One-Generated **WS5**-Algebras

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Abstract

We describe all finite one-generated **WS5**-algebras, and we describe and study the properties of free one-generated **WS5**-algebra. Using a splitting technique, we also prove that, in contrast to the variety of all **S5**-algebras, which is locally finite, and even though variety $\mathcal{M}$ of all **WS5**-algebras is finitely approximable, the variety $\mathcal{M}$ contains infinitely many non-finitely approximable subvarieties.

Keywords: Heyting algebra, Heyting algebra with additional operator,modal logic, weak logic WS5.

1 Introduction

In this paper, we study variety $\mathcal{M}$ of all **WS5**-algebras – the algebras that are models for modal logic WS5. Logic WS5 is known under different names for quite a long time [1]. It is a modal logic an assertoric part of which is the intuitionistic propositional logic, and modality is **S5**-type modality. This logic has attracted additional attention when it turned out that in Glivenko's theorem for intuitionistic modal logics, WS5 plays the same role as classical logic for intuitionistic logic (see [2]).

An algebraic semantic of WS5 are **WS5**–algebras. **WS5**-algebras are the Heyting algebras with an additional operator $\Box$ which satisfies the following condition:

(M_0) $\Box\mathbf{1} \approx \mathbf{1}$;
(M_1) $\Box x \to x \approx \mathbf{1}$;
(M_2) $\Box(x \to y) \to (\Box x \to \Box y) \approx \mathbf{1}$;
(M_3) $\Box x \to \Box\,\Box\, x \approx \mathbf{1}$;
(M_4) $\neg\,\Box\,\neg\,\Box\, x \approx \Box x$.

All **WS5**-algebras form a variety denoted by $\mathcal{M}$.

First, we recall all necessary facts about Heyting and **WS5**- algebras. Then, in Section 2, we describe all finite one-generated **WS5**-algebras, and using this description we prove that variety $\mathcal{M}$ of all **WS5**-algebras contains infinitely

[1] To be more precise, we use algebras that are models of logic L_4 from [10] – a unimodal logic on intuitionistic base.

many non-finitely approximable subvarieties. After this we turn to a description of the free one-generated algebra. In Section 4, we describe algebra $\mathbf{F}_{\mathcal{M}}(1)$ – the free one-generated algebra of $\mathcal{M}$, and we prove that $\mathbf{F}_{\mathcal{M}}(1)$ has a rather complex structure. Namely, $\mathbf{F}_{\mathcal{M}}(1)$ contains the infinite ascending and descending chains of open elements and the Heyting reduct of $\mathbf{F}_{\mathcal{M}}(1)$ is non-finitely generated as Heyting algebra.

1.1 Heyting Algebras

Algebra $\langle \mathsf{A}; \wedge, \vee, \rightarrow, \mathbf{0}, \mathbf{1}\rangle$, where $\langle \mathsf{A}; \wedge, \vee, \mathbf{0}, \mathbf{1}\rangle$ is a bounded distributive lattice, and $\rightarrow$ is a relative pseudo-complementation, is called a ***Heyting algebra***. We use the regular abbreviations: $\neg \mathsf{a} := \mathsf{a} \rightarrow \mathbf{0}$ and $\mathsf{a} \leq \mathsf{b} := \mathsf{a} \rightarrow \mathsf{b} = \mathbf{1}$. The variety of all Heyting algebras is denoted by $\mathcal{H}$, Heyting algebras are denoted by $\mathcal{A}, \mathcal{B}$, etc. (perhaps with indexes), and elements of Heyting algebras are denoted by $a, b, \dots$ (perhaps with indexes).

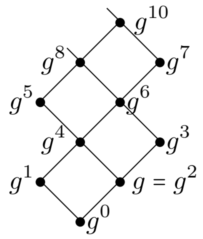

Fig.1.1. Degrees of an element.

For each element a of a Heyting algebra, we define the degrees of a as follows:

$$a^0 := \mathbf{0}, \quad a^1 := \neg a, \quad a^2 := a,$$
$$a^{2k+3} := a^{2k+1} \rightarrow a^{2k}, \quad a^{2k+4} := a^{2k+1} \vee a^{2k+2}$$

for all $k \geq 0$, and we let $a^\omega := \mathbf{1}$.

Let us recall that for each natural number m there exists a unique (up to isomorphism) one-generated Heyting m-element algebra which we denote by $\mathcal{Z}_m$. There is also a unique infinite one-generated Heyting algebra denoted by $\mathcal{Z}_\omega$. Thus, $\mathcal{Z}_2$ is a two-element Boolean algebra which we also denote by $\mathcal{B}_2$.

We will need the following simple observation.

Proposition 1.1 *Let $m > 1$ and element $g_m \in \mathcal{Z}_m$ be a generator of $\mathcal{Z}_m$. Then for every $k > 1$,*

(a) $g_m^m = \mathbf{1}$, and if $k < m$, then $g_m^k < \mathbf{1}$;

(b) if $k \geq m + 2$, then for every element $a \in \mathcal{Z}_m$, $a^k = \mathbf{1}$.

Proof. Proof can be done by a simple induction (or see Fig. 1.1). □

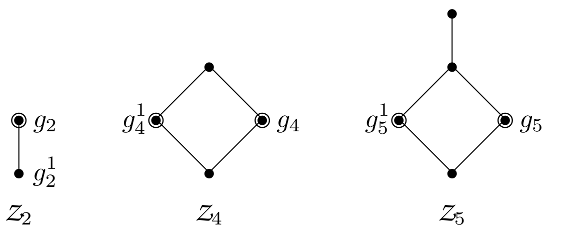

Fig. 1. One-generated Heyting algebras with distinct generators.

Let us observe that all one-generated Heyting algebras, except for $\mathcal{Z}_2, \mathcal{Z}_4$ and $\mathcal{Z}_5$, have a unique generator, namely, $g = g^2$, while algebras $\mathcal{Z}_2, \mathcal{Z}_4$ and $\mathcal{Z}_5$ have two distinct generators, namely $g = g^2$ and $\neg g = g^1$ (see Fig. 1). Let us note that only in these three one-generated Heyting algebras we have $g = \neg\neg g$, while in the rest of one-generated Heyting algebras $g \neq \neg\neg g$.

In the sequel, we use the notation $\mathbf{A}[\mathsf{a}]$ to underscore that we consider algebra $\mathbf{A}$ with generator a, for instance, $\mathcal{Z}_4[g^1]$ means that we consider $\mathcal{Z}_4$ as being generated by g^1.

Remark 1.2 It is worth noting that there is a significant difference between properties of generators of $\mathcal{Z}_2$ and generators of $\mathcal{Z}_4$ or $\mathcal{Z}_5$: for $\mathcal{Z}_4$ and $\mathcal{Z}_5$ the map $\varphi : g \mapsto \neg g$ can be extended to automorphism, while for $\mathcal{Z}_2$ it is not the case. In other words, for every pair of terms $t(x), r(x)$ for $\mathcal{Z}_4$ and $\mathcal{Z}_5$ we have

$$t(g) = r(g) \text{ if and only if } t(\neg g) = r(\neg g),$$

while for $\mathcal{Z}_2$, the above does not hold: take $t(x) = \neg x$ and $r(x) = \mathbf{1}$.

1.2 WS5-Algebras

Algebra $\langle \mathsf{A}; \wedge, \vee, \to, \mathbf{0}, \mathbf{1}, \Box \rangle$ where $\langle \mathsf{A}; \wedge, \vee, \to, \mathbf{0}, \mathbf{1} \rangle$ is a Heyting algebra and $\Box$ satisfies identities $(M_0) - (M_4)$, is called a ***WS5-algebra***. It is clear that the class of all **WS5**-algebras forms a variety which we denote by $\mathcal{M}$. All information about monadic Heyting algebras and, in particular about **WS5**-algebras that we use, can be found in [1]. To distinguish **WS5**-algebras from Heyting algebras, we denote **WS5**-algebras by $\mathbf{A}, \mathbf{B}, \mathbf{Z}$ perhaps with indexes, and elements of **WS5**-algebras are denoted by $\mathsf{a}, \mathsf{b}, \ldots$, perhaps with indexes.

An element a of a **WS5**-algebra $\mathbf{A}$ is called ***open*** if $\Box\mathsf{a} = \mathsf{a}$. Let us also observe that the following identities hold in $\mathcal{M}$ (see e.g. [10])

$$\begin{aligned} &\neg \Box x \approx \Box \neg \Box x; \\ &\Box x \to \Box y \approx \Box(\Box x \to \Box y); \\ &\Box (\Box x \vee y) \approx \Box x \vee \Box y. \end{aligned} \tag{1}$$

Hence, all open elements of **WS5**-algebra form a Boolean subalgebra. Thus, each **WS5**-algebra $\mathbf{A} := \langle \mathsf{A}; \wedge, \vee, \to, \mathbf{0}, \mathbf{1}, \Box \rangle$ can be viewed as a pair $\langle \mathcal{A}, \mathcal{B} \rangle$, where $\mathcal{A} = \langle \mathsf{A}; \wedge, \vee, \to, \mathbf{0}, \mathbf{1} \rangle$ is a Heyting algebra (a ***Heyting reduct*** of $\mathbf{A}$ – ***h-reduct*** for short), and $\mathcal{B}$ is the Boolean algebra of all open elements of $\mathcal{A}$.

1.3 WS5-Filters and Congruences

Definition 1.3 Recall (see e.g. [1]) that a subset $\mathsf{F} \subseteq \mathbf{A}$ is a **WS5-*filter*** of $\mathbf{A}$ if

(a) if $\mathsf{a}, \mathsf{b} \in \mathsf{F}$, then $\mathsf{a} \wedge \mathsf{b} \in \mathsf{F}$;

(b) if $\mathsf{a} \in \mathsf{F}$ and $\mathsf{a} \leq \mathsf{b}$, then $\mathsf{b} \in \mathsf{F}$;

(c) if $\mathsf{a} \in \mathsf{F}$, then $\Box\mathsf{a} \in \mathsf{F}$.

Since the meet of an arbitrary set of **WS5**-filters of a given **WS5**-algebra $\mathbf{A}$ forms a **WS5**–filter, for any subset of elements $\mathsf{D} \subseteq \mathbf{A}$ there is the least **WS5**-filter $[\mathsf{D})$ containing D. If $\mathsf{D} = \{\mathsf{f}\}$, i.e. if D consists of a single element f, **WS5**-filter $[\mathsf{D})$ is called ***principal***. A filter F is principal if and only if it has the smallest element.

There is a well known correspondence between congruences of $\mathbf{A}$ and **WS5**-filters of $\mathbf{A}$: if $\theta \in \mathsf{Con}(\mathbf{A})$, then the set $\mathsf{F}(\theta) := \{\mathsf{a} \in \mathbf{A} \,|\, \mathsf{a} \equiv \mathbf{1} \mod \theta\}$ is a **WS5**-filter of $\mathbf{A}$, and, given a **WS5**-filter F, the relation $\theta(\mathsf{F})$ defined by

$$\mathsf{a} \equiv \mathsf{b} \mod \theta(\mathsf{F}) \text{ if and only if } \mathsf{a} \leftrightarrow \mathsf{b} \in \mathsf{F}, \tag{2}$$

where $\mathsf{a} \leftrightarrow \mathsf{b} := (\mathsf{a} \to \mathsf{b}) \wedge (\mathsf{b} \to \mathsf{a})$, is a congruence of $\mathbf{A}$. And we write $\mathbf{A}/\mathsf{F}$ instead of $\mathbf{A}/\theta(\mathsf{F})$. Let us note that (2) establishes an isomorphism between lattice of filters of a algebra $\mathbf{A}$ and $\mathsf{Con}(\mathbf{A})$.

Because the set of all open elements of a **WS5**-algebra forms a Boolean algebra, it is not hard to demonstrate that a **WS5**-algebra is ***subdirectly irreducible*** (s.i. for short) if and only if it has exactly two open elements, namely $\mathbf{0}$ and $\mathbf{1}$, that is, s.i. **WS5**-algebra corresponds to a pair $\langle \mathcal{A}, \mathcal{B}_2 \rangle$, where $\mathcal{B}_2$ is the two-element Boolean algebra. Also, any nontrivial s.i. **WS5**-algebra $\mathbf{A}$ is ***simple***: $\mathbf{A}$ has precisely two congruences. By $\mathcal{M}_{fsi}$ we denotes the set of all finite s.i. **WS5**-algebras.

The following simple observation plays an important role in the sequel.

Proposition 1.4 *Let $\mathbf{A} = \prod_{i \in I} \mathbf{B}_i$ be a direct product of a family of nontrivial s.i. **WS5**-algebras and $\mathsf{F} \subset \mathbf{A}$ be a principal filter. Then there is a subset $J \subseteq I$ such that*

$$\mathbf{A}/\theta(\mathsf{F}) \cong \prod_{j \in J} \mathbf{B}_j. \tag{3}$$

Proof. Let $\mathbf{A} = \prod_{i \in I} \mathbf{B}_i$ be a direct product of nontrivial s.i. **WS5**-algebra. Hence, all algebras $\mathbf{B}_i$ are simple. Thus, each $\mathbf{B}_i$ contains precisely two open elements: $\mathbf{0}$ and $\mathbf{1}$. Hence, open elements of $\mathbf{A}$ are precisely the elements each projection of which belongs to $\{\mathbf{0}, \mathbf{1}\}$.

Because filter F is principal, it has the smallest element f. By property (c) of the definition of **WS5**-filter, f is an open element, and hence, its every projection belongs to $\{\mathbf{0}, \mathbf{1}\}$. Because $\mathbf{A}/\theta(\mathsf{F})$ is nontrivial, $\mathsf{f} \neq \mathbf{0}$, therefore, element f has distinct from $\mathbf{0}$ projections. Let

$$J = \{i \in I \,|\, \pi_i(\mathsf{f}) = \mathbf{1}\}, \text{ where } \pi_i \text{ is the projection to } \mathbf{B}_i.$$

Then, from (2) for any pair of elements $\mathsf{a}, \mathsf{b} \in \mathbf{A}$,

$$\mathsf{a} \equiv \mathsf{b} \mod \theta(\mathsf{F}) \text{ if and only if } \mathsf{a} \leftrightarrow \mathsf{b} \in \mathsf{F}.$$

That is,

$$\mathsf{a} \equiv \mathsf{b} \mod \theta(\mathsf{F}) \text{ if and only if } \pi_j(\mathsf{a}) = \pi_j(\mathsf{b}) \text{ for all } j \in J.$$

And this means that $\mathbf{A}/\theta(\mathsf{F}) \cong \prod_{j \in J} \mathbf{B}_j$. □

It is clear that all **WS5**-filters of any finite **WS5**-algebra are principal. Thus, the following holds.

Corollary 1.5 *Let $\mathbf{A} = \prod_{i \in I} \mathbf{A}_i$ be a finite* **WS5***-algebra and all $\mathbf{A}_i$ be simple. If $\mathbf{B}$ is a nontrivial homomorphic image of $\mathbf{A}$, then $\mathbf{B} \cong \prod_{i \in J} \mathbf{A}_j$ for some $J \subseteq I$.*

1.4 Some Properties of Variety $\mathcal{M}$

Variety $\mathcal{M}$ is well-behaved. First, we note that every nontrivial s.i. algebra is simple, that is, variety $\mathcal{M}$ is ***semisimple*** and hence, each nontrivial **WS5**-algebra is a subdirect product of simple algebras.

Next, we recall (e.g. [5]) that a term $t(x, y, z)$ is a ***discriminator*** on algebra $\mathbf{A}$ if for any $\mathsf{a}, \mathsf{b}, \mathsf{c} \in \mathbf{A}$

$$t(\mathsf{a}, \mathsf{b}, \mathsf{c}) = \begin{cases} \mathsf{a} & \text{if } \mathsf{a} \neq \mathsf{b} \\ \mathsf{c} & \text{if } \mathsf{a} = \mathsf{b}; \end{cases}$$

and a variety $\mathcal{V}$ is ***discriminator*** if it is generated by a class of algebras having the same of discriminator discriminator term.

It is not hard to verify that term

$$t(x, y, z) = (z \wedge \Box((x \vee y) \to (x \wedge y))) \vee (x \wedge (\Box((x \vee y) \to (x \wedge y)) \to \mathbf{0})$$

is discriminator on s.i. **WS5**-algebras (cf. [11, p. 571]) and, hence, $\mathcal{M}$ is a discriminator variety.

Every discriminator variety is congruence-distributive and congruence-permutable (see e.g. [5, Theorem 9.4]). Hence, $\mathcal{M}$ is congruence-distributive and congruence-permutable.

We will use the following property of congruence-permutable varieties.

Proposition 1.6 *[5, Corollary 10.2] Let $\mathbf{A}_1, \ldots, \mathbf{A}_k$ be simple algebras in a congruence-permutable variety $\mathcal{V}$. If $\mathbf{B} \leq \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ is a subdirect product, then $\mathbf{B} \cong \mathbf{A}_{i_1} \times \cdots \times \mathbf{A}_{i_k}$ for some $\{i_i, \ldots, i_k\} \subseteq \{1, \ldots, n\}$.*

Thus, in every discriminator variety $\mathcal{V}$ each finite algebra $\mathbf{A} \in \mathcal{V}$ is a direct product of simple algebras from $\mathcal{V}$. Moreover, Theorem 4.71 from [9] entails that each such decomposition contains the same number of factors.

Let us observe that for any **WS5**-algebra $\mathbf{A}$ and any elements $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d} \in \mathbf{A}$

$$\mathsf{c} \equiv \mathsf{d} \mod \theta(\mathsf{a}, \mathsf{b}) \text{ if and only if } \mathbf{A} \models (\Box(\mathsf{a} \leftrightarrow \mathsf{b}) \to \mathsf{c}) \approx (\Box(\mathsf{a} \leftrightarrow \mathsf{b}) \to \mathsf{d}),$$

where $\theta(\mathsf{a},\mathsf{b})$ is a congruence generated by elements a,b. Therefore (see [4]), variety $\mathcal{M}$ has equationally definable principal congruences (EDPC for short). We will also use that in each variety with EDPC (with finitely many fundamental operations), every nontrivial finite s.i. algebra is a splitting algebra ([3][Corollary 3.2], for Heyting algebras this property was observed in [7]), that is, for each nontrivial finite s.i. algebra $\mathbf{A}$ there is a term $s_{\mathbf{A}}$, such that for each algebra $\mathbf{B} \in \mathcal{M}$,

$$\begin{array}{l}\mathbf{B} \not\models s_{\mathbf{A}} \approx \mathbf{1} \text{ if and only if} \\ \mathbf{A} \text{ is embedded in some homomorphic image of } \mathbf{B}.\end{array} \tag{4}$$

If algebra $\mathbf{B}$ is s.i., and hence, it is simple and does not have nontrivial homomorphic images, we have

$$\mathbf{B} \not\models s_{\mathbf{A}} \approx \mathbf{1} \text{ if and only if } \mathbf{A} \text{ is embedded in } \mathbf{B}, \tag{5}$$

in particular, $\mathbf{A} \not\models s_{\mathbf{A}} \approx \mathbf{1}$.

2 One-generated WS5-algebras

It is clear that every one-generated **WS5**-algebra is a subdirect product of some one-generated s.i. algebras. This is why we start with describing all one-generated s.i. **WS5**-algebras.

2.1 Subdirectly Irreducible One-Generated WS5-algebras

Proposition 2.1 *An s.i.* **WS5***-algebra* $\mathbf{A} \in \mathcal{M}$ *is one-generated if and only if its h-reduct is a one-generated Heyting algebra.*

Proof. Suppose $\mathbf{A} \in \mathcal{V}$ is an s.i. one-generated **WS5**-algebra and g is a generator. Then for each element $\mathsf{a} \in \mathbf{A}$, there is a term $t(x)$ such that $\mathsf{a} = t(\mathsf{g})$. If $\Box r(x)$ is a subterm of $t(x)$, because $\mathbf{A}$ is s.i. and has just two open elements, either $\Box r(\mathsf{g}) = \mathbf{0}$, or $\Box r(\mathsf{g}) = \mathbf{1}$. In either case we can replace $r(x)$ respectively with $\mathbf{0}$ or $\mathbf{1}$, and obtain a new term $t'(x)$ such that $t'(\mathsf{g}) = \mathsf{a}$. In such a way we can reduce $t(x)$ to a Heyting term $t'(x)$ (which does not contain $\Box$) such that $t'(\mathsf{g}) = \mathsf{a}$.

Converse statement is trivial, because every generator of h-reduct is at the same time a generator of **WS5**-algebra. □

Thus, every nontrivial one-generated s.i. **WS5**-algebra is isomorphic to one of the following algebras: $\mathbf{Z}_k := \langle \mathcal{Z}_k, \mathcal{B}_2 \rangle, k = 2, 3, \ldots, \omega$.

Let us note that except for $\mathbf{Z}_2, \mathbf{Z}_4$ and $\mathbf{Z}_5$ all s.i. one-generated **WS5**-algebras have a unique generator.

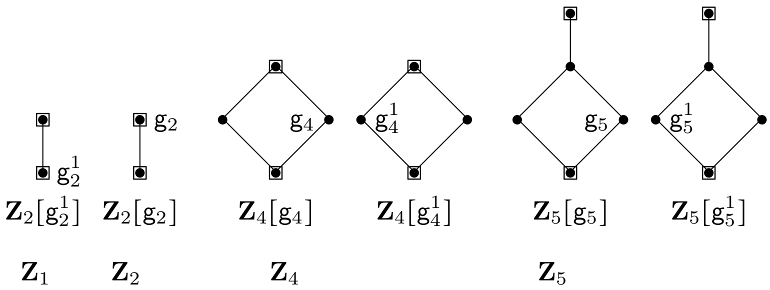

Fig. 2. Generators of $\mathbf{Z}_2, \mathbf{Z}_4$ and $\mathbf{Z}_5$.

WS5-algebras $\mathbf{Z}_2, \mathbf{Z}_4$ and $\mathbf{Z}_5$ have two distinct generators each, but the maps $\varphi : g_4 \mapsto g_4^1$ and $\psi : g_5 \mapsto g_5^1$ can be extended to automorphisms of respective algebras. Hence, if we consider algebras modulo automorphism, we can view $\mathbf{Z}_4$ and $\mathbf{Z}_5$ as having a unique generator. To simplify notation, we denote by $\mathbf{Z}_1$ an isomorphic copy of $\mathbf{Z}_2$ with generator $\mathbf{1}$, preserving $\mathbf{Z}_2$ for an isomorphic copy with the generator $\mathbf{0}$.

Let

$$\mathscr{Z} \coloneqq \{\mathbf{Z}_k, k = 1, 2, 3, \dots\},$$

i.e. $\mathscr{Z}$ is the list of all finite s.i. one-generated algebras and all these algebras are simple. For each $\mathbf{Z}_i$, by g_i we denote the generator of $\mathbf{Z}_i$.

The following algebras will play the central role in our study of one-generated algebras. We let

$$\mathbf{P} \coloneqq \prod_{i=1}^{\infty} \mathbf{Z}_i \tag{6}$$

and $\mathsf{g} \in \mathbf{P}$ be an element such that $\pi_i(\mathsf{g}) = \mathsf{g}_\mathsf{i}$ for all $i > 0$ and we take

$$\mathbf{Z} \leq \mathbf{P} \text{ to be a subalgebra of } \mathbf{P} \text{ generated by element } \mathsf{g}. \tag{7}$$

Let us note that $\mathbf{Z}$ is a subdirect product of algebras $\mathbf{Z}_i, i > 0$, for element $\pi_i(\mathsf{g})$ generates whole factor $\mathbf{Z}_i$ for each $i > 0$.

We will need the following technical lemma.

Lemma 2.2 *Let $\mathsf{s}_m \in \mathbf{P}$ be an element such that $\pi_m(\mathsf{s}_m) = \mathbf{1}$ and $\pi_j(\mathsf{s}_m) = \mathbf{0}$ for all $j \neq m$. Then, $s_m(\mathsf{g}) \in \mathbf{Z}$ for every $m > 0$.*

Proof. We need to prove that, given element

$$\mathsf{g} = (\mathsf{g}_1, \mathsf{g}_2, \dots),$$

for each $m > 2$ there is a term $s_m(x)$ such that

$$s_m(\mathsf{g}) = \mathsf{s}_m = (\underbrace{\mathbf{0}, \dots, \mathbf{0}}_{m-1 \text{ times}}, \mathbf{1}, \mathbf{0}, \dots).$$

In other words, we need to prove that each element s_m belongs to the subalgebra of **Z** generated by element g.

Indeed, we know from Proposition 1.1(a) that $g_m^m = \mathbf{1}$ and $g_k^m < \mathbf{1}$ for all $m < k < \omega$. Thus,

$$\mathsf{g}^m = (\mathsf{g}_1^m, \mathsf{g}_2^m, \dots, \mathsf{g}_m^m, \mathsf{g}_{m+1}^m, \mathsf{g}_{m+2}^m, \dots).$$

and

$$\Box\mathsf{g}^m = (\Box\mathsf{g}_1^m, \Box\mathsf{g}_2^m, \dots, \Box\mathsf{g}_m^m, \Box\mathsf{g}_{m+1}^m, \Box\mathsf{g}_{m+2}^m, \dots) = (\Box\mathsf{g}_1^m, \Box\mathsf{g}_2^m, \dots, \mathbf{1}, \mathbf{0}, \mathbf{0}, \dots).$$

Hence,

$$\pi_m(\Box(\mathsf{g}^m)) = \mathbf{1} \text{ and } \pi_j(\Box(\mathsf{g}^m)) = \mathbf{0} \text{ for all } j > m.$$

Let

$$\tilde{\mathsf{s}}_i = \bigvee_{j=1}^{m} \Box(\mathsf{g}^j) = (\underbrace{1, \dots, 1}_{m \text{ times}}, 0, \dots).$$

Then, $\mathsf{s}_1 = \tilde{\mathsf{s}}_1 = \Box\mathsf{g}$ and $\mathsf{s}_m = \tilde{\mathsf{s}}_m \wedge \neg\tilde{\mathsf{s}}_{m-1}$ for all $m > 1$, and this observation the proof □

With each set $I \subseteq \{1, 2, \dots\}$ we associate a congruence $\theta(I)$ of **Z**:

$$\mathsf{a} \equiv \mathsf{b} \mod \theta(I) \text{ if and only if } \pi_i(\mathsf{a}) = \pi_i(\mathsf{b}) \text{ for all } i \in I.$$

Corollary 2.3 *For any finite subset $\{i_1, \dots, i_n\} \subseteq \{1, 2, \dots\}$ and any elements $\mathsf{a}_i \in \mathbf{Z}_{i_j}, j = 1, \dots, n$, there is an element $\mathsf{a} \in \mathbf{Z}$ such that*

$$\pi_{i_j}(\mathsf{a}) = \mathsf{a}_{i_j}, \textit{ for all } j = 1, \dots, n. \tag{8}$$

Proof. Recall that each algebra $\mathbf{Z}_{i_j}$ is generated by g_{i_j}. Hence, for some numbers $k_1, \dots, k_n$, we have $\mathsf{a}_{i_j} = \mathsf{g}_{i_j}^{k_j}$. Hence

$$(\mathsf{g}^{k_1} \wedge \mathsf{s}_{i_1}) \vee \dots \vee (\mathsf{g}^{k_n} \wedge \mathsf{s}_{i_n})$$

is a desired element of **Z**. □

Corollary 2.4 *If $I \subseteq \{1, 2, \dots\}$ is a nonempty finite set of numbers, then direct product $\prod_{i \in I} \mathbf{Z}_i$ is (isomorphic to) $\mathbf{Z}/\theta(I)$.*

Proof. Suppose $I = \{i_1, \dots, i_n\}$. If $\mathsf{a} \in \mathbf{Z}$, by $\bar{\mathsf{a}}$ we denote $\theta(I)$-congruence class containing a. If $(\mathsf{a}_1, \dots, \mathsf{a}_n) \in \prod_{i \in I} \mathbf{Z}_i$, by virtue of Corollary 2.3, there is an element $\mathsf{a} \in \mathbf{Z}$ satisfying (8). Thus, the map $\phi : (\mathsf{a}_1, \dots, \mathsf{a}_n) \longrightarrow \bar{\mathsf{a}}$ is an isomorphism between $\mathbf{Z}/\theta(I)$ and $\prod_{i \in I} \mathbf{Z}_i$. □

2.2 Finite One-Generated WS5-algebras

Every finite one-generated **WS5**-algebra is a subdirect product of algebras from $\mathcal{Z}$. Hence, by virtue of Proposition 1.6, every finite one-generated **WS5**-algebra is a direct product of algebras from $\mathcal{Z}$. The following theorem gives a description of all finite one-generated **WS5**-algebras.

Theorem 2.5 *Finite* **WS5***-algebra* $\mathbf{A}$ *is one-generated if and only if there is a subset* $I \subseteq \{1, 2, , \dots\}$ *such that*

$$\mathbf{A} = \prod_{i \in I} \mathbf{Z}_i. \tag{9}$$

Proof. Suppose that $\mathbf{A}$ is finite one-generated **S5**-algebra. Then $\mathbf{A}$ is (isomorphic to) a direct product $\mathbf{B}_1 \times \cdots \times \mathbf{B}_n$ of algebras from $\mathcal{Z}$. First, let us verify that for every $k > 2$, algebra $\mathbf{Z}_i$ cannot occur in this decomposition more than once.

For contradiction: assume that $\mathbf{Z}_k$ occurs in the decomposition twice. Then, $\mathbf{Z}_k^2$ is a homomorphic image of $\mathbf{A}$, and hence, $\mathbf{Z}_k^2$ is one-generated. Recall that for every $k \geq 1$, algebra $\mathbf{Z}_k$ has a unique (up to automorphism) generator [2]. Note also that any projection of a generator of algebra $\mathbf{Z}_k \times \mathbf{Z}_k$ is a generator of a respective factor. Thus, if g is a generator of $\mathbf{Z}_k \times \mathbf{Z}_k$, then $\mathsf{g} = (\mathsf{g}_k, \mathsf{g}_k)$, where g_k is a generator of $\mathbf{Z}_k$. It is not hard to see that element $(\mathsf{g}_k, \mathsf{g}_k)$ generates just a diagonal of $\mathbf{Z}_k \times \mathbf{Z}_k$ and not the whole algebra: for instance, element $(\mathbf{0}, \mathbf{1})$ does not belong to the diagonal of $\mathbf{Z}_k \times \mathbf{Z}_k$. Thus, we have arrived to contradiction.

Conversely, let $\{i_1, \dots, i_n\} \subseteq \{1, 2, \dots\}$ and $\mathbf{A} = \mathbf{Z}_{i_1} \times \cdots \times \mathbf{Z}_{i_n}$. Then, by Corollary 2.4, $\mathbf{A}$ is a homomorphic image of algebra $\mathbf{Z}$, and $\mathbf{A}$ is one-generated, for $\mathbf{Z}$ being one-generated. □

Let us underscore that in Theorem 2.5, I is a set, and therefore, the factors of each direct decomposition, except for $\mathbf{Z}_2$, are unique. Also, it is not hard to see that if $\mathbf{A} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$, where all algebras $\mathbf{A}_i$ are s.i., then open elements of $\mathbf{A}$ form a Boolean algebra of cardinality 2^n.

We say that a direct product of simple algebras is ***non-repetitive*** if all its factors are mutually non-isomorphic.

Thus, Theorem 2.5 asserts that every finite nontrivial one-generated **WS5**-algebra $\mathbf{A}$ is of one of the following types:

(a) $\mathbf{A}$;

(b) $\mathbf{Z}_2 \times \mathbf{A}$,

(c) $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{A}$,

where $\mathbf{A}$ is a non-repetitive product of algebras $\mathbf{Z}_i, i > 2$ and such a representation is unique modulo rearranging the factors.

Example 2.6 There are precisely four (up to isomorphism) distinct one-generated **WS5**-algebras of cardinality 12, namely

$$(\mathcal{Z}_2 \times \mathcal{Z}_6, \mathcal{B}_2^2), \quad (\mathcal{Z}_2 \times \mathcal{Z}_2 \times \mathcal{Z}_3, \mathcal{B}_2^3), \quad (\mathcal{Z}_3 \times \mathcal{Z}_4, \mathcal{B}_2^2), \quad (\mathcal{Z}_{12}, \mathcal{B}_2).$$

These algebras are not pairwise isomorphic because only $(\mathcal{Z}_2 \times \mathcal{Z}_2 \times \mathcal{Z}_3, \mathcal{B}_2^3))$ and $(\mathcal{Z}_3 \times \mathcal{Z}_4, \mathcal{B}_2^2)$ have isomorphic h-reducts, but the former algebra has eight open elements, while the latter – just four.

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[2] This is where the distinction between $\mathbf{Z}_1$ and $\mathbf{Z}_2$ comes to play.

Example 2.7 There are precisely three (up to isomorphism) distinct one-generated **WS5**-algebras of cardinality 16, namely:

$$(\mathcal{Z}_2 \times \mathcal{Z}_8, \mathcal{B}_2^2), \quad (\mathcal{Z}_2 \times \mathcal{Z}_2 \times \mathcal{Z}_4, \mathcal{B}_2^3), \quad (\mathcal{Z}_{16}, \mathcal{B}_2).$$

Remark 2.8 In [6], Grigolia suggested an approach to description of a free monadic Heyting algebra with one free generator. From this description it does not immediately follow that a direct decomposition of a finite one-generated **WS5**-algebra, algebra $\mathbf{Z}_2$ can appear twice, while the rest of the factors are distinct modulo isomorphism.

3 Non-finitely approximable subvarieties of $\mathcal{M}$

If L is an extension of WS5, an ***assertoric fragment*** of L is an intermediate logic consisting of all assertoric formulas from L, that is, a logic consisting of all formulas of L without occurrences of $\Box$. The goal of this section is to show that there is an intermediate logic L enjoying the finite model property (f.m.p. for short) and such that it is an assertoric fragment of infinitely many extensions of WS5 without the f.m.p.

If $\mathcal{V}$ is a variety of **WS5**-algebras, then $\mathsf{L}(\mathcal{V})$ is an extension of WS5 consisting of all formulas valid in $\mathcal{V}$.

If $\mathcal{V}$ is a variety of **WS5**-algebras, by $\mathcal{V}^-$ we denote the variety of Heyting algebras generated by the Heyting reducts of all algebras from $\mathcal{V}$. Let us note that two logics $\mathsf{L}(\mathcal{V}_0)$ and $\mathsf{L}(\mathcal{V}_1)$ have the same assertoric fragment if and only iff $\mathcal{V}_0^- = \mathcal{V}_1^-$.

First, let us show that there are intermediate logics that are assertoric fragments of infinitely many extensions of WS5.

Proposition 3.1 *Let I be a set of natural numbers and let $\mathcal{V}_I$ be a variety generated by algebras $\{\mathbf{Z}_i, i \in I\}$ and $\mathbf{Z}_\omega$. Then, logics $\mathsf{L}(\mathcal{V}_I)$ and $\mathsf{L}(\mathcal{V}_\varnothing)$ have the same assertoric fragment.*

Proof. Indeed, it is clear that $\mathcal{V}_\varnothing \subseteq \mathcal{V}_I$, and hence, $\mathcal{V}_\varnothing^- \subseteq \mathcal{V}_I^-$. On the other hand, for each $i \in I$, $\mathcal{Z}_i$ is a homomorphic image of $\mathcal{Z}_\omega$, and hence, $\mathcal{V}_I^- \subseteq \mathcal{V}_\varnothing^-$. $\Box$

Let us recall that a variety $\mathcal{V}$ is said to be ***finitely approximable*** if $\mathcal{V}$ is generated by its finite algebras. If $\mathcal{V}$ is a finitely approximable variety, then each of its free algebras $\mathbf{F}_\mathcal{V}(n)$ is finitely approximable, that is, $\mathbf{F}_\mathcal{V}(n)$ is a subdirect product of the finite algebras (see [8, Chapter VI Theorem 5]). Variety $\mathcal{V}$ is finitely approximable if and only if $\mathsf{L}(\mathcal{V})$ enjoys the f.m.p.

Proposition 3.2 *The variety $\mathcal{V}_I^-$ from Proposition 3.1 is finitely approximable.*

Proof. The proof follows from the observation that variety $\mathcal{V}_\varnothing^-$ is generated by $\mathcal{Z}_\omega$ which is a subdirect product of finite algebras $\mathcal{Z}_i, i > 0$. And this means that variety $\mathcal{V}_\varnothing^-$ is generated by finite algebras, and thus, it is finitely approximable. $\Box$

A variety is ***locally finite*** if each of its finitely generated algebra is finite. A variety $\mathcal{V}$ is locally finite if and only if free algebra $\mathbf{F}(n)$ is finite for every finite n (see [5, Theorm 10.15]). It is clear that every locally finite variety is finitely approximable, and that every subvariety of locally finite variety is locally finite.

For instance, a variety of all **S5**-algebras is locally finite, while variety $\mathcal{M}$ of all **WS5**-algebras is not locally finite but it is nevertheless finitely approximable [1, Theorem 42].

For each $m > 6$, let $\mathcal{V}_m$ be a variety generated by algebras $\{\mathbf{Z}_k, 6 < k \leq m\}$ and $\mathbf{Z}_\omega$. From Proposition 3.1 we know that $\mathcal{V}_m^- = \mathcal{V}_n^-$ for all $m, n > 6$.

Theorem 3.3 *For each $m > 6$, variety $\mathcal{V}_m$ is not finitely approximable.*

Proof. First, we observe that $\mathcal{V}_m$ contains the infinite one-generated algebra $\mathbf{Z}_\omega$. Hence, algebra $\mathbf{F}_{\mathcal{V}_m}(1)$ is infinite. Thus, to prove that $\mathcal{V}_m$ is non-finitely approximable, it suffices to demonstrate that $\mathbf{F}_{\mathcal{V}_m}(1)$ is not a subdirect product of finite algebras. Because $\mathbf{F}_{\mathcal{V}_m}(1)$ is one generated, every subdirect factor of $\mathbf{F}_{\mathcal{V}_m}(1)$ is one-generated too. Therefore, to prove that $\mathbf{F}_{\mathcal{V}_m}(1)$ is non-finitely approximable, it is enough to show that $\mathbf{F}_{\mathcal{V}_m}(1)$ does not belong to subvariety $\mathcal{V}_m^{(1)} \subseteq \mathcal{V}_m$ generated by all finite one-generated s.i. algebras from $\mathcal{V}_m$.

Next, we note that for any finite s.i. algebra $\mathbf{A} \in \mathcal{V}_m$, $\mathbf{A}$ is (isomorphic to) a subalgebra of one of the algebra $\mathbf{Z}_k$ for some $k \in [7, m] \cup \{\omega\}$.

Indeed, by (5), there is a term s such that $\mathbf{A} \not\models s \approx \mathbf{1}$ and for every $\mathbf{B} \in \mathcal{V}_m$, $\mathbf{B} \not\models s \approx \mathbf{1}$ entails that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$. Because algebras $\mathbf{Z}_k$, where $k \in [7, m] \cup \{\omega\}$ generate variety $\mathcal{V}$, for some $k \in [7, m] \cup \{\omega\}$, identity $s \approx \mathbf{1}$ fails in $\mathbf{Z}_k$.

Also, the only proper subalgebras of algebra $\mathbf{Z}_k, k \in [7, m] \cup \{\omega\}$ are $\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_5$. Indeed, each algebra $\mathbf{Z}_k, k \geq 7$ contains just three types of elements: if $\mathsf{a} \in \mathbf{Z}_k$, then

$$\neg\neg\mathsf{a} = \begin{cases} \mathsf{a} \\ \mathbf{1} \\ \text{neither } \mathsf{a}, \text{ nor } \mathbf{1} \end{cases}$$

If $\neg\neg\mathsf{a} = \mathsf{a}$ and $\mathsf{a} \in \{\mathbf{0}, \mathbf{1}\}$, then a generates (a subalgebra isomorphic to) $\mathbf{Z}_2$, otherwise, a generates $\mathbf{Z}_5$. If $\neg\neg\mathsf{a} = \mathbf{1}$, then a generates $\mathbf{Z}_3$. And, if $\neg\neg\mathsf{a} \neq \mathsf{a}$ and $\neg\neg\mathsf{a} \neq \mathbf{1}$, element a is the generator of $\mathbf{Z}_k$, and hence, a generates whole algebra $\mathbf{Z}_k$.

Thus, variety $\mathcal{V}_m$ contains finite s.i. one-generated algebras only from $\{\mathbf{Z}_k, k \in [7, m] \cup \{2, 3, 5\}\}$. That is, variety $\mathcal{V}_m^{(1)}$ is generated by finite set of finite algebras. Then, variety $\mathcal{V}_m^{(1)}$ is locally finite (see [5, Theorem 10.16]), and therefore, $\mathcal{V}_m^{(1)}$ does not contain infinite finitely-generated algebras. Thus, $\mathbf{F}_{\mathcal{V}_m}(1) \notin \mathcal{V}_m^{(1)}$. □

Corollary 3.4 *The logic of Heyting algebra $\mathcal{Z}_\omega$ enjoys the f.m.p. and it is an assertoric fragment of infinitely many extensions of* WS5 *without the f.m.p.*

4 Free WS5-Algebra on one free generator

The goal of this section is to prove that algebra $\mathbf{Z}$, introduced earlier (see (7)), is free in $\mathcal{M}$, and then to give an alternative intrinsic description of this algebra.

Theorem 4.1 *Algebra* $\mathbf{Z}$ *is freely generated in variety* $\mathcal{M}$ *by element* g*, that is* $\mathbf{Z} \cong \mathbf{F}_{\mathcal{M}}(1)$.

Proof. First, recall from [8, Section 12.2 Theorem 1] that for any variety $\mathcal{V}$, any algebra $\mathbf{A} \in \mathcal{V}$ generated by element $\mathsf{g} \in \mathbf{A}$, if for any identity $t(x) \approx r(x)$,

$$t(\mathsf{g}) = r(\mathsf{g}) \text{ yields } \mathcal{V} \models t(x) \approx r(x),$$

then g is a free generator.

To prove the contrapositive, we take any identity $t(x) \approx r(x)$, such that $\mathcal{M} \not\models t(x) \approx r(x)$. Because variety $\mathcal{M}$ is finitely approximable, there is a finite algebra $\mathbf{A} \in \mathcal{M}$ such that $\mathbf{A} \not\models t(x) \approx r(x)$, that is, for some $\mathsf{a} \in \mathbf{A}$, $t(\mathsf{a}) \neq r(\mathsf{a})$. We can safely assume that a generated $\mathbf{A}$.

By Birkhoff's theorem, algebra $\mathbf{A}$ is (isomorphic to) a subdirect product $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ of s.i. **WS5**-algebras. Hence $\mathbf{A}_k \not\models t(x) \approx r(x)$ for some $1 \leq k \leq n$ and $t(\mathsf{a}_k) \neq r(\mathsf{a}_k)$, where $\mathsf{a}_k = \pi_k(\mathsf{a})$. Let us also note that element a_k generates $\mathbf{A}_k$, for element a generates $\mathbf{A}$ and $\mathbf{A}_k$ is a subdirect factor of $\mathbf{A}$.

Thus, algebra $\mathbf{A}_k$ is finite and s.i. and it is generated by element a_k. Hence, for some $m > 0$, $\mathbf{A}_k \cong \mathbf{Z}_m \in \mathcal{Z}$ and $t(\mathsf{g}_m) \neq r(\mathsf{g}_m)$. Recall that $\mathbf{Z}_m$ is a subdirect factor of algebra $\mathbf{Z}$ and that $\mathsf{g}_m = \pi_m(\mathsf{g})$, which entails $t(\mathsf{g}) \neq r(\mathsf{g})$ and this observation completes the proof. □

Let us give a criterion for an element of $\mathbf{P}$ to belong to $\mathbf{Z}$.

4.1 Leveled Elements

We regard elements of $\mathbf{P}$ as infinite vectors, and let $\pi_j, j > 0$ be a j-th component (j-th projection). For instance, if g is a generator defined by (7), then $\pi_j(\mathsf{g}) = \mathsf{g}_j$.

Definition 4.2 Let $k > 0$ and $m \in \{0, 1, \ldots, \omega\}$. An element $\mathsf{a} \in \mathbf{P}$ is called (k,m)***-leveled***, if for all $i \geq k$,

$$\pi_i(\mathsf{a}) = \mathsf{g}_i^m.$$

Thus, an element $\mathsf{a} \in \mathbf{P}$ is (k,m)-leveled if, starting from k-th component, each component of a is equal to the same degree of the respective generator, that is, a is of form

$$(\mathsf{a}_1, \ldots, \mathsf{a}_{k-1}, \mathsf{g}_k^m, \mathsf{g}_{k+1}^m, \ldots)$$

For instance, g is $(2,2)$-leveled, because each of its components, starting with $\pi_2(\mathsf{g})$, is the 2-nd degree of the respective generator (recall that $\mathsf{a}^2 = \mathsf{a}$).

Definition 4.3 An element $\mathsf{a} \in \mathbf{P}$ is ***leveled***, if it is (k,m)-leveled for some $k > 0$ and $m \in \{0, 1, \ldots, \omega\}$.

For instance, if a is a ***binary element***, that is each component of a is $\mathbf{0}$ or $\mathbf{1}$, then a is leveled if and only if it contains either a finite number of $\mathbf{0}$-components, or a finite number of $\mathbf{1}$-components.

It is obvious that if an element a is (k,m)-leveled, it is (k',m)-leveled for every $k' > k$. We will need a bit more stronger property of leveled elements.

Proposition 4.4 *Let* $\mathsf{a} \in \mathbf{P}$ *be a leveled element. Then for some* $k > 0$ *and* $m \in \{0, 1, \dots, \omega\}$*, either element* a *is* (k, ω)*-leveled, or it is* (k, m)*-leveled and* $\pi_i(\mathsf{a}) < \mathbf{1}$ *for all* $i \geq k$*.*

Proof. Let $\mathsf{a} \in \mathbf{P}$ be a (k, m)-leveled element and $m < \omega$. Without loss of generality, we can assume that $k \geq m + 2$. Then, by Proposition 1.1(b), for all $j \geq k$, we have $\pi_j(\mathsf{a}) < \mathbf{1}$.. □

Proposition 4.5 *Let* $\mathbf{L} \subseteq \mathbf{P}$ *be a subset of all leveled elements of* $\mathbf{P}$*.* $\mathbf{L}$ *forms a subalgebra of* $\mathbf{P}$*.*

Proof. First, let us recall that, by the definition, $\mathsf{a}^0 = \mathbf{0}$ and $\mathsf{a}^\omega = \mathbf{1}$ for any element a. Hence, elements $\mathbf{0_P} = (\mathbf{0}, \mathbf{0}, \dots)$ and $\mathbf{1_P} = (\mathbf{1}, \mathbf{1}, \dots)$ are $(1, 0)$- and $(1, \omega)$-leveled. Thus, $\mathbf{0_P}, \mathbf{1_P} \in \mathbf{L}$, and we need to check only that $\mathbf{L}$ is closed under $\wedge, \vee, \to$ and $\Box$.

Let us start with $\Box$. Suppose that $\mathsf{a} \in \mathbf{L}$ and a is (k, m)-leveled. Then, by Proposition 4.4, we can assume that either $m = \omega$, or for each $j \geq k$, $\pi_j(\mathsf{a}) < \mathbf{1}$. Hence, either $\Box\mathsf{a}$ is (k, ω)-leveled, or $\Box\mathsf{a}$ is $(k, 0)$-leveled (because each $\mathbf{Z}_j$ is s.i., and therefore, $\Box\mathsf{b} = \mathbf{0}_{\mathbf{Z}_j}$ as long as $\mathsf{b} < \mathbf{1}_{\mathbf{Z}_j}$).

Now, suppose that $\mathsf{a}, \mathsf{b} \in \mathbf{L}$ are leveled elements. Without loss of generality we can assume that a is (k, m)-leveled and b is (k, n)-leveled, where $k > 1$ and $m, n \geq k + 2$. If $m = \omega$ or $n = \omega$, the statement is trivial.

Suppose that $m \neq \omega$, $n \neq \omega$ and $\circ \in \{\wedge, \vee, \to\}$. Then for each $j \geq k$ we have

$$\pi_j(\mathsf{a} \circ \mathsf{b}) = \mathsf{g}_j^m \circ \mathsf{g}_j^n,$$

and for some s, because g_j is a generator of $\mathcal{Z}_j$, we have

$$\mathsf{g}_j^m \circ \mathsf{g}_j^n = \mathsf{g}_j^s.$$

Hence, $\mathsf{a} \circ \mathsf{b}$ is (k, s)-leveled, and this observation completes the proof. □

Now we are in a position to give a intrinsic description of $\mathbf{Z}$.

Theorem 4.6 *Algebra* $\mathbf{Z}$ *is a subalgebra of* $\mathbf{P}$ *consisting of all leveled elements.*

Proof. Let $\mathbf{L} \subseteq \mathbf{P}$ be a set of all leveled elements of $\mathbf{P}$. The generator g is leveled and, therefore $\mathsf{g} \in \mathbf{L}$. By Proposition 4.5, $\mathbf{L}$ is a subalgebra of $\mathbf{P}$, hence, $\mathbf{Z} \subseteq \mathbf{L}$.

To prove $\mathbf{L} \subseteq \mathbf{Z}$ it is sufficient to demonstrate that every leveled element can be expressed via g. More precisely, we will demonstrate that for every element $\mathsf{a} \in \mathbf{L}$ there is a term $t(x)$ such that $\mathsf{a} = t(\mathsf{g})$, where g is a generator of $\mathbf{Z}$.

Let us consider elements $\mathsf{s}_m, m = 1, 2, \dots$ defined as in Lemma 2.2:

$$\pi_j(\mathsf{s}_m) = \begin{cases} \mathbf{1}, & \text{if } j = m; \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

It is clear that each element s_m is leveled, that is, $\mathsf{s}_m \in \mathbf{L}$ as well as $\mathsf{g} \in \mathbf{L}$. On the other hand, by virtue of Lemma 2.2, $\mathsf{s}_m \in \mathbf{Z}$ and $\mathsf{g} \in \mathbf{Z}$. Because $\mathbf{Z}$ is closed under WS5-operations, to prove $\mathbf{L} \subseteq \mathbf{Z}$ it is sufficient to show that each element $\mathsf{a} \in \mathbf{L}$ can be expressed via g and elements from $\{\mathsf{s}_k, k > 0\}$.

Let $\mathsf{a} \in \mathbf{L}$. Recall that $\mathsf{g} = (\mathsf{g}_1, \mathsf{g}_2, \dots)$ is a generator of $\mathbf{Z}$. Because a is leveled, a is (k, m)-leveled for some k and m. Hence, a is an element of form

$$\mathsf{a} = (\mathsf{g}_1^{m_1}, \dots, \mathsf{g}_{k-1}^{m_{k-1}}, \mathsf{g}_k^m, \mathsf{g}_{k+1}^m, \dots).$$

Let $\mathsf{s}_i \in \mathbf{P}, i > 0$, that is,

$$\mathsf{s}_i = (\underbrace{\mathbf{0}, \dots, \mathbf{0}}_{i-1 \text{ times}}, \mathbf{1}, \mathbf{0}, \dots).$$

Hence, for each $k > 0$,

$$\mathsf{s}_1 \vee \dots \vee \mathsf{s}_k = (\underbrace{\mathbf{1}, \dots, \mathbf{1}}_{k \text{ times}}, \mathbf{0}, \dots) \text{ and } \neg(\mathsf{s}_1 \vee \dots \vee \mathsf{s}_k) = (\underbrace{\mathbf{0}, \dots, \mathbf{0}}_{k \text{ times}}, \mathbf{1}, \dots).$$

Therefore we can express a in the following way:

$$\mathsf{a} = (\mathsf{s}_1 \wedge \mathsf{g}^{m_1}) \vee \dots \vee (\mathsf{s}_{k-1} \wedge \mathsf{g}^{m_{k-1}}) \vee (\neg(\mathsf{s}_1 \vee \dots \vee \mathsf{s}_{k-1}) \wedge \mathsf{g}^m). \tag{10}$$

And this completed the proof. □

4.2 Some Properties of $\mathbf{F}_{\mathcal{M}}(1)$

First, let as take a closer look at h-reduct of $\mathbf{F}_{\mathcal{M}}(1)$.

We say that a **WS5**-algebra $\mathbf{A}$ is ***finitely h-generated*** if h-reduct of $\mathbf{A}$ is finitely generated as Heyting algebra. In particular, any finite **WS5**-algebra is finitely h-generated.

Let us observe that (10) entails that elements g and $\mathsf{s}_i, i > 0$ generate h-reduct of $\mathbf{Z}$ as Heyting algebra. On the other hand, the following holds.

Corollary 4.7 $\mathbf{Z}$ *is non-finitely h-generated.*

Proof. Indeed, let us take any finite set of elements $\mathsf{a}_1, \dots, \mathsf{a}_n \in \mathbf{Z}$. Then, for every $0 < i \leq n$ every element a_i is (k_i, m_i)-leveled. Let $k = \max(k_i)$. Then every element a_i is (k, m_i)-leveled. Any Heyting operation over (k, m_i)-leveled elements yields a (k, m')-leveled element. So, for instance, element $\mathsf{s}_k \in \mathbf{Z}$ is not (k, m)-leveled for any m (it is $(k+1, 0)$-leveled) and, hence, it cannot by expressed via $\mathsf{a}_i, i \leq n$ and Heyting operations. Thus, elements $\mathsf{a}_1, \dots, \mathsf{a}_n$ do not generate h-reduct of $\mathbf{Z}$. □

Now, let us note that

$$\mathsf{s}_1 < \mathsf{s}_1 \vee \mathsf{s}_2 < \mathsf{s}_1 \vee \mathsf{s}_2 \vee \mathsf{s}_3 \dots, \text{ and subsequently, } \neg \mathsf{s}_1 > \neg(\mathsf{s}_1 \vee \mathsf{s}_2) > \neg(\mathsf{s}_1 \vee \mathsf{s}_2 \vee \mathsf{s}_3) \dots$$

Hence, the following holds.

Corollary 4.8 *Algebra* $\mathbf{F}_{\mathcal{M}}(1)$ *has infinite ascending and descending chains of open elements.*

An element $\mathsf{a} \in \mathbf{A}$ is said to be an ***atom*** of $\mathbf{A}$ if $\mathbf{0} < \mathsf{a}$ and there are no elements strongly between $\mathbf{0}$ and a. An algebra $\mathbf{A}$ is said to be ***atomic*** if there is a set $\mathsf{A} \subseteq \mathbf{A}$ of atoms such that for every $\mathsf{b} \in \mathbf{A}$ if $\mathsf{b} > \mathbf{0}$, then $\mathsf{b} \geq \mathsf{a}$ for some $\mathsf{a} \in \mathsf{A}$.

Clearly, every finite algebra is atomic. Let us observe that algebra $\mathbf{Z}_2$ has the only atom: $\mathbf{1}$, while algebra $\mathbf{Z}_3$ has the only atom: its generator. Each algebra $\mathbf{Z}_m, m > 3$ has exactly two atoms: its generator g_m and $\neg\mathsf{g}_m$. Hence, the following holds.

Corollary 4.9 (comp. [6, Theorem 5.2]) *Algebra* $\mathbf{F}_{\mathcal{M}}(1)$ *is atomic and has infinitely many atoms.*

Proof. The elements $\mathsf{s}_1, \mathsf{s}_2, \mathsf{s}_3 \wedge \mathsf{g}$ and $\mathsf{s}_m \wedge \mathsf{g}, \mathsf{s}_m \wedge \neg\mathsf{g}, m > 3$ form the complete list of atoms of $\mathbf{Z}$. It is not hard to see that for any element $\mathsf{a} \in \mathbf{Z}$ (and even for every element from $\mathbf{P}$), if $\mathsf{a} > 0$, there is an atom a' from the above list such that $\mathsf{a}' \leq \mathsf{a}$. □

Corollary 4.10 *Algebra* $\mathbf{F}_{\mathcal{M}}(1)$ *contains a single s.i. subalgebra, namely,* $\mathbf{Z}_2$.

Proof. Suppose that $\mathbf{A}$ is an s.i. subalgebra of $\mathbf{Z}$ and $\mathbf{A}$ has more than two elements. Let $\mathsf{a} \in \mathbf{A}$ and $\mathbf{0} < \mathsf{a} < \mathbf{1}$. There are just two possibilities: either $\neg\mathsf{a} = \mathbf{0}$, or $\neg\mathsf{a} > \mathbf{0}$.

Assume that $\neg\mathsf{a} = \mathbf{0}$. Observe, that $\pi_1(\mathsf{a}) \in \{\mathbf{0}, \mathbf{1}\}$ and, hence, $\neg\mathsf{a} = \mathbf{0}$ entails $\pi_1(\mathsf{a}) = \mathbf{1}$. Therefore, $\pi_1(\Box\mathsf{a}) = \Box(\pi_1(\mathsf{a})) = \mathbf{1}$, that is, $\Box\mathsf{a} \neq \mathbf{0}$, which is impossible, for $\mathbf{A}$ is s.i. and $\mathsf{a} < \mathbf{1}$.

Assume that $\neg\mathsf{a} > \mathbf{0}$. Let us note that $\neg\mathsf{a} < \mathbf{1}$, for $\mathsf{a} > \mathbf{0}$. Thus, we have $\mathbf{0} < \mathsf{a} < \mathbf{1}$ and $\mathbf{0} < \neg\mathsf{a} < \mathbf{1}$ and, hence, $\mathbf{0} \leq \Box\mathsf{a} < \mathbf{1}$ and $\mathbf{0} \leq \Box\neg\mathsf{a} < \mathbf{1}$. If we prove that $\Box\mathsf{a} > \mathbf{0}$ or $\Box\neg\mathsf{a} > \mathbf{0}$, we will be able to conclude that $\mathbf{A}$ is not s.i., because it contains more than two open elements.

Indeed, let us consider the first two projections of a. Note that $\pi_1(\mathsf{a}), \pi_2(\mathsf{a}) \in \{\mathbf{0}, \mathbf{1}\}$. Hence, there are just four distinct combinations of their values, so we have

$\Box(\pi_1(\mathsf{a}), \pi_2(\mathsf{a}), \dots)$	$\Box(\neg\pi_1(\mathsf{a}), \neg\pi_2(\mathsf{a}), \dots)$
$(\mathbf{0}, \mathbf{0}, \dots)$	$(\mathbf{1}, \mathbf{1}, \dots)$
$(\mathbf{0}, \mathbf{1}, \dots)$	$(\mathbf{1}, \mathbf{0}, \dots)$
$(\mathbf{1}, \mathbf{0}, \dots)$	$(\mathbf{0}, \mathbf{1}, \dots)$
$(\mathbf{1}, \mathbf{1}, \dots)$	$(\mathbf{0}, \mathbf{0}, \dots)$

It is clear that in any case either $\Box\mathsf{a} > \mathbf{0}$, or $\Box\neg\mathsf{a} > \mathbf{0}$. □

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