \wedge @_n F @_m \phi$	(Nec$_@$+K$_@$+T1)
	$\vdash @_n \hat{F} m \wedge @_m \phi \to @_n \hat{F}(m \wedge @_m \phi)$	(by modal reasoning: $\Box A \wedge \Diamond B \to \Diamond(A \wedge B)$)
	$\vdash @_n \hat{F} m \wedge @_m \phi \to @_n \hat{F} \phi$	(by T2: $\vdash m \wedge @_m \phi \to \phi$)

(T9)	$\vdash @_n F\psi \wedge @_n \hat{F} m \to @_m \psi$.	
	$\vdash @_n F\psi \wedge @_n \hat{F} m \to @_n \hat{F}(m \wedge \psi)$	(modal reasoning: $\Box A \wedge \Diamond B \to \Diamond(A \wedge B)$)
	$\vdash m \wedge \psi \to @_m \psi$	(T2)
	$\vdash @_n F\psi \wedge @_n \hat{F} m \to @_n \hat{F} @_m \psi$	(two above lines)
	$\vdash \hat{F} @_m \psi \to @_m \psi$	(dual of Back)
	$\vdash @_n F\psi \wedge @_n \hat{F} m \to @_m \psi$	(two above lines plus (T1))

Before showing (R1), let us show this rule:

(Name')	If $\vdash \phi \to @_m\psi$ and m is fresh, then $\vdash \phi \to \psi$.	
	$\vdash \phi \to @_m\psi$	(Premise)
	$\vdash @_m\phi \to @_m@_m\psi$	(Nec$_@$+K$_@$)
	$\vdash @_m\phi \to @_m\psi$	(Agree)
	$\vdash @_m(\phi \to \psi)$	(T5)
	$\vdash \phi \to \psi$	(Name)

With this:

(R1)	If $\vdash @_n\hat{F}m \wedge @_m\phi \to \psi$, then $\vdash @_n\hat{F}\phi \to \psi$, with $m \neq n$ fresh in ϕ and ψ.	
	$\vdash @_n\hat{F}m \wedge @_m\phi \to \psi$	(Premise)
	$\vdash @_i@_n\hat{F}m \wedge @_i@_m\phi \to @_i\psi$	(Nec$_@$+K$_@$, i fresh)
	$\vdash @_n\hat{F}m \wedge @_m\phi \to @_i\psi$	(T1)
	$\vdash @_n\hat{F}m \wedge @_m\phi \to @_m@_i\psi$	(Nec$_@$+K$_@$+T1)
	$\vdash @_n\hat{F}m \to @_m(\phi \to @_i\psi)$	(T5)
	$\vdash @_nF(\phi \to @_i\psi)$	(BG)
	$\vdash @_n\hat{F}\phi \to @_n\hat{F}@_i\psi$	($\Box(A \to B) \to (\Diamond A \to \Diamond B)$)
	$\vdash @_n\hat{F}\phi \to @_n@_i\psi$	(Back)
	$\vdash @_n\hat{F}\phi \to @_i\psi$	(T1)
	$\vdash @_n\hat{F}\phi \to \psi$	(Name')

Proof sketch of Thm. 4.13. Like Thm. 3.2, this amounts to showing that, given a model satisfying a formula ϕ_0, there is a finite submodel satisfying it.

We define nom_{ϕ_0} to be the (finite) set of nominal variables occuring in ϕ_0, we define $\mathbf{R}, \mathbf{S}$ as in Thm. 3.2 and, for $n \in \mathsf{nom}_{\phi_0}$, we let $(w, a)\mathbf{A}_n(w', a')$ iff $w = w'$ and $a' = \underline{n}$. Given a formula ϕ, we let mod ϕ be the total number of K, F and $@_n$ modalities occurring in ϕ and we let $N =$ mod ϕ_0. We construct a finite set F of chains of length at most N, with the property that, for each relation $\mathbf{T} \in \{\mathbf{R}, \mathbf{S}, \mathbf{A}_n : n \in \mathsf{nom}_{\phi_0}\}$, and each $\alpha \in F$ of length less than N, at least one $\mathbf{T}$-successor of α per equivalence class occurs in F.

Then we consider $\mathfrak{M}^f$ to the the corresponding restriction of $\mathfrak{M}$ to F and we prove that, given a chain α of length $k \leq N$ and a subformula ψ of ψ_0 with mod $\psi \leq N-k$, it is the case that $\mathfrak{M}, \mathsf{last}\,\alpha \models \psi$ iff $\mathfrak{M}^f, \mathsf{last}\,\alpha \models \psi$. This is almost identical to the proof of Lemma 3.3 with the addition of a straightforward induction step for the case $\psi = @_n\theta$. This finishes the proof. □

Proof of Prop. 5.6. First we note that if a formula has no free variables, the assignment g does not play a role in the semantics (and thus $\mathfrak{M}, \Gamma, [n], g \models \psi$ iff $\mathfrak{M}, \Gamma, [n], g' \models \psi$ for any g, g') and, with this in mind, we first prove:

If ψ is a sentence, then $\mathfrak{M}, \Gamma, [n], g \models \psi$ iff $@_n\psi \in \Gamma$. (*)

This suffices to prove our result: let $x_1, ..., x_k$ be all the free variables of ϕ. Then $\mathfrak{M}, \Gamma, [n], g \models \phi$ if and only if (by Lemma 5.5, noting that $g(x_i) = [n_i] = \underline{n}_i$)

$$\mathfrak{M}, \Gamma, [n], g \models \phi[x_i/n_i]_{i=1}^k,$$

if and only if (by the result we just proved, noting that $\phi[x_i/n_i]_{i=1}^k$ has no free variables) $@_n\phi[x_i/n_i]_{i=1}^k \in \Gamma$.

We prove (*) by induction on the length of ψ. It is exactly like the proof of Lemma 4.11, with one extra induction step:
$@_n{\downarrow}x.\psi \in \Gamma$ if and only if (by the (DA) axiom) $@_n\psi[x/n] \in \Gamma$, if and only if (by induction hypothesis, since $\psi[x/n]$ has no free variables) $\mathfrak{M}, \Gamma, [n], g \models \psi[x/n]$, if and only if (because the choice of g does not affect the truth value of a sentence) $\mathfrak{M}, \Gamma, [n], g_n^x \models \psi[x/n]$, if and only if (by Lemma 5.5) $\mathfrak{M}, \Gamma, [n], g_n^x \models \psi$, which is the same as $\mathfrak{M}, \Gamma, [n], g \models {\downarrow}x.\psi$.

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