On Kripke Completeness of Some Modal Predicate Logics with the Density Axiom

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Abstract

We prove completeness for some normal modal predicate logics in the standard Kripke semantics with expanding domains. We consider quantified versions of propositional logics with the axiom of density plus some others (transitivity, confluence).
The method of proof modifies the technique developed for other cases (without density) by S. Ghilardi, G. Corsi and D. Skvorstov; but now we arrange the whole construction in a game-theoretic style.

Keywords: modal predicate logic, Kripke semantics, Kripke completeness, canonical model, model construction games, density axiom.

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1 Modal logics and Kripke frames

Let us recall some basic definitions and notation; most of them are the same as in the book [3].

Atomic formulas are constructed from predicate letters P_k^n (countably many for each arity $n \geq 0$) and a countable set of individual variables Var, without constants and function letters. Also we do not use equality. *Modal (predicate) formulas* are obtained from atomic formulas by applying classical propositional connectives ($\supset, \bot$), the quantifier $\forall$ and the modal operator $\Box$. All other connectives (and $\exists$) are derived.

In *modal propositional formulas* only the proposition letters (P_k^0) are used as atoms.

A *modal propositional logic* is a set of modal propositional formulas containing classical propositional tautologies, the axiom of **K** ($\Box(p \supset q) \supset (\Box p \supset \Box q)$, where p, q are proposition letters) and closed under the basic inference rules: Modus Ponens, $\Box$-introduction, and (propositional) Substitution.

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As usual $\mathbf{K}$ denotes the minimal modal propositional logic, $\Lambda + A$ is the smallest logic containing a logic Λ and a formula A, and $\mathbf{K4} := \mathbf{K}{+}\Box\mathbf{p} \supset \Box\Box\mathbf{p}$.

Recall that Kripke semantics for propositional modal logics is given by *(propositional) Kripke frames* of the form (W, R), where $W \neq \varnothing$, $R \subseteq W \times W$. The set of all propositional formulas valid in a frame F (the *modal logic of* F) is denoted by $\mathbf{ML}(\mathbf{F})$. The class of all frames validating a propositional logic Λ (Λ-*frames*) is denoted by $\mathbf{V}(\Lambda)$.

A *p-morphism* from (W, R) onto (W', R') is a surjective map $f : W \longrightarrow W'$ such that for any $x \in W$ $f[R(x)] = R'(f(x))$. In this case $\mathbf{ML}(W, R) \subseteq \mathbf{ML}(W', R')$ (the *p-morphism lemma*).

A *cone in* $F = (W, R)$ *with root* u (denoted by $F{\uparrow}u$) is the restriction of F to the smallest subset V containing u and such that $R(V) \subseteq V$; obviously, $V = R(u) \cup \{u\}$ if R is transitive. If $F = F{\uparrow}u$, F itself is called *rooted* (or a *cone*). So a transitive frame (W, R) is rooted with root u if $W = R(u)$, or equivalently, if it has a first cluster.

A *modal predicate logic* is a set of modal predicate formulas containing classical predicate axioms, the axiom of $\mathbf{K}$ and closed under Modus Ponens, Generalization, $\Box$-introduction, and (predicate) Substitution.

$\mathbf{Q}\Lambda$ denotes the smallest predicate logic containing the propositional logic Λ (*the predicate version* of Λ).

For predicate formulas we use the standard Kripke semantics. Recall that a *predicate Kripke frame* over a propositional Kripke frame $F = (W, R)$ is a pair $\mathbf{F} = (\mathbf{F}, \mathbf{D})$, in which $D = (D_u)_{u \in W}$, $D_u \neq \varnothing$ and such that $D_u \subseteq D_v$ whenever uRv.

For a class of propositional frames $\mathcal{C}$, the class of all predicate frames (F, D) with $F \in \mathcal{C}$ is denoted by $\mathcal{KC}$.

A *valuation* ξ in $\mathbf{F}$ is a function sending every predicate letter P^n_k to a family of n-ary relations on the domains:

$$\xi(P^n_k) = (\xi_u(P^n_k))_{u \in W},$$

where $\xi_u(P^n_k) \subseteq D^n_u$ for $n = 0$ and $\xi_u(P^0_k) \in \{0, 1\}$.

The pair $M = (\mathbf{F}, \xi)$ is a *Kripke model* over $\mathbf{F}$. The definition of truth in a Kripke model is standard. So at every point $u \in W$ we evaluate *modal* D_u-*sentences*, i.e., modal formulas, in which all parameters (free variables) are replaced with elements of D_u; $M, u \vDash A$ means that A is true at u in M. Then

$$\begin{aligned}
&M, u \vDash P^n_k(a_1, \ldots, a_n) \text{ iff } (a_1, \ldots, a_n) \in \xi_u(P^n_k),\\
&M, u \vDash P^0_k \text{ iff } \xi_u(P^0_k) = 1,\\
&M, u \vDash A \supset B \text{ iff } (M, u \nvDash A \text{ or } M, u \vDash B),\\
&M, u \nvDash \bot,\\
&M, u \vDash \forall x A(x) \text{ iff } \forall a \in D_u\ M, u \vDash A(a),\\
&M, u \vDash \Box A \text{ iff } \forall v \in R(u)\ M, v \vDash A.
\end{aligned}$$

A modal formula $A(x_1, \ldots, x_n)$ is called *true in* M (in symbols, $M \vDash A(x_1, \ldots, x_n)$) if $M, u \vDash A(\mathbf{a})$ for every $u \in W$ and $\mathbf{a} \in D^n_u$.

A modal formula A is *valid* in a frame $\mathbf{F}$ (in symbols, $\mathbf{F} \vDash \mathbf{A}$) if it is true in every Kripke model over $\mathbf{F}$. $\mathbf{ML(F)} := \{\mathbf{A} \mid \mathbf{F} \vDash \mathbf{A}\}$ is the *modal logic of* $\mathbf{F}$.

The *modal logic of a class of frames* $\mathcal{C}$ (or the logic *determined by* $\mathcal{C}$) is $\mathbf{ML}(\mathcal{C}) := \bigcap\{\mathbf{ML(F)} \mid \mathbf{F} \in \mathcal{C}\}$. Logics of this form are called *Kripke complete.*

A modal predicate logic L is *strongly Kripke complete* if every L-consistent theory (a set of sentences) is satisfied at a point of some Kripke model over a frame validating L.

Similar definitions are given for modal propositional logics. Also recall that a modal propositional logic *has the finite model property (fmp)* if it is determined by some class of finite frames.

From the definitions it follows that for a predicate frame (F, D) and a propositional formula A,

$$(F, D) \vDash A \text{ iff } F \vDash A.$$

So for a propositional logic Λ and a predicate frame $\mathbf{F}$

$$\mathbf{F} \vDash \mathbf{Q}\Lambda \text{ iff } \mathbf{F} \in \mathcal{K}\mathbf{V}(\Lambda).$$

2 Completeness and incompleteness in modal predicate logic

In modal predicate logic there are too many examples of incompleteness, and proofs of completeness can be rather nontrivial. For instance, for a propositional modal logic $\Lambda \supseteq \mathbf{S4}$, $\mathbf{Q}\Lambda$ is complete only if $\mathbf{S5} \subseteq \Lambda$ or $\Lambda \subseteq \mathbf{S4.3}$ (cf. [5]). Still some logics $\mathbf{Q}\Lambda$ are complete, in particular, for the well-known modal logics $\Lambda = \mathbf{K}$, $\mathbf{K4}$, $\mathbf{S4}$, $\mathbf{S5}$, $\mathbf{S4.2}$, $\mathbf{S4.3}$ (cf. [3], theorems 6.1.29, 6.6.7, 6.7.12). These results were obtained by different authors — S. Kripke, D. Gabbay, S. Ghilardi, G. Corsi and others.

In this paper we are mainly interested in the logic $\mathbf{K4}Ad := \mathbf{K4}+Ad$, where

$$Ad := \Box\Box p \supset \Box p$$

is the axiom of density; $(W, R) \vDash Ad$ iff R is dense, i.e., $R \subseteq R \circ R$.

An extension of $\mathbf{K4}Ad$ is $\mathbf{D4.3}Ad$ obtained by adding the axiom of non-branching (.3) and seriality ($\Diamond\top$). It is well-known that $\mathbf{D4.3}Ad = \mathbf{ML}(\mathbb{Q}, <)$, where $\mathbb{Q}$ denotes the set of rationals. Moreover, completeness transfers to the predicate version [1]:

$$\mathbf{Q(D4.3Ad)} = \mathbf{ML}(\mathcal{K}(\mathbb{Q}, <)).$$

3 Trees and unravelling

A *tree* is a frame (W, R) with a root u_0 such that $R^{-1}(u_0) = \varnothing$ and $R^{-1}(x)$ is a singleton for any $x \neq u_0$. A *transitive tree* is a transitive closure of a tree, so it is a strictly ordered set $(W, <)$ with the least element such that every subset $\{y \mid y < x\}$ is linearly ordered and finite.

Lemma 3.1 *Every rooted transitive frame is a p-morphic image of a transitive tree.*

A well-known proof is by unravelling: for a rooted frame $F = (W, R)$ with root u we construct a tree $F^\sharp = (W^\sharp, <)$, where $W^\sharp$ is the set of all finite paths from u to points of W (i.e., finite sequences $x_0 x_1 \dots x_n$ such that $x_0 = u$ and $x_i R x_{i+1}$ for any $i < n$), and $\alpha < \beta$ iff β prolongs α. The required p-morphism sends every path to its last point.

Hence we have

Proposition 3.2 K4 *is determined by the class of all (at most) countable trees.*

This follows from lemma 3.1, the p-morphism lemma and the fmp of **K4**; note that unravelling of a finite frame is finite or countable.

Definition 3.3 Let $(W, <)$ be a tree, and consider a frame $(W, <')$, in which $<'$ is obtained from $<$ by making some points reflexive. Then $(W, <')$ is called a *semireflexive tree*.

One can easily check that a semireflexive tree $(W, <')$ validates Ad iff its irreflexive points can have only reflexive immediate successors. [3] Such a semireflexive tree is called *dense*.

Proposition 3.4 K4Ad *is determined by the class of all (at most) countable dense semireflexive trees.*

Proof. A standard filtration argument shows that **K4**Ad has the fmp, so it is determined by finite rooted **K4**Ad-frames (cf. [6]). Finite **K4**-frames consist of clusters, some of which can be degenerate (i.e., irreflexive singletons), while in finite **K4**Ad-frames successors of degenerate clusters are non-degenerate.

Now let us unravel a finite **K4**Ad-frame $F = (W, R)$ with root u more carefully than in lemma 3.1. Call a path $x_0 \dots x_n$ *long* if

$$\forall i < n \, \forall y \in F (x_i R y R x_{i+1} \Rightarrow y R x_i \vee x_{i+1} R y).$$

Consider the set W_1 of all long paths from u to points in F and take the restriction $F_1 := F^\sharp | W_1$. This frame is a tree, and the map f sending a path to its last point is still a p-morphism $F_1 \longrightarrow F$. This is because every two R-related points can be connected by a long path.

Now we extend the relation in F_1 by making reflexive every point a such that $f(a)$ is reflexive. We obtain a semireflexive tree F_2 and again f is a p-morphism $F_2 \longrightarrow F$.

F_2 is a dense semireflexive tree. In fact, if in F_2 we have an irreflexive a and its successor b, then a is a long path in F ending at an irreflexive point $f(a)$, and the cluster of $f(b)$ is a successor of $f(a)$. So $f(b)$ is reflexive, and thus b is reflexive in F_2. ■

To obtain a class of irreflexive transitive frames determining **K4**Ad we can use Segerberg's bulldozing method (cf. [6]). Viz., given a dense semireflexive tree F_2, we can replace each its reflexive point with a strict dense linear order

[3] Henceforth by a 'successor' we mean an 'immediate successor'.

without the last element (e.g., the non-negative rationals $\mathbf{Q}_+$). Then we obtain $\mathbf{K4}Ad$-frame F_3, and there is a p-morphism from F_3 sending every irreflexive point from F_2 to itself and every copy of $\mathbf{Q}_+$ to the corresponding reflexive point in F_2. We call such a frame F_3 a *sprouting tree*. So we have

Proposition 3.5 $\mathbf{K4}Ad$ *is determined by the class of sprouting trees.*

Remark 3.6 It is not clear if predicate frames over sprouting trees determine the predicate logic **QK4Ad**. The completeness proof proposed below yields more complicated frames.

4 Completeness of QK4Ad

To prove completeness for **QK4Ad** we use a method originating from G. Corsi's paper [1] and further developed by D. Skvortsov [8]; also cf. [3], sec. 6.4.

The main idea is to extract an appropriate submodel from a canonical model of a given logic L and to make a sort of unravelling which leads to a frame validating L. More exactly, this frame is obtained as a direct limit of a sequence of finite trees. This sequence can be constructed by induction, or equivalently, by playing a game.

First we recall some definitions from [3], sections 6.1, 6.3, with little changes.

We fix a denumerable set of extra constants S^*. A subset $S' \subseteq S^*$ is called *small* if the complement $(S^* - S')$ is infinite.

Definition 4.1 For a modal predicate logic L, an *L-place* is a maximal L-consistent theory (i.e, a set of sentences) Γ in the basic language with extra constants from S^* with the *Henkin property*: for any formula $\varphi(x)$ with at most one parameter x there exists a constant c such that $(\exists x\varphi(x) \supset \varphi(c)) \in \Gamma$. An L-place is *small* if the set of its constants is small.

It is well-known that every L-consistent theory with a small set of constants can be extended to a small L-place ([3], Lemma 6.1.9).

Definition 4.2 The *canonical model* VM_L is (VP_L, R_L, D_L, ξ_L), where

- VP_L is the set of all small L-places,
- $\Gamma R_L \Delta$ iff $\Box^-\Gamma \subseteq \Delta$, where $\Box^-\Gamma := \{A \mid \Box A \in \Gamma\}$,
- $(D_L)_\Gamma$ (also denoted by D_Γ) is the set of constants occurring in Γ,
- $(\xi_L)_\Gamma(P^m_k) := \{\mathbf{c} \in (\mathbf{D}_\Gamma)^\mathbf{m} \mid \mathbf{P^m_k(c)} \in \Gamma\}$
  for $m > 0$, and
  $(\xi_L)_\Gamma(P^0_k) := 1$ iff $P^0_k \in \Gamma$.

Note that $\Box^-\Gamma \subseteq \Delta$ implies $D_\Gamma \subseteq D_\Delta$; this holds, since $\Box(P^1_1(c) \supset P^1_1(c)) \in \Gamma$ for any $c \in D_\Gamma$, so $(P^1_1(c) \supset P^1_1(c)) \in \Delta$.

Then for any D_Γ-sentence A

$$VM_L, \Gamma \vDash A \text{ iff } A \in \Gamma$$

(the *Canonical model theorem*).

Note that for arbitrary L-places an analogue of this theorem does not hold, but we still need them for further considerations. So put $VM_L^+ := (VP_L^+, R_L, D_L, \xi_L)$, where VP_L^+ is the set of all L-places, and R_L, D_L, ξ_L are the same as above.[4] This VM_L^+ is actually a submodel of a canonical model for some larger set of extra constants.

Definition 4.3 Let L be a predicate logic, $F = (W, R)$ a propositional frame. An *L-network over F* is a monotonic map from F to (VP_L^+, R_L), i.e. a map $h : W \longrightarrow VP_L^+$ such that for any $u, v \in W$

$$uRv \Rightarrow h(u) R_L h(v).$$

The frame F is denoted by $dom(h)$ and called the *domain* of h. An L-network h is *small* if every $h(u)$ is small and *transitive* if $dom(h)$ is transitive.

With every L-network h we associate a predicate Kripke frame $\mathbf{F(h)} := (\mathbf{dom(h)}, \mathbf{D})$, where $D_u = (D_L)_{h(u)}$ for $u \in W$, and a Kripke model $M(h) := (\mathbf{F(h)}, \xi(\mathbf{h}))$, where

$$\xi(h)_u(P_k^m) := \{\mathbf{c} \in \mathbf{D_u^m} \mid \mathbf{P_k^m(c)} \in \mathbf{h(u)}\}$$

for $m > 0$ and

$$\xi(h)_u(P_k^0) := 1 \text{ iff } P_k^0 \in h(u).$$

We define the partial order on networks.

$$h \leq h' := dom(h) \text{ is a subframe of } dom(h') \text{ and } \forall u \in dom(h)\ h(u) \subseteq h'(u).$$

Definition 4.4 A *defect* in a network h over a frame (W, R) is a pair (u, A) such that $u \in W$ and $\Diamond A \in h(u)$. A defect (u, A) is *eliminated* in h if there exists $v \in R(u)$ such that $A \in h(v)$.

Henceforth in this section we assume that L contains **QK4**, so L-frames are transitive.

We will call a transitive L-network h *finite* if it is small and $dom(h)$ is a finite transitive tree.

Lemma 4.5 *(On elimination of defects) Let h be a finite L-network with a defect (u, A). Then there is a finite L-network $h' \geq h$ eliminating this defect.*

Proof. If h eliminates (u, A), take $h' = h$. Otherwise extend $dom(h)$ by adding a new successor v of u (such that v has no successors). Since $\Diamond A \in h(u)$, by the properties of the canonical model VM_L, there exists a small L-place Γ such that $A \in \Gamma$ and $h(u) R_L \Gamma$. So we can put $h'(v) := \Gamma$. ■

If Γ, Δ are L-places, $\Gamma \restriction \Delta$ denotes the restriction of Γ to the language of Δ.

Lemma 4.6 *(Skvortsov's extension lemma)*

[4] More exactly, R_L is extended to $VP_L^+ \times VP_L^+$, etc.

(1) *Let Γ, Δ be L-places, $\Gamma_0 = \Gamma \upharpoonright \Delta$ and suppose that $\Box^-\Gamma_0 \subseteq \Delta$. Then there exists an L-place $\Delta' \supseteq \Delta$ such that $\Gamma R_L \Delta'$. Δ' can be chosen small if Γ, Δ are small.*

(2) *Let h be a finite L-network over a transitive tree F with root v, and let Γ be an L-place, $\Gamma_0 = \Gamma \upharpoonright h(v)$, and suppose that $\Box^-\Gamma_0 \subseteq h(v)$. Let F' be the transitive tree obtained by adding a root u below F. Then there exists a finite L-network $h' \geq h$ over F' such that $\Gamma = h'(u)$.*

Proof. This is a reformulation of Lemma 6.4.28 from [3], and the proof follows the same lines.

(1) The assumptions imply that the theory $\Box^-\Gamma \cup \Delta$ is consistent (see the details in [3]); so it extends to an L-place Δ'.

(2) We can argue by induction on the cardinality of F. By (1) there exists an L-place $\Delta' \supseteq h(v)$ such that $\Gamma R_L \Delta'$. If v has no successors (i.e., F is a singleton), we are done: take h' defined on the chain $\{u, v\}$ such that $h'(u) = \Gamma,\ h'(v) = \Delta'$.

Suppose v has successors $v_1, \dots v_n$, $F_i = F{\uparrow}v_i$. h_i is the restriction of h to F_i. Since we can rename the constants from $D_{\Delta'} - D_{h(v)}$, we may assume that they do not occur in any $h(v_i)$; thus $h(v) = \Delta' \upharpoonright h(v_i)$, and $\Box^-h(v) \subseteq h(v_i)$. Now by IH there exists $h_i' \geq h_i$ defined on the tree F_i with the added bottom element v such that $h_i'(v) = \Delta'$. Then we define the following network h' over F':

$$h'(u) = \Gamma,\ h'(v) = \Delta',\ h'|F_i = h_i'.$$

■

Now we assume that L contains **QK4Ad**.

Lemma 4.7 *(On inserts) Let h be a finite L-network, and let v be a successor of u in dom(h). Then there exists a finite L-network $h' > h$ such that v is not a successor of u in dom(h').*

Proof. Suppose $h(u) = \Gamma,\ h(v) = \Delta$, and let $\Delta_0 = \Delta \upharpoonright \Gamma$. It follows that the set $\Gamma' := \Box^-\Gamma \cup \{\Diamond A \mid A \in \Delta_0\}$ is L-consistent. In fact, otherwise there exist $B \in \Box^-\Gamma$ and $A \in \Delta_0$ such that $\{B, \Diamond A\}$ is inconsistent (since the sets $\Box^-\Gamma$, Δ_0 are closed under conjunction and $\Diamond(A_1 \wedge A_2)$ implies $\Diamond A_1 \wedge \Diamond A_2$). So $\vdash_L B \supset \neg\Diamond A$, or equivalently, $\vdash_L B \supset \Box\neg A$. Hence by the monotonicity of $\Box$, $\vdash_L \Box B \supset \Box\Box\neg A$; thus $\vdash_L \Box B \supset \Box\neg A$ by *Ad*. Since $\Box B \in \Gamma$ and A is in the language of Γ, this implies $\Box\neg A \in \Gamma$. Since $\Gamma R_L \Delta$, it follows that $\neg A \in \Delta$, which is a contradiction.

Then Γ' can be extended to an L-place Θ (with new unused constants). Let $\Theta_0 = \Theta \upharpoonright \Delta$ ($= \Theta \upharpoonright \Delta_0$, since new constants of Θ do not occur in Δ).

It follows that $\Box^-\Theta_0 \subseteq \Delta_0$. In fact, $\neg A \in \Delta_0$ implies $\Diamond\neg A \in \Gamma' \subseteq \Theta$, so $\Box A \notin \Theta_0,\ A \notin \Box^-\Theta_0$.

Consider the tree F' obtained from $F = dom(h)$ by adding a new point z between u and v. By Lemma 4.6 there exists a finite network h^1 over $F'{\uparrow}z$ such that $h^1(z) = \Theta$ and $h^1 \geq h$ on $F{\uparrow}v$. Now we can define h' on F', which coincides with h^1 on $F'\uparrow z$ and coincides with h at all other points. This is a network, since $\Box^-\Gamma \subseteq \Theta$, i.e., $h'(u)R_L h'(z)$. ■

Definition 4.8 Let Γ_0 be a small L-place. The *selective game* $SG_L(\Gamma_0)$ is played by two players, $\forall$ (the first) and $\exists$ (the second). A position after the n-th turn is a finite network h_n over a transitive tree $F_n = (W_n, R_n)$. We also assume[5] that $W_n \subseteq \omega$.

At the initial position F_0 is an irreflexive singleton 0 and $h_0(0) = \Gamma_0$.

For the $(n+1)$-th move the player $\forall$ has two options.

1. Selecting a *defect*, i.e., a pair (u, A) such that $u \in W_n$ and $\Diamond A \in h_n(u)$.

2. A query for an *insert*, i.e., a pair (u, v) such that $uR_n v$ and there are no points between u and v.

The player $\exists$ should respond with a network $h_{n+1} \geq h_n$ such that

1. If the move of $\forall$ was a defect (u, A), then there exists v such that $uR_{n+1}v$ and $A \in h_{n+1}(v)$.

2. If the move of $\forall$ was a query for an insert (u, v), then then there exists w such that $uR_{n+1}wR_{n+1}v$.

The player $\exists$ wins if the play continues infinitely or $\forall$ cannot make his move.

Note that $\forall$ cannot make the $(n+1)$th move in the only case when $n = 0$ and h_0 has no defects. This happens if Γ_0 is an endpoint in VM_L, i.e., $R_L(\Gamma_0) = \varnothing$.

Every infinite play of the game generates a sequence of networks $h_0 \leq h_1 \leq \ldots$ Then we define the resulting network h_ω, with $dom(h_\omega) = F_\omega := (W_\omega, R_\omega)$, $W_\omega := \bigcup_n W_n$, $R_\omega := \bigcup_n R_n$, $h_\omega(u) := \bigcup_{n \geq m} h_n(u)$ for $u \in W_m$. One can easily check that this is really a network (not necessarily finite or small).

Lemma 4.9 *$\exists$ has a winning strategy in $SG_L(\Gamma_0)$.*

Proof. If $\forall$ cannot make the first move, there is nothing to prove. If the $(n+1)$-th move of $\forall$ is a defect, $\exists$ can eliminate it by her next move according to Lemma 4.5. If the move of $\forall$ is a query for an insert, $\exists$ can respond according to Lemma 4.7. ■

Lemma 4.10 *If Γ_0 is not an endpoint in VM_L, then there exists a play of $SG_L(\Gamma_0)$ generating a sequence of networks such that $F_\omega \vDash$ **K4**Ad and for any u, for any $D_{h_\omega(u)}$-sentence A*

$$M(h_\omega), u \vDash A \text{ iff } A \in h_\omega(u).$$

Proof. A *dense tree* is a rooted strictly ordered set $(W, \prec)$, in which every subset $\{u \mid u \prec w\}$ is a dense chain. Let us construct an infinite play such that F_ω is a dense tree.

At the initial position $F_0 = (0, \varnothing)$ and $h_0(0) = \Gamma_0$.

Let us choose the further strategy for $\forall$ as follows. Fix an enumeration of the countable set $\omega \times \omega$, and an enumeration of $\omega \times \Phi$, where Φ is the set of all modal sentences with constants from S^*. An odd move $(n+1)$ of $\forall$ chooses

[5] This technical detail is needed for the further proofs.

the first new pair (u, A), which is a defect in h_n. An even move $(n+1)$ of $\forall$ chooses the first new pair $(u, v) \in \omega \times \omega$, which is a query for an insert in h_n.

By lemma 4.9 there is a winning strategy for $\exists$. For the resulting network we have

$$M(h_\omega), u \vDash A \text{ iff } A \in h_\omega(u).$$

This is checked by induction. The atomic case holds by the definition of $\xi(h)$; the cases of propositional connectives and quantifiers hold by the properties of L-places.

Let us consider the case $A = \Box B$. Suppose $M(h_\omega), u \nvDash A$; then $M(h_\omega), v \nvDash B$ for some $v \in R_\omega(u)$. Since A is in the language of $h_\omega(u)$ and h_ω is a network, we have $h_\omega(u) R_L h_\omega(v)$, so A (and B) is also in the language of $h_\omega(v)$. By IH it follows that $B \notin h_\omega(v)$; hence $A = \Box B \notin h_\omega(u)$ by the definition of R_L.

The other way round, suppose $A \notin h_\omega(u)$; then $\Diamond\neg B \in h_\omega(u)$, so $\Diamond\neg B \in h_n(u)$ (i.e., $(u, \Diamond\neg B)$ is a defect in h_n) for some finite n. Choose the minimal such n; so $(u, \Diamond\neg B)$ is a defect in h_m for all $m > n$. Since the defects subsequently appear as odd moves of $\forall$, there exists m such that $(u, \Diamond\neg B)$ is his $(m+1)$-th move. By the response of $\exists$, we have $\neg B \in h_{m+1}(v)$ for some $v \in R_{m+1}(u)$. Hence $\neg B \in h_\omega(v)$, $v \in R_\omega(u)$. By IH, we have $M(h_\omega), v \nvDash B$. Thus $M(h_\omega), u \nvDash A$.

To check the density for F_ω, we can use even moves. In fact, if $uR_\omega v$, there exists n such that $uR_n v$. If v is a successor of u in R_n, the pair (u, v) must show up as a later even move of $\forall$. By the response of $\exists$ we obtain w such that $uR_\omega wR_\omega v$. ■

Definition 4.11 A modal predicate logic L is strongly Kripke complete if every L-consistent set of sentences is satisfiable at some point of a Kripke model over a frame validating L.

Theorem 4.12 **QK4Ad** *is strongly Kripke complete.*

Proof. Every L-consistent theory Γ without constants can be extended to a small L-place Γ_0. If Γ_0 is an endpoint in VM_L, then for any A in its language

$$VM_L, \Gamma_0 \vDash A \text{ iff } A \in \Gamma_0$$

by the canonical model theorem. Since Γ_0 is an endpoint, the truth at this point reduces to the truth in a model over an irreflexive singleton.

In all other cases we can apply lemma 4.10. So there exists a model $M(h_\omega)$ such that $M(h_\omega), u_0 \vDash \Gamma_0$ and $F_\omega \vDash \mathbf{K4}Ad$. Hence $\mathbf{F}(\mathbf{h}_\omega) \vDash \mathbf{L}$. ■

Theorem 4.13 *If* Π *is a set of closed (i.e., constructed only from* $\bot$, $\Box$ *and* $\supset$*) propositional formulas, then* $\mathbf{QK4Ad} + \Pi$ *is strongly Kripke complete.*

Proof. By the same argument as in the previous theorem. In this case $\Pi \subset \Gamma$ for all L-places Γ (where $L := \mathbf{QK4Ad} + \Pi$), so $M(h_\omega) \vDash \Pi$. Hence $F_\omega \vDash \Pi$, and thus $F(h_\omega) \vDash \Pi$. ■

5 Logics with n-density

Let us first notice that for the logic $\mathbf{QKAd} := \mathbf{QK}+Ad$ one can use the same method as in the previous section. Now we only need finite networks over non-transitive frames. If (W, R) is a tree, R^+ is the transitive closure of R and $R \subseteq R_1 \subseteq R^+$, then (W, R_1) is called an *almost transitive tree.* Lemmas 4.5, 4.6, 4.7 are transferred to almost transitive trees and proved by the same arguments.

The analogue of lemma 4.10 also holds for $\mathbf{QKAd}$. The same proof constructs a frame F_ω validating $\mathbf{KAd}$ (but this frame should not be called a "dense tree").

Thus we obtain

Theorem 5.1 *If* Π *is a set of closed propositional formulas, then* $\mathbf{QKAd} + \Pi$ *is strongly Kripke complete.*

Now recall the n-density axiom Ad_n generalizing Ad:

$$Ad_n := \bigwedge_{i=1}^{n} \Diamond p_i \supset \Diamond(\bigwedge_{i=1}^{n} \Diamond p_i).$$

This is a Sahlqvist formula, so for the logic $\mathbf{KAd_n} := \mathbf{K}+Ad_n$ we have

Proposition 5.2 $\mathbf{KAd_n}$ *is canonical and determined by the following first-order condition on frames:*

$$\forall x, y_1, \ldots, y_n\, (\bigwedge_{i=1}^{n} xRy_i \supset \exists z\, (xRz \wedge \bigwedge_{i=1}^{n} zRy_i)).$$

Lemma 5.3 *(On inserts) For L containing* $\mathbf{QKAd_n}$ *let h be a finite L-network over a frame (W, R) and suppose $uRv_1, \ldots, uRv_n$. Then there exists a finite L-network $h' > h$ and z such that $uR'z,\ zR'v_1, \ldots, zR'v_n$, where R' is the relation in $dom(h')$.*

Proof. The same argument as in 4.7, with slight changes.

Let $h(u) = \Gamma,\ h(v_i) = \Delta_i,\ \Delta_{i0} = \Delta_i \upharpoonright \Gamma$. Then the set

$$\Gamma' := \Box^-\Gamma \cup \bigcup_{i=1}^{n} \{\Diamond A \mid A \in \Delta_{i0}\}$$

is L-consistent.

For, otherwise there exist $B \in \Box^-\Gamma$ and $A_i \in \Delta_{i0}$ such that $\{B, \Diamond A_1, \ldots, \Diamond A_n\}$ is L-inconsistent, i.e., $\vdash_L B \supset \neg \bigwedge_i \Diamond A_i$. Hence

$$\vdash_L \Box B \supset \Box\neg \bigwedge_i \Diamond A_i;$$

thus

$$\vdash_L \Box B \supset \neg\Diamond \bigwedge_i \Diamond A_i,$$

and

$$\vdash_L \Box B \supset \neg \bigwedge_i \Diamond A_i,$$

by Ad_n. However, $\Box B \in \Gamma$, so $\neg \bigwedge_i \Diamond A_i \in \Gamma$. On the other hand, every A_i is in the language of Γ, $A_i \in \Delta_i$, and $\Gamma R_L \Delta_i$, which implies $\Diamond A_i \in \Gamma$. Hence $\bigwedge_i \Diamond A_i \in \Gamma$, which is a contradiction.

Then Γ' can be extended to an L-place Θ such that $D_\Theta - D_{\Gamma'}$ contains only new constants. So we have $\Gamma' = \Theta \upharpoonright \Delta_i$, $\Box^-\Gamma' \subseteq \Delta_i$.

Consider the tree F' obtained from $F = dom(h)$ by adding a new unique successor z of u below all the v_i. Let $F_i := F\uparrow v_i$, $h_i := h|F_i$. Since $\Box^-(\Theta \upharpoonright \Delta_i) \subseteq \Delta_i$, by Lemma 4.6 there exists a finite network $h'_i \geq h_i$ defined on F_i with the added root z such that $\Theta = h'_i(z)$. Then we can define the finite network h' over F' such that $h'(z) = \Theta$, $h'|F_i = h_i$ and $h'(x) = h(x)$ for all $x \notin R(u)$. This is a network, since $\Box^-\Gamma \subseteq \Theta$, i.e., $h'(u) = h(u) R_L h'(z)$. ∎

Now let $L = \mathbf{QKAd_n} + \Pi$, where Π is a set of closed propositional formulas.

Definition 5.4 The selective game $SG_L(\Gamma_0)$ is defined as in definition 4.8, but now a query for an insert at the $(m+1)$-th move is a tuple $(u, v_1, \ldots, v_n)$ such that $uR_m v_1, \ldots, uR_m v_n$ and there is no z with $uR_m z R_m v_i$ for all i.

In a response for this move there must be w such that

$$uR_{m+1}w,\ wR_{m+1}v_1, \ldots, wR_{m+1}v_n.$$

Now we have analogues of lemmas 4.9, 4.10.

Lemma 5.5 *$\exists$ has a winning strategy in $SG_L(\Gamma_0)$.*

Proof. By applying lemmas 4.5, 5.3. ∎

Lemma 5.6 *If Γ_0 is not an endpoint in VM_L, then there exists a play of $SG_L(\Gamma_0)$ generating a sequence of networks such that $\mathbf{F(h_\omega)} \vDash \mathbf{L}$ and for any u, for any $D_{h_\omega(u)}$-sentence A*

$$M(h_\omega), u \vDash A \textit{ iff } A \in h_\omega(u).$$

Proof. The same as for lemma 4.10, with the following change.

An odd move $(m+1)$ of $\forall$ is the first new tuple from ω^{n+1} which is an insert query in h_m. These moves guarantee the n-density for F_ω. ∎

Theorem 5.7 *If Π is a set of closed propositional formulas, the logic $\mathbf{QKAd_n} + \Pi$ is strongly Kripke complete.*

Proof. Similar to theorem 4.13. If an L-place Γ_0 is not an endpoint in the canonical model, we apply lemma 5.6 to obtain a model $M(h_\omega)$ satisfying Γ_0, with $\mathbf{F(h_\omega)} \vDash \mathbf{L}$. ∎

A similar result holds for the transitive case; note that $\mathbf{K4} + Ad_2 \vdash Ad_n$ for any n.

Theorem 5.8 *If Π is a set of closed propositional formulas, the logic $\mathbf{QK4}$+$Ad_2 + \Pi$ is strongly Kripke complete.*

6 Logics with confluence and density

Now let us consider logics containing the confluence ("Church–Rosser") axiom

$$A2 := \Diamond\Box p \supset \Box\Diamond p.$$

The semantical characterization of $A2$ is well-known:

Proposition 6.1 *The logic* $\mathbf{K2} := \mathbf{K}+A2$ *is canonical and determined by the following condition on frames:*

$$\forall x, y, z\,(xRy \wedge xRz \supset \exists u\,(yRu \wedge zRu)).$$

For completeness proofs in this section we also need transitivity. So we will consider extensions of $\mathbf{QK4.2} := \mathbf{QK4}+A2$.

Lemma 6.2 $\mathbf{K2} \vdash \Box\Diamond\top$.

Proof. On the one hand, it is clear that $\mathbf{K} \vdash \Diamond\top \supset \Diamond\Box\top$, so $\mathbf{K2} \vdash \Diamond\top \supset \Box\Diamond\top$.

On the other hand, $\mathbf{K} \vdash \Box\bot \supset \Box\Diamond\top$; hence the statement follows. ■

In this section we deal with finite networks over transitive trees and infinite networks over other frames (sums of trees).

Definition 6.3 A finite network h over a transitive tree (W, R) is called *rich* if its satisfies the following condition.

Let $u_1, \ldots, u_n$ be R-incomparable, and let v be their maximal common predecessor. Then the sets $D_{h(u_i)} - D_{h(v)}$ are disjoint.

Lemma 6.4 *Let* $\Delta, \Gamma_1, \Gamma_2$ *be* L*-places for* $L \supseteq \mathbf{QK4.2}$ *such that* $\Delta R_L \Gamma_1$, $\Delta R_L \Gamma_2$ *and* $D_{\Gamma_1} \cap D_{\Gamma_2} = D_\Delta$. *Then the set* $\Box^-\Gamma_1 \cup \Box^-\Gamma_2$ *is* L*-consistent.*

Proof. Suppose not. Since $\Box$ distributes over conjunction, then there exist $\Box B_1 \in \Gamma_1,\ B_2 \in \Gamma_2$ such that $\vdash_L \neg(B_1 \wedge B_2)$. Every B_i can be presented as $A_i(\mathbf{a_i}, \mathbf{b})$ for a list $\mathbf{a_i}$ of constants from $D_{\Gamma_i} - D_\Delta$, and a list $\mathbf{b}$ of constants from D_Δ. By assumption, $\mathbf{a_1}, \mathbf{a_2}$ are disjoint.

By predicate logic, it follows that

$$\vdash_L \forall\mathbf{x_1}\forall\mathbf{x_2}\neg(\mathbf{A_1}(\mathbf{x_1}, \mathbf{b}) \wedge \mathbf{A_2}(\mathbf{x_2}, \mathbf{b}))$$

for disjoint lists of variables $\mathbf{x_1}, \mathbf{x_2}$; hence

$$\vdash_L \neg(\exists\mathbf{x_1}\mathbf{A_1}(\mathbf{x_i}, \mathbf{b}) \wedge \exists\mathbf{x_2}\mathbf{A_2}(\mathbf{x_2}, \mathbf{b})).$$

Claim The rule $A/\Box\Diamond A$ is admissible in L.

In fact, $\vdash_L A$ implies $\vdash_L \top \supset A$, and thus $\vdash_L \Box\Diamond\top \supset \Box\Diamond A$, and finally $\vdash_L \Box\Diamond A$ by lemma 6.2.

Now by the Claim we have

$$\vdash_L \Box\Diamond\neg(\exists\mathbf{x_1}\mathbf{A_1}(\mathbf{x_i}, \mathbf{b}) \wedge \exists\mathbf{x_2}\mathbf{A_2}(\mathbf{x_2}, \mathbf{b})),$$

and so

$$\vdash_L \neg\Diamond\Box(\exists \mathbf{x_1 A_1}(\mathbf{x_i}, \mathbf{b}) \wedge \exists \mathbf{x_2 A_2}(\mathbf{x_2}, \mathbf{b})),$$

$$\neg\Diamond\Box(\exists \mathbf{x_1 A_1}(\mathbf{x_i}, \mathbf{b}) \wedge \exists \mathbf{x_2 A_2}(\mathbf{x_2}, \mathbf{b})) \in \Delta. \tag{$*$}$$

On the other hand, by confluence and transitivity we have

$$\mathbf{K4.2} \vdash \Diamond\Box p_1 \wedge \Diamond\Box p_2 \supset \Diamond\Box(p_1 \wedge p_2),$$

thus

$$\vdash_L \Diamond\Box\exists \mathbf{x_1 A_1}(\mathbf{x_1}, \mathbf{b}) \wedge \Diamond\Box\exists \mathbf{x_2 A_2}(\mathbf{x_2}, \mathbf{b}) \supset \Diamond\Box(\exists \mathbf{x_1 A_1}(\mathbf{x_1}, \mathbf{b}) \wedge \exists \mathbf{x_2 A_2}(\mathbf{x_2}, \mathbf{b})). \tag{$**$}$$

Since $\Delta R_L \Gamma_i$ and $\Box A_i(\mathbf{a_i}, \mathbf{b}) \in \mathbf{\Gamma_i}$, it also follows that $\Box\exists \mathbf{x_i A_i}(\mathbf{x_i}, \mathbf{b}) \in \mathbf{\Gamma_i}$, $\Diamond\Box\exists \mathbf{x_i A_i}(\mathbf{x_i}) \in \Delta$, so from (**) we obtain

$$\Diamond\Box(\exists \mathbf{x_1 A_1}(\mathbf{x_1}, \mathbf{b}) \wedge \exists \mathbf{x_2 A_2}(\mathbf{x_2}, \mathbf{b})) \in \Delta$$

contradicting (*). ■

Lemma 6.5 *(Cf. [3], Lemma 6.6.5) Let h be a rich finite small L-network over a nontrivial tree (W, R) for $L \supseteq$ **QK4.2**. Then there exists a small Θ such that $h(w) R_L \Theta$ for any $w \in W$.*

Proof. By induction on the cardinality of W.

If (W, R) is a two-element chain: $W = \{u, v\}$, uRv, then we can apply lemma 6.4 to $\Delta = h(u)$, $\Gamma_1 = \Gamma_2 = h(v)$ and construct $\Theta \supseteq \Box^- h(v)$.

The same argument goes through for any finite chain with the first element u and the last element v.

So for the induction step we may assume that (W, R) has maximal points $u_1, \ldots, u_n$, $n > 1$. By IH there exists Θ such that $h(u_1) R_L \Theta, \ldots, h(u_{n-1}) R_L \Theta$, and by renaming constants we may also assume that all new constants in D_Θ do not occur in $h(u_n)$. Let $v = inf(u_1, \ldots, u_n)$. Since h is rich, it follows that $D_\Theta \cap D_{h(u_n)} = D_{h(v)}$. So lemma 6.4 is applicable, which gives us a small Θ' such that $\Theta R_L \Theta'$, $h(u_n) R_L \Theta'$. It remains to note the R_L is transitive. ■

Now let us consider the logics $L :=$ **QK4.2**$+Ad$ or $L :=$ **QK4.2**$+Ad+\Diamond\top$. [6]

To define an appropriate game we need an increasing sequence $(S_n)_{n\geq 1}$ of subsets of the set of constants S^* such that S_1 and all the sets $(S_{n+1} - S_n)$ are infinite.

Definition 6.6 Let Γ_0 be an S_1-small L-place. The selective game $SG_L(\Gamma_0)$ is defined as in 4.8, with the some changes.

1. The length of the game is ω^2.
2. A position after the turn $\alpha = \omega \cdot m + n$ is a rich network h_α over a finite tree $F_\alpha = (W_\alpha, R_\alpha)$ such that $W_\alpha \subseteq \omega$ and all L-places $h_\alpha(u)$ are S_{m+1}-small.
3. Every tree $F_{\omega \cdot m}$ is an irreflexive singleton 0, $h_0(0) = \Gamma_0$.

[6] The method also works for the logic **QS4.2** (its completeness was first proved in [2]).

4. If $\alpha = \omega \cdot m + n$, the player $\forall$ has the same two options for the move $\alpha + 1$ as in definition 4.8: selecting a defect or a query for an insert in h_α. A response of $\exists$ is also described in 4.8; it yields a network $h_{\alpha+1} \geq h_\alpha$.

5. A limit move $\alpha = \omega \cdot (m+1)$ of the player $\forall$ is just waiting for the response of $\exists$. For the response $\exists$ should construct the limit network h^*_α over $F^*_\alpha := (W^*_\alpha, R^*_\alpha)$, where

$$W^*_\alpha := \bigcup_n W_{\omega\cdot m+n},\ R^*_\alpha := \bigcup_n R_{\omega\cdot m+n},\ h^*_\alpha(u) := \bigcup_{n\geq k} h_{\omega\cdot m+n}(u) \text{ for } u \in W_{\omega\cdot m+k};$$

then she should choose an S_{m+2}-small L-place Γ_α such that $h^*_\alpha(u) R_L \Gamma_\alpha$ for any $u \in W^*_\alpha$. The resulting position would be the network $h_\alpha : 0 \mapsto \Gamma_\alpha$.

6. The player $\exists$ wins if the play is of length ω^2 or if $\forall$ cannot make one of his moves.

In this game a position, at which $\forall$ cannot make the next move, may occur only at the stage 0 if $\Box\bot \in \Gamma_0$. In fact, otherwise at every non-limit stage we have $\Diamond\top \in h_\alpha(0)$ and also $\Diamond\top \in h_\alpha(u)$ for any $u \neq 0$ (since $\Box\Diamond\top \in \Gamma_0$ by lemma 6.2 and $h_\alpha(0) R_L h_\alpha(u)$); so $\forall$ can select a defect.

An ω^2-play generates a sequence of networks $h^*_\omega \leq h^*_{\omega\cdot 2} \leq \ldots$.

The resulting network h^+ is then defined as the sum $\sum_{m\in\omega} h^*_{\omega\cdot(m+1)}$. So $dom(h^+) = F^+ = (W^+, R^+) := \sum_{m\in\omega} F^*_{\omega\cdot(m+1)}$ (the ordered sum), i.e.,

$$W^+ := \bigcup_{m\geq 1} W^*_{\omega\cdot m} \times \{m\},\ (x,m) R^+ (y,l) \text{ iff } (m < l \vee m = l \ \&\ x R^*_{\omega\cdot m} y),$$

and

$$h^+(x,m) := h^*_{\omega\cdot m}(x) \text{ for } x \in W^*_{\omega\cdot m}.$$

One can easily see that h^+ is really a network. In fact, it coincides with $h^*_{\omega\cdot m}$ on each component. To show that $(x,m) R^+ (x,l)$ implies $h^+(x,m) R_L h^+(y,l)$ for $m < l$, it is sufficient to consider the case $l = m+1$. In this case we have

$$h^+(x,m) = h^*_{\omega\cdot m}(x) R_L h^*_{\omega\cdot(m+1)}(0) = h^+(0, m+1) R_L h^+(y, m+1).$$

Lemma 6.7 *$\exists$ has a winning strategy in $SG_L(\Gamma_0)$.*

Proof. For non-limit moves use lemmas 4.5, 4.7 with an extra observation that the networks can be always kept rich by choosing new constants.

For a limit move $\alpha = \omega \cdot (m+1)$ a response of $\exists$ also exists. In fact, F^*_α has the root 0, so $h^*_\alpha(0) R_L h^*_\alpha(u)$ for any $u \in W^*_\alpha,\ u \neq 0$. All these L-places $h^*_\alpha(u)$ are S_{m+1}-small.

We claim that the theory

$$\Sigma := \bigcup \{\Box^- h(u) \mid u \in W^*_\alpha,\ u \neq 0\}$$

is L-consistent.

In fact, otherwise the set

$$S := \Box^{-}h(u_1) \cup \ldots \cup \Box^{-}h(u_n)$$

is L-inconsistent for some finite n. Then there exist $\alpha = \omega \cdot m + k$ such that $u_1, \ldots, u_n \in domh^*_\alpha$. The network h^*_α is finite and rich, so by lemma 6.5 there exists Θ such that $h(u_i)R_L\Theta$ for every i. So Θ contains S, which is a contradiction.

Note that the set of constants of Σ is S_{m+2}-small, so this theory can be extended to an S_{m+2}-small L-place Γ_α. It follows that $h^*_\alpha(u)R_L\Gamma_\alpha$ for any $u \in W^*_\alpha$, $u \neq 0$, and $h^*_\alpha(0)R_L\Gamma_\alpha$ by transitivity. ■

Lemma 6.8 *If Γ_0 is not an endpoint in VM_L, then there exists a play of $SG_L(\Gamma_0)$ of length ω^2 generating a network h^+ such that $\mathbf{F}(\mathbf{h}^+) \vDash \mathbf{L}$ and for any u, for any $D_{h^+(u)}$-sentence A*

$$M(h^+), u \vDash A \textit{ iff } A \in h^+(u).$$

Proof. Similar to lemma 4.10. Such a play is provided by the winning strategy of $\exists$ used against the following strategy of $\forall$.

At the initial position $F_0 = (0, \varnothing)$ and $h_0(0) = \Gamma_0$.

The further strategy for $\forall$ will be the same as in lemma 4.10 for every ω-sequence of moves $\omega \cdot m + 1, \omega \cdot m + 2, \ldots$.

So we fix an enumeration of the countable set $\omega \times \omega$, and an enumeration of $\omega \times \Phi$, where Φ is the set of all S_{m+1}-sentences.

An odd move $(\omega \cdot m + n + 1)$ of $\forall$ chooses the first new (for this sequence of moves) pair (u, A), which is a defect in $h_{\omega \cdot m+n}$. An even move $(\omega \cdot m + n + 1)$ of $\forall$ chooses the first new (again for this sequence) pair $(u, v) \in \omega \times \omega$, which is a query for an insert in $h_{\omega \cdot m+n}$.

Let $\exists$ apply her winning strategy (lemma 6.7). We claim that the resulting network h^+ satisfies the statement of lemma 6.8.

In fact, the equivalence

$$u \vDash A \text{ iff } A \in h^+(u)$$

is again checked by induction. In the case $A = \Box B$ 'if' follows easily, since h^+ is a network.

For 'only if' suppose $A \notin h^+(u)$, $u = (x, m)$, $x \in W^*_{\omega \cdot m}$. Then the defect $(u, \Diamond\neg B)$ appears as some move $\omega \cdot m + n$ of $\forall$, and by the strategy of $\exists$ we obtain $v \in R^+(u)$ such that $\neg B \in h^+(v)$. Then $v \nvDash B$ by IH, so $u \nvDash A$.

The density of F^+ in every its component $F^*_{\omega \cdot m}$ is provided by even moves. For the points $u = (x, m)$, $v = (y, m')$ in different components ($m < m'$) we have uR^+v, and there is always an intermediate point — any point accessible from u in the same m-th component.

F^+ is confluent, since the points $(x, m), (y, m')$ with $m \leq m'$ both see the root $(0, m' + 1)$ of a later component. ■

Theorem 6.9 *The logics* $\mathbf{QK4.2}+Ad$, $\mathbf{QK4.2}+Ad+\Diamond\top$ *are strongly Kripke complete.*

Proof. As above, either an L-place Γ_0 is an endpoint in VM_L or by lemma 6.8 we can construct $M(h^+)$ satisfying Γ_0. ■

Theorem 6.10 *The logics* $\mathbf{QK4.2}$, $\mathbf{QK4.2} + \Diamond\top$ *are strongly Kripke complete.*

Proof. By applying the same method as in the previous theorem. The game $SG_L(\Gamma_0)$ is the same as in definition 6.6, but now at non-limit moves $\forall$ can only select defects. An analogue of lemma 6.7 still holds, so we can construct an appropriate network h^+. ■

Theorem 6.11 *The logics* $\mathbf{QK4.2} + Ad_2$, $\mathbf{QK4.2} + \Diamond\top + Ad_2$ *are strongly Kripke complete.*

Proof. We can use the same method. Now definition 6.6 changes for even moves — they are queries for 2-inserts (cf. definition 5.4 for $n = 2$).

Then the resulting frame $\mathbf{F}(\mathbf{h}^+)$ is 2-dense: the 2-density of each component is guaranteed by even moves, and points in a later component $F^*_{\omega\cdot m}$ have a common predecessor, the root $(0, m)$. ■

7 Final remarks

Axiomatizing modal predicate logics of specific frames is usually a nontrivial problem. In particular, we can be interested in predicate logics of relativistic time. The only clear case is the following.

Theorem 7.1 *Let F be the Minkowski lower halfspace with the causal future relation: aRb iff a signal can be sent from a to b. Them* $\mathbf{ML}(\mathcal{K}F) = \mathbf{QS4}$.

Proof. Every cone in F can be mapped p-morphically onto the infinite reflexive binary tree IT_2 [6]. It is also well-known that $\mathbf{ML}(\mathcal{K}IT_2) = \mathbf{QS4}$ (cf. [3], section 6.4). Hence the claim follows. ■

However, the method does not work for the logic of chronological future. Its propositional version was axiomatized in [7], this is $\Lambda = \mathbf{K4.2} + Ad_2 + \Diamond\top$. It is hardly probable that $\mathbf{Q}\Lambda$ fits for the predicate case, and we do not know how to play a game constructing a chronological order on the Minkowski plane.

Also note that our method is inapplicable to the case of constant domains. Moreover, the corresponding logic $L' := \mathbf{QK4Ad}+Ba$, where

$$Ba := \forall x \Box P(x) \supset \Box \forall x P(x)$$

is the Barcan axiom, may be Kripke incomplete. In fact, incompleteness is known for the logic $\mathbf{QKAd} + \Diamond\top + Ba$ (cf. [4]), and it probably extends to L' (although the proof from [4] does not fit for L', because of transitivity).

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