Fully Arbitrary Public Announcements

Hans van Ditmarsch

LORIA - CNRS / University of Lorraine

Wiebe van der Hoek

University of Liverpool

Louwe B. Kuijer

University of Liverpool
LORIA - CNRS / University of Lorraine

Abstract

In Arbitrary Public Announcement Logic (APAL) an operator $\Box$ is used. The intended meaning of $\Box\varphi$ is "for every ψ we have $[\psi]\varphi$." However, for technical reasons the semantics of APAL do not entirely match the intended meaning: in APAL the formula $\Box\varphi$ holds if and only if for every $\Box$-free ψ we have $[\psi]\varphi$. Here we introduce Fully Arbitrary Public Announcement Logic (F-APAL), where the semantics do match the intended meaning: in F-APAL the formula $\Box\varphi$ holds if and only if for every ψ we have $[\psi]\varphi$.

Keywords: Dynamic Epistemic Logic, Public Announcements, Arbitrary Public Announcements.

1 Introduction

One line of research in dynamic epistemic logic is to add "arbitrary" versions of dynamic operators. Examples of such logics include Arbitrary Public Announcement Logic [4,5], Group Announcement Logic [1], Arbitrary Action Model Logic [14], Refinement Modal Logic [9] and Arbitrary Arrow Update Logic [11]. The intuition behind these arbitrary operators is that they represent universal quantification over their non-arbitrary counterpart.

As the title suggests, we will focus on Arbitrary Public Announcement Logic (APAL). APAL is based on Public Announcement Logic (PAL) [6], but adds an extra operator $\Box$. [1] Intuitively, $\Box\varphi$ is intended to mean "after every public

[1] The symbol $\Box_a$ is also often used for "agent a knows that..." Here we use K_a to denote the knowledge operator, and reserve $\Box$ for arbitrary public announcement operators.

announcement ψ, φ will hold." So we would like $\Box$ to satisfy the following property:

$$\mathcal{M}, w \models \Box\varphi \Leftrightarrow \forall\psi \in \mathcal{L}_{APAL} : \mathcal{M}, w \models [\psi]\varphi \tag{1}$$

where $\mathcal{L}_{APAL}$ is the language of APAL. Unfortunately, (1) is circular: if we want to use it to determine the value of $\Box\varphi$, then we have to determine whether $[\psi]\varphi$ holds for all $\psi \in \mathcal{L}_{APAL}$. So in particular, we have to determine whether $[\Box\varphi]\varphi$ holds, which in turn requires us to determine whether $\Box\varphi$ holds. But that's where we started out; if we want to find out whether $\Box\varphi$ holds, we first have to know whether $\Box\varphi$ holds. In order to avoid this circularity, [4,5] define $\Box$ not by (1) but by

$$\mathcal{M}, w \models \Box\varphi \Leftrightarrow \forall\psi \in \mathcal{L}_{PAL} : \mathcal{M}, w \models [\psi]\varphi \tag{2}$$

where $\mathcal{L}_{PAL}$ is the set of formulas that do not themselves contain the operator $\Box$. Since (2) is non-circular, it can be used as a definition of $\Box$.

If we could have used (1) as definition of $\Box$, then $\Box$ would trivially have satisfied (1). But we cannot. Still, not all hope is lost. Even though (1) is not suitable as a *definition* of $\Box$, it might be a *property* of $\Box$ under another definition. Unfortunately, as shown in [16], (2) is not such a definition: in APAL, $\Box$ does not satisfy (1).

Let us reflect briefly on what it means for $\Box$ not to satisfy (1). The operator $\Box$ is called an "arbitrary public announcement." This name is justified by the intuition that $\Box\varphi$ is supposed to hold if and only if $[\psi]\varphi$ holds for every public announcement $[\psi]$. So $\Box$ is supposed to represent any announcement $[\psi]$; an arbitrary announcement indeed. But there are φ and ψ in APAL such that $\Box\varphi$ holds but $[\psi]\varphi$ does not. In other words, $\Box$ in APAL does not represent every possible announcement $[\psi]$, so it is not a fully arbitrary public announcement.

In this paper, we introduce Fully Arbitrary Public Announcement Logic (F-APAL). The idea behind F-APAL is that our $\Box$ operator, unlike the APAL one, will represent a fully arbitrary public announcement. More precisely, we will define a logical language $\mathcal{L}$ that has $\Box$ as an operator and semantics for $\mathcal{L}$ such that

$$\mathcal{M}, w \models \Box\varphi \Leftrightarrow \forall\psi \in \mathcal{L} : \mathcal{M}, w \models [\psi]\varphi \tag{*}$$

is satisfied. The price we pay for this fully arbitrary announcement operator $\Box$ is that our language $\mathcal{L}$ contains multiple auxiliary operators. If fact, we use a proper class of indexed auxiliary operators: $\{\Box_\alpha \mid \alpha \text{ is an ordinal}\}$.

The remainder of this paper is structured as follows. First, in Section 2, we briefly discuss the problems with circularity, and show that the circularity in (*) is vicious, so (*) cannot be used as a definition of $\Box$. Then, in Section 3, we introduce the language and semantics of our logic F-APAL. In Section 4, we show that $\Box$ satisfies (*) in F-APAL. In Section 5, we prove a few other properties of F-APAL. Finally, in Section 6, we discuss the relation between F-APAL and fixed points.

2 Circularity

The easiest way to obtain semantics for $\Box$ that satisfy (*), would be to use (*) as a definition of $\Box$. Certainly, (*) is circular, and circular definitions are generally frowned upon. But not all circularity is vicious, and non-viciously circular properties can be used as definitions. For example, many fixed points can be given a circular definition. The question, then, is whether the circularity in (*) is vicious.

We are working in a two-valued modal logic. So a property can be used as a definition for an operator X of arity n if and only if, for each pointed model $\mathcal{M}, w$ and each tuple $\varphi_1, \cdots, \varphi_n$, exactly one of $\mathcal{M}, w \models X(\varphi_1, \cdots, \varphi_n)$ and $\mathcal{M}, w \not\models X(\varphi_1, \cdots, \varphi_n)$ satisfies the property. There are two ways in which a property can fail to be suitable as a definition. Firstly, the property can be *inconsistent*, and allow neither truth value for a sentence in some pointed model. A typical example of an inconsistent property is the (modal) liar:[2]

$$\text{“This sentence is false (in this pointed model).”} \tag{3}$$

If we suppose that (3) is true, then the claim made in (3) is is true, so (3) is false. In two-valued logic this is a contradiction, so (3) cannot be true. If, on the other hand, we suppose that (3) is false, then the claim made in (3) is false, so (3) is not false. This too is a contradiction, so (3) cannot be false either. It follows that (3) allows neither truth value, so it is inconsistent.

Secondly, a property can be *underdetermined*, and allow both truth values for a sentence in some pointed model. A typical example of an underdetermined sentence is the (modal) truth teller:

$$\text{“This sentence is true (in this pointed model).”} \tag{4}$$

If we suppose that (4) is true, then the claim made in (4) is true, so (4) is true. This does not lead to a contradiction, so we can consistently say that (4) is true. If, on the other hand, we suppose that (4) is false, then the claim made in (4) is false, so (4) is not true and therefore false. Again, we do not arrive at a contradiction, so we can consistently say that (4) is false. We are working in a two-valued logic, so we cannot assign (4) both truth values at the same time. We are, however, free to choose either of the truth values. So (4) is underdetermined.

If a circular property is inconsistent or underdetermined, then the circularity is vicious. But that does not mean that the two kinds of vicious circularity are equally bad. In both cases, we cannot use the circular property as a definition. But with an underdetermined property we can try to find a different definition that satisfies the property, whereas with an inconsistent property we have no choice but to give up.

Sadly, the circularity in (*) turns out to be vicious. Fortunately, however, it exhibits the less problematic kind of viciousness: (*) is underdetermined but

[2] Note that, in the terminology used above, the liar sentence is a nullary operator.

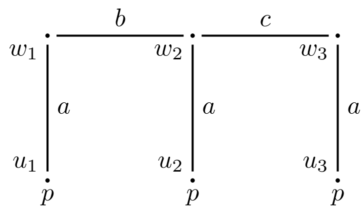

Fig. 1. The S5 model $\mathcal{M}_{Un}$. Reflexive arrows are not drawn, for reasons of clarity.

consistent. So while we cannot use (*) as a definition of $\Box$, we may be able to find semantics for $\Box$ that satisfy (*). We prove the consistency of (*) in Sections 3 and 4, by constructing semantics that satisfy it. In this section, we show that (*) is underdetermined.

A slight complication is that whether (*) is underdetermined may depend on the other connectives that are present in the language. We have not defined the language $\mathcal{L}$ yet, so we will use a smaller language in this section. We assume that our language uses only propositional atoms and the connectives $\neg, \vee, K_a, [\psi]$ and $\Box$. All except $\Box$ are given the usual semantics,[3] we make no assumptions about the semantics of $\Box$. Additionally, for ease of notation, we use $\wedge, \hat{K}_a, \langle\psi\rangle$ and $\Diamond$ as duals of $\vee, K_a, [\psi]$ and $\Box$, respectively.

In this section we do not consider the auxiliary operators $\Box_\alpha$. This is only for reasons of clarity of presentation, however. If we added the $\Box_\alpha$ operators (with the semantics as given in Section 3), then the proofs given in this section would still work with only very minor modifications.

Consider the S5 model $\mathcal{M}_{Un}$ shown in Figure 1 and let $\xi := \hat{K}_a p \wedge \hat{K}_b K_a \neg p \wedge \hat{K}_c K_a \neg p$. So ξ holds if and only if (1) there is an a-accessible p world, (2) there is a b-accessible world where there is no a-accessible p world and (3) there is a c-accessible world where there is no a-accessible p world. Regardless of which worlds of $\mathcal{M}_{Un}$ we retain or eliminate, the only world in which ξ can possibly hold is w_2. And for ξ to hold in w_2 it must be the case that exactly w_1, w_2, w_3 and u_2 are retained while u_1 and u_3 are eliminated.

In the next two lemmas, we show that $\mathcal{M}_{Un}, w_2 \not\models \Diamond\xi$ and $\mathcal{M}_{Un}, w_2 \models \Diamond\xi$ are both consistent with (*). In order to do this, we use the following observation. By the definition of public announcements, we have $\mathcal{M}, w \models [\psi]\varphi$ if and only if $\mathcal{M}, w \models \psi$ and $\mathcal{M}_\psi, w \models \varphi$, where $\mathcal{M}_\psi$ is the restriction of $\mathcal{M}$ to those worlds satisfy ψ. This implies that, for every ψ_1, ψ_2 that have the same extension, $\mathcal{M}, w \models [\psi_1]\varphi$ if and only if $\mathcal{M}, w \models [\psi_2]\varphi$. Now, consider the right hand side of (*): $\forall \psi \in \mathcal{L} : \mathcal{M}, w \models [\psi]\varphi$. Because it is only the extension of ψ that matters, that is equivalent to

$$\forall x \in \{[\![\psi]\!]_{\mathcal{M}} \mid \psi \in \mathcal{L}\} : \mathcal{M}_x, w \models \varphi,$$

where $\mathcal{M}_x$ is the restriction of $\mathcal{M}$ to x and, by convention, $\mathcal{M}_x, w \models \varphi$ for

[3] We assume that the reader is familiar with the standard semantics for these operators. If not, see Section 3.

every φ if $w \notin x$.[4] So (*) is equivalent to

$$\mathcal{M}, w \models \Box\varphi \Leftrightarrow \forall x \in \{[\![\psi]\!]_{\mathcal{M}} \mid \psi \in \mathcal{L}\} : \mathcal{M}_x, w \models \varphi. \tag{**}$$

In the following two lemmas, we start by defining a set X. We then define the semantics for $\Box$ by

$$\mathcal{M}, w \models \Box\varphi \Leftrightarrow \forall x \in X : \mathcal{M}_x, w \models \varphi. \tag{5}$$

We then show that, under these semantics, X is exactly the set of extensions on $\mathcal{M}$, so $X = \{[\![\psi]\!]_{\mathcal{M}} \mid \psi \in \mathcal{L}\}$. It follows immediately that (**) is satisfied, and therefore (*) is satisfied as well.

Lemma 2.1 *There is a valuation for $\Diamond\xi$ that is consistent with* (*) *such that $\mathcal{M}_{Un}, w_2 \not\models \Diamond\xi$.*

Proof. Take $X = \{\varnothing, \{w_1, w_2, w_3\}, \{u_1, u_2, u_3\}, \{w_1, w_2, w_3, u_1, u_2, u_3\}\}$, and define $\Box$ by (5). We show that X is exactly the set of extension on $\mathcal{M}_{Un}$. First, note that for every $x \in X$ there is a formula ψ such that $x = [\![\psi]\!]$: we have $\varnothing = [\![p \wedge \neg p]\!]$, $\{w_1, w_2, w_3\} = [\![\neg p]\!]$, $\{u_1, u_2, u_3\} = [\![p]\!]$ and $\{w_1, w_2, w_3, u_1, u_2, u_3\} = [\![p \vee \neg p]\!]$.

Left to show is that for every formula ψ, there is some $x \in X$ such that $x = [\![\psi]\!]$. So we need to show that ψ cannot distinguish between the three columns of the model, i.e. there is no formula that can distinguish between w_1, w_2 and w_3 or between u_1, u_2 and u_3. We do this by induction. First, as base case, note that there is no atomic formula that can distinguish between the three columns. Then, assume as induction hypothesis that ψ is not atomic and that no strict subformula of ψ can distinguish between the columns. We continue by case distinction on the main connective of ψ.

- Suppose the main connective of ψ is not $\Box$. Then ψ can only distinguish between the columns if at least one of its strict subformulas can. By the induction hypothesis this is not the case.
- Suppose $\psi = \Box\psi'$. Then ψ can distinguish between two columns only if ψ' distinguishes between the columns in $(\mathcal{M}_{Un})_x$ for some $x \in X$. For $x = \varnothing$ this is trivially not the case, by convention we have $(\mathcal{M}_{Un})_\varnothing, v \models \psi$ for every world v, since $v \notin \varnothing$.

  For $x = \{w_1, w_2, w_3\}$ and $x = \{u_1, u_2, u_3\}$ we have that all worlds in the resulting model $(\mathcal{M}_{Un})_x$ agree on all propositional variables. So no formula using atoms, $\neg, \vee$ and K_a can distinguish between any two worlds. Furthermore, this property is retained in submodels of $(\mathcal{M}_{Un})_x$, so the operators $[\chi]$ and $\Box$ cannot help distinguish between any worlds either. In particular, ψ' cannot distinguish between the columns.

  Finally, for $x = \{w_1, w_2, w_3, u_1, u_2, u_3\}$ we have $(\mathcal{M}_{Un})_x = \mathcal{M}_{Un}$. So by the induction hypothesis ψ' cannot distinguish between the columns.

[4] This convention corresponds to the convention in public announcement logic that $\mathcal{M}, w \models [\psi]\varphi$ for all φ if $\mathcal{M}, w \not\models \psi$.

For every $x \in X$, we have seen that ψ' cannot distinguish between the columns of $(\mathcal{M}_{Un})_x$. So $\psi = \Box\psi'$ cannot distinguish between the columns of $\mathcal{M}_{Un}$.

So in both cases ψ cannot distinguish between the columns of the model. This completes the induction step, thereby showing that for every ψ there is an $x \in X$ such that $[\![\psi]\!] = x$. It follows that X is indeed the set of extensions on $\mathcal{M}_{Un}$, so (*) is satisfied.

Furthermore, we have $\mathcal{M}_{Un}, w_2 \models \Diamond\xi$ if and only if $\{w_1, w_2, w_3, u_2\} \in X$. This is not the case, so $\mathcal{M}_{Un}, w_2 \not\models \Diamond\xi$ is consistent with (*).

□

Lemma 2.2 *There is a valuation for $\Diamond\xi$ that is consistent with* (*) *such that $\mathcal{M}_{Un}, w_2 \models \Diamond\xi$.*

Proof. Take $X = 2^{\{w_1,w_2,w_2,u_1,u_2,u_3\}}$, and define $\Box$ by (5). We show that X is the set of extensions on $\mathcal{M}_{Un}$. First, note that for every ψ we trivially have $[\![\psi]\!] \in X$. Left to show is that for every $x \in X$ there is some formula ψ such that $x = [\![\psi]\!]$.

We have $\mathcal{M}_{Un}, w_2 \models \Diamond\xi$, since $\{w_1, w_2, w_3, u_2\} \in X$. As discussed above, w_2 is also the only world where $\Diamond\xi$ holds. But then every world v of $\mathcal{M}_{Un}$ can be uniquely identified by some formula δ_v. Specifically, we have $\delta_{w_1} = \neg\Diamond\xi \wedge \hat{K}_b\Diamond\xi$, $\delta_{w_2} = \Diamond\xi$, $\delta_{w_3} = \neg\Diamond\xi \wedge \hat{K}_c\Diamond\xi$ and $\delta_{u_i} = p \wedge \hat{K}_a\delta_{w_i}$ for $i \in \{1, 2, 3\}$. Every $x \in X$ is then the extension of the appropriate disjunction of such δ_v. So X is indeed the set of extensions, which implies that $\mathcal{M}_{Un}, w_2 \models \Diamond\xi$ is consistent with (*). □

Corollary 2.3 *The characterization* (*) *is underdetermined.*

We should note that, although $\mathcal{M}_{Un}, w_2 \not\models \Diamond\xi$ and $\mathcal{M}_{Un}, w_2 \models \Diamond\xi$ are both consistent with (*), this does not mean that we consider both to be equally good solutions. The worlds w_1, w_2 and w_3 in $\mathcal{M}_{Un}$ are bisimilar to each other. In a well behaved modal logic we would therefore expect them to satisfy the same formulas. If we take $\mathcal{M}_{Un}, w_2 \not\models \Diamond\xi$ then they do indeed satisfy the same formulas. But if we take $\mathcal{M}_{Un}, w_2 \models \Diamond\xi$ then $\Diamond\xi$ distinguishes between w_2 on the one hand and w_1 and w_3 on the other. There seems to be no compelling reason to allow the $\Diamond$ operator to break bisimilarity, so we prefer $\mathcal{M}_{Un}, w_2 \not\models \Diamond\xi$ over $\mathcal{M}_{Un}, w_2 \models \Diamond\xi$. Fortunately, it will turn out that in our semantics for F-APAL we have $\mathcal{M}_{Un}, w_2 \not\models \Diamond\xi$.

3 Language and Semantics

In the previous section we showed that (*) is underdetermined, and therefore not suitable as a definition. If we want $\Box$ to satisfy (*), we will have to find deterministic semantics for $\Box$ that satisfy (*). In this section we introduce such semantics, in the next section we prove that the semantics satisfy (*). Our logic F-APAL uses ordinals, so before defining the language and semantics of F-APAL we give a brief reminder of the properties of ordinals that we need. A more thorough introduction to ordinals can be found in most textbooks about set theory, see for example [15]. [5] The following two definitions are as usual.

Definition 3.1 A set x is *transitive* if for all $y \in x$ and $z \in y$ we have $z \in x$.

Definition 3.2 A set α is an *ordinal number* if α is transitive and, for all $\beta \in \alpha$, β is a transitive set. The class $\{\alpha \mid \alpha \text{ is an ordinal number}\}$ of all ordinal numbers is denoted Ord.

If α and β are ordinals numbers, we write $\alpha < \beta$ if $\alpha \in \beta$ and $\alpha \leq \beta$ if $\alpha < \beta$ or $\alpha = \beta$. Furthermore, we write $\alpha + 1$ for the set $\{\alpha\} \cup \alpha$.

We often omit the word "number" and speak simply of an ordinal α. We also follow the usual convention of using the natural numbers to denote the finite ordinals; 0 represents $\varnothing$, 1 represents $\varnothing + 1 = \{\varnothing\}$, and so on.

We will use a few relatively well known properties of ordinal numbers that we state here without proof.

Lemma 3.3 *The following properties hold.*

- Ord *is not a set, it is a proper class,*
- *for any* $\alpha, \beta \in \mathrm{Ord}$*, either* $\alpha \leq \beta$ *or* $\beta \leq \alpha$*,*
- *for any set* X *of ordinals, the set* $\sup(X) := \bigcup_{\alpha \in X} \alpha$ *is an ordinal, and* $\alpha \leq \sup(X)$ *for all* $\alpha \in X$*,*
- *for any class* X *of ordinals, there is an ordinal* $\min(X) \in X$ *such that* $\min(X) \leq \alpha$ *for all* $\alpha \in X$*.*

This finishes our very brief discussion of ordinals. Let us continue by defining our language $\mathcal{L}$.

Definition 3.4 Let a countable set $\mathcal{P}$ of propositional variables and a finite set $\mathcal{A}$ of agents be given. The language $\mathcal{L}(\mathcal{P}, \mathcal{A})$ is given by the following normal form:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid K_a\varphi \mid [\varphi]\varphi \mid \Box_\alpha\varphi \mid \Box\varphi$$

where $p \in \mathcal{P}$, $a \in \mathcal{A}$ and $\alpha \in \mathrm{Ord}$.

For $\alpha \in \mathrm{Ord}$, the sub-language $\mathcal{L}_\alpha(\mathcal{P}, \mathcal{A})$ is the class of formulas that contain neither $\Box$ nor $\Box_\beta$ with $\beta \geq \alpha$.

We write $\mathcal{L}$ and $\mathcal{L}_\alpha$ for $\mathcal{L}(\mathcal{P}, \mathcal{A})$ and $\mathcal{L}_\alpha(\mathcal{P}, \mathcal{A})$ respectively where this should not cause confusion. Furthermore, we use $\wedge$, $\hat{K}_a$, $\langle\varphi\rangle$, $\Diamond_\alpha$ and $\Diamond$ in the usual way as abbreviations.

Note that $\mathcal{L}$ is a proper class. It is unusual for a logic to have a proper class of formulas, and it has certain consequences that F-APAL does. For example, the validities of F-APAL are trivially not recursively enumerable—or enumerable at all. For the results presented in this paper, however, the fact that $\mathcal{L}$ is a proper class does not give any trouble. With the exception of the proofs that depend on $\mathcal{L}$ being a proper class, all proofs proceed in the same way as they would have if $\mathcal{L}$ had been a set.

[5] Alternatively, see [7] for an introduction to ordinals and circularity.

We evaluate our language on S5-models. The choice of this class of models is more for historical reasons than practical ones: the original APAL papers [4,5] use S5 models and we follow their example, but very few of the proofs in this paper depend on the fact that we are working in S5.

Definition 3.5 A *model* $\mathcal{M}$ is a triple $\mathcal{M} = (W, R, V)$ where W is a set of worlds, $R : \mathcal{A} \to \wp(W \times W)$ assigns to each agent an equivalence relation on W and $V : \mathcal{P} \to \wp W$. A pointed model is a pair $\mathcal{M}, w$ where $\mathcal{M} = (W, R, V)$ is a model and $w \in W$. We write $R_a(w)$ for $\{w' \mid (w, w') \in R(a)\}$.

The semantics for F-APAL are as follows.

Definition 3.6 The satisfaction relation $\models$ is given recursively by

$$\begin{array}{lll}
\mathcal{M}, w \models p & \Leftrightarrow & w \in V(p), \\
\mathcal{M}, w \models \neg\varphi & \Leftrightarrow & \mathcal{M}, w \not\models \varphi, \\
\mathcal{M}, w \models \varphi_1 \vee \varphi_2 & \Leftrightarrow & \mathcal{M}, w \models \varphi_1 \text{ or } \mathcal{M}, w \models \varphi_2, \\
\mathcal{M}, w \models K_a\varphi & \Leftrightarrow & \mathcal{M}, w' \models \varphi \text{ for all } w' \in R_a(w), \\
\mathcal{M}, w \models [\varphi_1]\varphi_2 & \Leftrightarrow & \mathcal{M}, w \not\models \varphi_1 \text{ or } \mathcal{M}_{\varphi_1}, w \models \varphi_2, \\
\mathcal{M}, w \models \Box_\alpha\varphi & \Leftrightarrow & \mathcal{M}, w \models [\psi]\varphi \text{ for all } \psi \in \mathcal{L}_\alpha, \\
\mathcal{M}, w \models \Box\varphi & \Leftrightarrow & \mathcal{M}, w \models \Box_\alpha\varphi \text{ for all } \alpha \in \mathrm{Ord}.
\end{array}$$

Where $\mathcal{M}_\varphi$ is given by $\mathcal{M}_\varphi = (W_\varphi, R_\varphi, V_\varphi)$, $W_\varphi = \{w \in W \mid \mathcal{M}, w \models \varphi\}$, $R_\varphi(a) = R(a) \cap (W_\varphi \times W_\varphi)$ and $V_\varphi(p) = V(p) \cap W_\varphi$.

We write $\mathcal{M} \models \varphi$ if $\mathcal{M}, w \models \varphi$ for every world w of $\mathcal{M}$, and $\models \varphi$ if $\mathcal{M} \models \varphi$ for every model $\mathcal{M}$.

Note that $\mathcal{L}_0$ is the language without any $\Box_\alpha$ or $\Box$ operators, so $\mathcal{L}_0 = \mathcal{L}_{PAL}$. The operator $\Box_0$ quantifies over all announcements from $\mathcal{L}_0 = \mathcal{L}_{PAL}$, so $\Box_0$ has the same semantics as the APAL operator from the original APAL papers [4,5]. Thus, APAL is embedded in F-APAL as the fragment $\mathcal{L}_1$.

4 Fully Arbitrary Public Announcements

In this section we show that $\Box$, as defined above, is a fully arbitrary public announcement. So we show that $\Box$ satisfies (*). Before we can do so, however, we need a few auxiliary definitions and one lemma.

Definition 4.1 Let $\varphi \in \mathcal{L}$ and $\beta \in \mathrm{Ord}$. Then $\downarrow_\beta(\varphi) \in \mathcal{L}_{\beta+1}$ is the formula obtained by replacing all occurrences of $\Box$ and $\Box_\alpha$ where $\alpha > \beta$ by $\Box_\beta$.

Definition 4.2 Let $\mathcal{M} = (W, R, V)$ be a model, $\varphi \in \mathcal{L}$ and let α be an ordinal. We say that α *approximates* Ord for φ on $\mathcal{M}$ if for every submodel $\mathcal{M}' = (W', R', V')$ of $\mathcal{M}$, every $w \in W'$, every subformula ψ of φ and every $\beta \geq \alpha$ we have

$$\mathcal{M}', w \models \psi \Leftrightarrow \mathcal{M}', w \models \downarrow_\beta(\psi).$$

We write $\mathrm{Approx}(\mathcal{M}, \varphi)$ for the class of ordinals that approximate Ord on $\mathcal{M}$ for φ.

Lemma 4.3 (Approximation Lemma) *For every model $\mathcal{M} = (W, R, V)$ and every $\varphi \in \mathcal{L}$, the class $\mathrm{Approx}(\mathcal{M}, \varphi)$ is non-empty.*

Proof. By induction on the construction of φ. If φ is atomic then it does not contain any boxes, so the lemma is trivial. Suppose therefore as induction hypothesis that φ is not atomic and that the lemma holds for all strict subformulas of φ. Since, by assumption, $\text{Approx}(\mathcal{M},\psi)$ is nonempty for every strict subformula ψ of φ, let $\alpha_\psi \in \text{Approx}(\mathcal{M},\psi)$.

As usual, we continue by a case distinction on the main connective of φ. Most of the cases are quite trivial, so we do not give their proofs in much detail.

- Suppose $\varphi = \neg\psi$. Then $\alpha_\psi \in \text{Approx}(\mathcal{M},\varphi)$.
- Suppose $\varphi = \psi_1 \vee \psi_2$. Then $\max(\alpha_{\psi_1},\alpha_{\psi_2}) \in \text{Approx}(\mathcal{M},\varphi)$.
- Suppose $\varphi = K_a\psi$. Then $\alpha_\psi \in \text{Approx}(\mathcal{M},\varphi)$, since α_ψ approximates Ord on every world of $\mathcal{M}$.
- Suppose $\varphi = [\psi_1]\psi_2$. Let $\alpha = \max(\alpha_{\psi_1},\alpha_{\psi_2})$. Then, for every $\beta \geq \alpha$, $\mathcal{M}_{\psi_1} = \mathcal{M}_{\downarrow_\beta(\psi_1)}$. Furthermore, α_{ψ_2} approximates Ord not just on $\mathcal{M}$ but also on all of its submodels, so ψ_2 is equivalent to $\downarrow_\beta(\psi_2)$ on the updated model. Clearly, $\downarrow_\beta([\psi_1]\psi_2) = [\downarrow_\beta(\psi_1)]\downarrow_\beta(\psi_2)$, so it follows that $\downarrow_\beta([\psi_1]\psi_2)$ is equivalent to $[\psi_1]\psi_2$ on $\mathcal{M}$. So $\alpha \in \text{Approx}(\mathcal{M},\varphi)$.
- Suppose $\varphi = \Box_\gamma\psi$. Then $\max(\gamma,\alpha_\psi) \in \text{Approx}(\mathcal{M},\varphi)$.
- Suppose $\varphi = \Box\psi$. Let $\mathfrak{M}$ be the set of pointed models $\mathcal{M}',w$ such that $\mathcal{M}'$ is a submodel of $\mathcal{M}$. We can partition $\mathfrak{M}$ into $\mathfrak{M}^+ := \{\mathcal{M}',w \in \mathfrak{M} \mid \mathcal{M}',w \models \Box_\alpha\psi \text{ for all ordinals } \alpha\}$ and $\mathfrak{M}^- := \{\mathcal{M}',w \in \mathfrak{M} \mid \mathcal{M}',w \not\models \Box_\alpha\psi \text{ for some ordinal } \alpha\}$.

  For $\mathcal{M}',w \in \mathfrak{M}$, let $\alpha_{\mathcal{M}',w}$ be given by

  $$\alpha_{\mathcal{M}',w} := \begin{cases} 0 & \text{if } \mathcal{M}',w \in \mathcal{M}^+ \\ \min\{\alpha \mid \mathcal{M}',w \not\models \Box_\alpha\psi'\} & \text{if } \mathcal{M}',w \in \mathcal{M}^- \end{cases}$$

  For every $\beta \geq \alpha_{\mathcal{M}',w}$ we have $\mathcal{M}',w \models \Box\psi \Leftrightarrow \mathcal{M}',w \models \Box_\beta\psi$.

  Now, let $\alpha_{\mathfrak{M}} := \sup\{\alpha_{\mathcal{M}',w} \mid \mathcal{M}',w \in \mathfrak{M}\}$. This supremum exists and is itself an ordinal, because $\mathfrak{M}$ is a set. Take $\alpha = \max(\alpha_{\mathfrak{M}},\alpha_\psi)$. For every $\mathcal{M}',w$ and every $\beta \geq \alpha$ we have $\mathcal{M}',w \models \Box\psi \Leftrightarrow \mathcal{M}',w \models \Box_\beta\psi$ since $\beta \geq \alpha_{\mathfrak{M}}$ and therefore $\beta \geq \alpha_{\mathcal{M}',w}$. Furthermore, $\mathcal{M}',w \models \Box_\beta\psi \Leftrightarrow \mathcal{M}',w \models \downarrow_\beta(\Box_\beta\psi)$ since $\beta \geq \alpha_\psi$ (and $\beta \geq \beta$, so $\downarrow_\beta(\Box_\beta\psi) = \Box_\beta\downarrow_\beta(\psi)$). It follows that $\mathcal{M}',w \models \Box\psi \Leftrightarrow \mathcal{M}',w \models \downarrow_\beta(\Box\psi)$. This holds for every $\mathcal{M}',w$ and every $\beta \geq \alpha$, so $\alpha \in \text{Approx}(\mathcal{M},\varphi)$.

$\Box$

Remark 4.4 Alternatively, we could have proven the Approximation Lemma by using the pigeonhole principle: a model $\mathcal{M} = (W,R,V)$ has at most $|2^W|$ different extensions, so for any α with $|\alpha| > |2^W|$ and any φ we have $\mathcal{M} \models \varphi \leftrightarrow \downarrow_\alpha(\varphi)$. This alternative proof is more complicated than the one given above, however, which is why we gave this one.

Theorem 4.5 *For every pointed model $\mathcal{M},w$ and every $\varphi \in \mathcal{L}$, we have $\mathcal{M},w \models \Box\varphi \Leftrightarrow \forall\psi \in \mathcal{L} : \mathcal{M},w \models [\psi]\varphi$.*

Proof. First, suppose $\forall\psi \in \mathcal{L} : \mathcal{M},w \models [\psi]\varphi$. Then, in particular, for every ordinal α we have $\forall\psi \in \mathcal{L}_\alpha : \mathcal{M}, w \models [\psi]\varphi$ and therefore, by the semantics of $\Box_\alpha$, $\mathcal{M}, w \models \Box_\alpha\varphi$. Since this holds for every ordinal α, we have $\mathcal{M}, w \models \Box\alpha$.

We continue the proof by contraposition, so suppose there is some $\psi \in \mathcal{L}$ such that $\mathcal{M}, w \not\models [\psi]\varphi$. By the Approximation Lemma, there is an ordinal α such that ψ is equivalent to $\downarrow_\alpha(\psi)$ on all submodels of $\mathcal{M}$. In particular, ψ and $\downarrow_\alpha(\psi)$ are equivalent on $\mathcal{M}$. This implies that $\mathcal{M}, w \not\models [\downarrow_\alpha(\psi)]\varphi$. We have $\downarrow_\alpha(\psi) \in \mathcal{L}_{\alpha+1}$, so $\mathcal{M}, w \not\models \Box_{\alpha+1}\varphi$. That, finally, implies $\mathcal{M}, w \not\models \Box\varphi$. □

5 Properties of F-APAL

Here we briefly discuss a few properties of F-APAL. None of these properties are particularly surprising, this section could be seen as a "sanity check" for $\Box$, showing that it has about the properties one would expect.

Proposition 5.1 *F-APAL is invariant under bisimulation.*

Proof. First, we show that $\mathcal{L}_\alpha$ is invariant under bisimulation for all $\alpha \in \text{Ord}$. We do this by induction on α. As base case, suppose $\alpha = 0$. Then $\mathcal{L}_\alpha$ is public announcement logic, which is known to be invariant under bisimulation. Suppose then as induction hypothesis that $\mathcal{L}_\beta$ is invariant under bisimulation for all $\beta < \alpha$.

Now, take any $\varphi \in \mathcal{L}_\alpha$. We show that φ is invariant under bisimulation by a secondary induction on the construction of φ. If φ is atomic, then it is trivially invariant under bisimulation. Suppose as secondary induction hypothesis that all strict subformulas of φ are invariant under bisimulation. Take any two pointed models $\mathcal{M}_1, w_1$ and $\mathcal{M}_2, w_2$ that are bisimilar. We proceed by case distinction on the main connective of φ, but most of the cases are trivial so we omit them. The one case that we do consider in detail is $\varphi = \Box_\beta\psi$. Since $\varphi \in \mathcal{L}_\alpha$, we have $\beta < \alpha$. By the primary induction hypothesis, this implies that, for every $\chi \in \mathcal{L}_\beta$, we have that χ is invariant under bisimulation. As a result, for every such χ, the pointed models $(\mathcal{M}_1)_\chi, w_1$ and $(\mathcal{M}_2)_\chi, w_2$ are bisimilar (if they exist). By the secondary induction hypothesis, this implies that ψ cannot distinguish between $(\mathcal{M}_1)_\chi, w_1$ and $(\mathcal{M}_2)_\chi, w_2$. It follows that $\Box_\beta\psi$ cannot distinguish between $\mathcal{M}_1, w_1$ and $\mathcal{M}_2, w_2$. this holds for any two bisimilar pointed models, so φ is invariant under bisimulation. This completes the induction step of the secondary and primary inductions, so $\mathcal{L}_\alpha$ is invariant under bisimulation for all $\alpha \in \text{Ord}$.

Left to show is that $\mathcal{L}$ is invariant under bisimulation. Take any $\varphi \in \mathcal{L}$. Once again, we use induction on the construction of φ to show that it is invariant under bisimulation. As base case, suppose φ is atomic. Then it is trivially invariant under bisimulation. Suppose therefore as induction hypothesis that all strict subformulas of φ are invariant under bisimulation. We proceed by a case distinction on the main connective of φ. Most cases are trivial, so we omit them. The cases that we do consider are $\varphi = \Box_\alpha\psi$ and $\varphi = \Box\psi$.

Suppose $\varphi = \Box_\alpha\psi$. Then we reason as before: the formulas that $\Box_\alpha$ quantifies over are invariant under bisimulation, as is ψ. It follows that $\Box_\alpha\psi$ is also invariant under bisimulation.

Suppose $\varphi = \Box\psi$. As shown in the previous case, $\Box_\alpha\psi$ is invariant under bisimulation for all $\alpha \in \text{Ord}$. By definition, $\Box\psi$ holds if and only if $\Box_\alpha\psi$ holds for all $\alpha \in \text{Ord}$. So $\Box\psi$ is also invariant under bisimulation. This completes the induction step and thereby the proof. $\Box$

Recall that in Section 2 we showed that (*) is under-determined by showing that in the model $\mathcal{M}_{Un}$, see Figure 1, we can assign two different valuations for $\Diamond\xi$ that are both consistent with (*). In one of these valuations, $\Diamond\xi$ is false in the world w_2. In the other valuation, $\Diamond\xi$ is true in w_2 because it is self-fulfilling.

By defining our semantics for F-APAL, we made a choice between these two valuations. It follows easily from the fact that F-APAL is invariant under bisimulation that we have chosen the valuation $\mathcal{M}_{Un}, w_2 \not\models \Diamond\xi$. As discussed in Section 2, we prefer $\mathcal{M}_{Un}, w_2 \not\models \Diamond\xi$ over $\mathcal{M}_{Un}, w_2 \models \Diamond\xi$. So we are satisfied that F-APAL makes $\Diamond\xi$ false in $\mathcal{M}_{Un}, w_2$.

Proposition 5.2 *Let $\varphi, \varphi' \in \mathcal{L}$ and $\alpha \in \text{Ord}$. Then*

(i) *if $\models \varphi$, then $\models \Box\varphi$ and $\models \Box_\alpha\varphi$,*

(ii) *$\models \Box(\varphi \to \varphi') \to (\Box\varphi \to \Box\varphi')$ and $\models \Box_\alpha(\varphi \to \varphi') \to (\Box_\alpha\varphi \to \Box_\alpha\varphi')$,*

(iii) *$\models \Box\varphi \to \varphi$ and $\models \Box_\alpha\varphi \to \varphi$,*

(iv) *$\models \varphi \to \Diamond\varphi$ and $\models \varphi \to \Diamond_\alpha\varphi$,*

(v) *$\models \Box\varphi \leftrightarrow \Box\Box\varphi$ and $\models \Box_\alpha\varphi \leftrightarrow \Box_\alpha\Box_\alpha\varphi$.*

Proof. Let $\mathcal{M} = (W, R, V)$ be any model and let $w \in W$.

- Suppose $\models \varphi$. Then, in particular, for every ψ such that $\mathcal{M}, w \models \psi$, we have $\mathcal{M}_\psi, w \models \varphi$ and therefore $\mathcal{M}, w \models [\psi]\varphi$. For every ψ such that $\mathcal{M}, w \not\models \psi$ we trivially have $\mathcal{M}, w \models [\psi]\varphi$. So for all ψ, we have $\mathcal{M}, w \models [\psi]\varphi$. This implies that $\mathcal{M}, w \models \Box\varphi$, and therefore also the weaker statement $\mathcal{M}, w \models \Box_\alpha\varphi$.
- If $\mathcal{M}, w \models \Box(\varphi \to \varphi')$ and $\mathcal{M}, w \models \Box\varphi$, then for every ψ such that $\mathcal{M}, w \models \psi$ we have $\mathcal{M}_\psi, w \models \varphi \to \varphi'$ and $\mathcal{M}_\psi, w \models \varphi$ and therefore $\mathcal{M}_\psi, w \models \varphi'$. It follows that $\mathcal{M}, w \models [\psi]\varphi'$ and therefore $\models \Box(\varphi \to \varphi') \to (\Box\varphi \to \Box\varphi')$.

  The same holds if we restrict to $\psi \in \mathcal{L}_\alpha$ instead of all ψ, so $\models \Box_\alpha(\varphi \to \varphi') \to (\Box_\alpha\varphi \to \Box_\alpha\varphi')$.
- If $\mathcal{M}, w \models \Box\varphi$ or $\mathcal{M}, w \models \Box_\alpha\varphi$ then, in particular, $\mathcal{M}, w \models [\top]\varphi$ and therefore $\mathcal{M}, w \models \varphi$.
- If $\mathcal{M}, w \models \varphi$ then $\mathcal{M}, w \models \langle\top\rangle\varphi$ and therefore $\mathcal{M}, w \models \Diamond\varphi$ and $\mathcal{M}, w \models \Diamond_\alpha\varphi$.
- Suppose $\mathcal{M}, w \models \Box\varphi$. Then, in particular, for every ψ_1, ψ_2 we have $\mathcal{M}, w \models [\psi_1 \wedge [\psi_1]\psi_2]\varphi$. That is equivalent to $\mathcal{M}, w \models [\psi_1][\psi_2]\varphi$. This holds for every ψ_1, ψ_2, so $\mathcal{M}, w \models \Box\Box\varphi$.

  Similarly, if $\mathcal{M}, w \models \Box_\alpha\varphi$ then $\mathcal{M}, w \models [\psi_1 \wedge [\psi_1]\psi_2]\varphi$ for all $\psi_1, \psi_2 \in \mathcal{L}_\alpha$. It follows that $\mathcal{M}, w \models [\psi_1][\psi_2]\varphi$ for all $\psi_1, \psi_2 \in \mathcal{L}_\alpha$, so $\mathcal{M}, w \models \Box_\alpha\Box_\alpha\varphi$.

  The other side of the bi-implications follows from $\models \Box\varphi \to \varphi$ and $\models \Box_\alpha\varphi \to \varphi$.

$\Box$

In particular, this shows that $\Box$ and $\Box_\alpha$ are S4 operators.

Before proving the next proposition, we need an auxiliary lemma. This lemma and the next proposition are the only places in this paper where we use the fact that we use the class S5 of models, all other results apply to K as well. The lemma uses a technique very similar to the one used in [10] to show that every formula is "whether-knowable," i.e. for every formula φ, every agent a and pointed model $\mathcal{M}, w$ there is a formula ψ such that either $\mathcal{M}, w \models [\psi]K_a\varphi$ or $\mathcal{M}, w \models [\psi]K_a\neg\varphi$.

Lemma 5.3 *Let $\varphi \in \mathcal{L}$, and let P be the propositional variables that occur in φ. There is a function $f : 2^P \to \{\top, \bot\}$ such that for every model $\mathcal{M}$ and every $P' \subseteq P$: if $\mathcal{M} \models p$ for all $p \in P'$ and $\mathcal{M} \models \neg p$ for all $p \in P \setminus P'$, then $\mathcal{M} \models \varphi$ if $\mathcal{M} \models f(P')$ and $\mathcal{M} \models \neg\varphi$ if $\mathcal{M} \models \neg f(P')$.*

Proof. By induction on the construction of φ. As base case, suppose φ is atomic. Then $\varphi = p$ for some $p \in \mathcal{P}$. The function given by $f(\varnothing) = \bot$ and $f(\{p\}) = \top$ then satisfies the lemma. Suppose then as induction hypothesis that φ is not atomic and that the lemma holds for all strict subformulas of φ. Given such a strict subformula ψ, let f_ψ be the function associated with ψ. The proof continues by a case distinction on the main connective of φ.

- Suppose $\varphi = \neg\psi$. Then the function given by $f(P') = \neg f_\psi(P')$ satisfies the lemma.
- Suppose $\varphi = \psi_1 \vee \psi_2$. Then the function $f(P') = f_{\psi_1}(P') \vee f_{\psi_2}(P')$ satisfies the lemma.
- Suppose $\varphi = K_a\psi$. For every χ, we have $\mathcal{M} \models \chi \Rightarrow \mathcal{M} \models K_a\chi$. Furthermore, since we are working in S5, we have $\mathcal{M} \models \neg\chi \rightarrow \mathcal{M} \models \neg K_a\chi$. As a result, the function $f = f_\psi$ satisfies the lemma.
- Suppose $\varphi = [\psi_1]\psi_2$. Let f be given by $f(P') = f_{\psi_1}(P') \rightarrow f_{\psi_2}(P')$. Let $\mathcal{M}$ be any model such that $\mathcal{M} \models p$ for all $p \in P'$ and $\mathcal{M} \models \neg p$ for all $p \in P \setminus P'$. If $f_{\psi_1}(P') = \bot$ then $\mathcal{M} \models \neg\psi_1$, so trivially $\mathcal{M} \models [\psi_1]\psi_2$. Note that in this case $f(P') = \top$, so the lemma is satisfied. If $f_{\psi_1}(P') = \top$, then $\mathcal{M}_{\psi_1} = \mathcal{M}$, so φ is equivalent to ψ_2 on $\mathcal{M}$. Note that $f(P') = f_{\psi_2}(P')$ in this case, so the lemma is satisfied.
- Suppose $\varphi = \Box_\alpha\psi$ or $\varphi = \Box\varphi$. For every updated model $\mathcal{M}_\chi$ and every $p \in P$, we have $\mathcal{M} \models p \Rightarrow \mathcal{M}_\chi \models p$ and $\mathcal{M} \models \neg p \Rightarrow \mathcal{M}_\chi \models \neg p$. So if we take $f = f_\psi$, then the lemma is satisfied.

$\Box$

Proposition 5.4 *Let $\varphi \in \mathcal{L}$ and $\alpha \in$* Ord. *Then*

(i) $\models \Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$ *and* $\models \Box_\alpha\Diamond_\alpha\varphi \rightarrow \Diamond_\alpha\Box_\alpha\varphi$,

(ii) $\models \Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$ *and* $\models \Diamond_\alpha\Box_\alpha\varphi \rightarrow \Box_\alpha\Diamond_\alpha\varphi$.

Proof. Let a pointed model $\mathcal{M}, w$ be given. Let $P = \{p_1, \cdots, p_n\}$ be the set of propositional variables that occur in φ. Furthermore, let $P' = \{p \in P \mid \mathcal{M}, w \models p\}$ and let $\zeta = \bigwedge_{p \in P'} p \wedge \bigwedge_{p \in P \setminus P'} \neg p$. Finally, let $f_\varphi : 2^P \to \{\top, \bot\}$ be the function from Lemma 5.3.

Now, consider the models $\mathcal{M}_\zeta$. For every $p \in P'$ we have $\mathcal{M}_\zeta \models p$ and for every $p \in P \setminus P'$ we have $\mathcal{M}_\zeta \models \neg p$. The same holds for every submodel of $\mathcal{M}_\zeta$. Lemma 5.3 therefore implies that, for all submodels $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4$ of $\mathcal{M}_\zeta$, we have $\mathcal{M}_1, w \models \varphi \Leftrightarrow \mathcal{M}_2, w \models \Diamond\varphi \Leftrightarrow \mathcal{M}_3, w \models \Box\varphi \Leftrightarrow \mathcal{M}_4, w \models f_\varphi(P')$.

Suppose now that $\mathcal{M}, w \models \Box\Diamond\varphi$. Then, in particular, $\mathcal{M}, w \models [\zeta]\Diamond\varphi$. It follows that $\mathcal{M}_\zeta, w \models \Diamond\varphi$ and therefore $\mathcal{M}_\zeta \models \Box\varphi$. As such, we have $\mathcal{M}, w \models \langle\zeta\rangle\Box\varphi$ and therefore $\mathcal{M}, w \models \Diamond\Box\varphi$.

Suppose then that $\mathcal{M}, w \models \Diamond\Box\varphi$. Then there is some ψ such that $\mathcal{M}_\psi, w \models \Box\varphi$. In particular, $\mathcal{M}_\psi, w \models [\zeta]\varphi$, so $(\mathcal{M}_\psi)_\zeta, w \models \varphi$. But $(\mathcal{M}_\psi)_\zeta$ is a submodel of $\mathcal{M}_\zeta$, so we have $\mathcal{M}', w \models \varphi$ for every submodel of $\mathcal{M}_\zeta$ (that includes w). For every ψ, $(\mathcal{M}_\psi)_\zeta$ is a submodel of $\mathcal{M}_\zeta$, so $(\mathcal{M}_\psi)_\zeta, w \models \varphi$. It follows that, for every ψ, $\mathcal{M}, w \models [\psi]\langle\zeta\rangle\varphi$ and therefore $\mathcal{M}, w \models \Box\Diamond\varphi$.

Finally, since $\zeta \in \mathcal{L}_0$, the same reasoning holds for $\Box_\alpha$ and $\Diamond_\alpha$ instead of $\Box$ and $\Diamond$, so $\models \Box_\alpha\Diamond_\alpha\varphi \to \Diamond_\alpha\Box_\alpha\varphi$ and $\models \Diamond_\alpha\Box_\alpha\varphi \to \Box_\alpha\Diamond_\alpha\varphi$. □

Proposition 5.4.(i) is known as the Church-Rosser schema and characterizes the property known as convergence or confluence. In our terms: if in a given model $\mathcal{M}, w$ you make two different (truthful) announcements φ and ψ, you get two typically different (non-bisimilar) model restrictions $\mathcal{M}_\varphi, w$ and $\mathcal{M}_\psi, w$. Proposition 5.4.(i) then says that in such a case there are announcements φ' and ψ' such that $(\mathcal{M}_\varphi)_{\varphi'}, w$ is bisimilar to $(\mathcal{M}_\psi)_{\psi'}, w$.

Proposition 5.4.(ii) is also known as the McKinsey schema. In the presence of the schema $\Box\varphi \to \Box\Box\varphi$ (which is valid in F-APAL, see Proposition 5.2) this characterizes so-called *atomicity*. Intuitively, given any model $\mathcal{M}, w$ and any set P of propositional variables, there is a ζ such that on $\mathcal{M}_\zeta, w$ we have $\varphi \leftrightarrow \Box\varphi$ for all φ that contain only propositional variables from P. So in $\mathcal{M}_\zeta, w$ you already know all there is to know about P, any further model restriction is uninformative. Even more intuitively, Proposition 5.4.(ii) says that given any model $\mathcal{M}, w$ and any formula φ, you can make a most informative announcement with respect to the propositional variables occurring in φ; namely the ζ above.

For a more detailed description of these properties, see [8].

6 F-APAL and fixed points

The semantics of $\Box$ are reminiscent of fixed point constructions. Indeed, we can define $\Box$ as a fixed point. The relation between F-APAL and fixed points is not straightforward, however. There are also several open questions regarding the fixed points related to F-APAL. Let us therefore discuss the relation between F-APAL and fixed points in detail. The auxiliary operators $\Box_\alpha$ are not important to the fixed point behavior of $\Box$, so in this section we work with a language $\mathcal{L}'$ that includes $\neg, \vee, K_a, [\psi]$ and $\Box$ but not $\Box_\alpha$.

Let us start by recalling the definitions of (*), (**) and (5) (with $\mathcal{L}'$ substituted for $\mathcal{L}$):

$$\mathcal{M}, w \models \Box\varphi \Leftrightarrow \forall\psi \in \mathcal{L}' : \mathcal{M}, w \models [\psi]\varphi \qquad (*')$$

$$\mathcal{M}, w \models \Box\varphi \Leftrightarrow \forall x \in \{[\![\psi]\!]_{\mathcal{M}} \mid \psi \in \mathcal{L}'\} : \mathcal{M}_x, w \models \varphi. \tag{**$'$}$$

$$\mathcal{M}, w \models \Box\varphi \Leftrightarrow \forall x \in X : \mathcal{M}_x, w \models \varphi. \tag{5}$$

Also, recall that (*$'$) and (**$'$) are equivalent. Now, let $\mathcal{M} = (W, R, V)$ and suppose that we would define $\Box$ not as in Definition 3.6 but by (5) for some $X \subseteq 2^W$. This alternative definition only applies to the specific model $\mathcal{M}$, so we will assume that $\Box$ is defined in some way on other models.

Because we define $\Box$ by (5), we have (**$'$) and (*$'$) if X is the set of extensions on $\mathcal{M}$. (See Section 2 for a discussion of why this is so.) In order to emphasize that our semantics, and therefore our set of extensions, depend on X let us write $[\![\mathcal{L}']\!]^X_{\mathcal{M}}$ for the set of all extensions on $\mathcal{M}$. So (*$'$) is satisfied if

$$X = [\![\mathcal{L}']\!]^X_{\mathcal{M}}.$$

In other words, (*$'$) is satisfied if X is a fixed point of $f : X \mapsto [\![\mathcal{L}']\!]^X_{\mathcal{M}}$. Unfortunately, f is not monotone,[6] nor do we have $X \subseteq f(X)$. The standard methods for proving the existence of a fixed point therefore do not apply; whether f has a fixed point (on every model) is, to the best of our knowledge, an open question.

Alternatively, we can consider the function $g : X \mapsto X \cup [\![\mathcal{L}']\!]^X_{\mathcal{M}}$. This function is also not monotone, but it does by construction satisfy $X \subseteq g(X)$. As a result, g is guaranteed to have a fixed point. For example, 2^W is a fixed point of g. More importantly, $\{[\![\psi]\!]_{\mathcal{M}} \mid \psi \in \mathcal{L}\}$ is a fixed point of g. In fact, $\{[\![\psi]\!]_{\mathcal{M}} \mid \psi \in \mathcal{L}\} = \lim_{\alpha\in\mathrm{Ord}} g^\alpha(\varnothing)$.[7]

The construction of $\{[\![\psi]\!]_{\mathcal{M}} \mid \psi \in \mathcal{L}\}$ is therefore an instance of a general kind of fixed point construction: let S be any complete lattice, and let $h : S \to S$ be any function that satisfies $s \leq h(s)$ for all $s \in S$. Then $\lim_{\alpha\in\mathrm{Ord}} h^\alpha(0)$ is guaranteed to exist and be a fixed point of h, where 0 is the least element of S and $h^\alpha(0)$ is defined as $\sup_{\beta<\alpha} h^\beta(\varnothing)$ when α is a limit ordinal.

However, unless h is monotone, it is not guaranteed that $\lim_{\alpha\in\mathrm{Ord}} h^\alpha(0)$ is the *least* fixed point of h. As such, we cannot immediately conclude that $\{[\![\psi]\!]_{\mathcal{M}} \mid \psi \in \mathcal{L}\}$ is the least fixed point of g. Whether it is in fact the least fixed point is an open question.

The fact that $\{[\![\psi]\!]_{\mathcal{M}} \mid \psi \in \mathcal{L}\} = \lim_{\alpha\in\mathrm{Ord}} g^\alpha(\varnothing)$ shows that we could have defined the semantics of $\Box$ using a fixed point, instead of using $\mathcal{M}, w \models \Box\varphi \Leftrightarrow \mathcal{M}, w \models \Box_\alpha\varphi$ for all $\alpha \in \mathrm{Ord}$. It should be noted, however, that although the auxiliary operators $\Box_\alpha$ do not occur in the fixed point definition of $\Box$, we do still need them in order to satisfy (*).

As noted before, any fixed point of f satisfies (*$'$). So it can be used to define a fully public arbitrary announcement for the language $\mathcal{L}'$, the language

[6] i.e. it is not the case that, for all X, Y, if $X \subseteq Y$ then $f(X) \subseteq F(Y)$.

[7] Where $g^\alpha(\varnothing)$ is defined as $\bigcup_{\beta<\alpha} g^\beta(\varnothing)$ if α is a limit ordinal, and we use the discrete topology (i.e. $\lim_{\alpha\in\mathrm{Ord}} Z_\alpha = l \Leftrightarrow \exists\beta\forall\alpha > \beta : Z_\alpha = l$).

without auxiliary operators. A fixed point of g is not guaranteed to share this property, however. If X is a fixed point of g, then we have $X = X \cup [\![\mathcal{L}']\!]^X_{\mathcal{M}}$. But then there might be $x \in X \setminus [\![\mathcal{L}']\!]^X_{\mathcal{M}}$, i.e. there might be some sets that are quantified over by $\Box$ that are not the extension of any formula.

For the specific fixed point $\lim_{\alpha\in\mathrm{Ord}} f^\alpha(\varnothing)$ of g, we can solve this problem using the auxiliary operators. While some elements of $\lim_{\alpha\in\mathrm{Ord}} f^\alpha(\varnothing)$ are not the extension of any formula of $\mathcal{L}'$, each of them is the extension of some formula of $\mathcal{L}$.

We hope that this section has clarified the relation between F-APAL and fixed points. In particular, we hope that it explains why we define the semantics of F-APAL the way we do, instead of as a fixed point.

7 Conclusion

We introduced a logic F-APAL, in which the connective $\Box$ represents a fully arbitrary public announcement, i.e. we have

$$\mathcal{M}, w \models \Box\varphi \Leftrightarrow \forall\psi \in \mathcal{L} : \mathcal{M}, w, \models [\psi]\varphi$$

for all $\varphi \in \mathcal{L}$ and every pointed model $\mathcal{M}, w$. The price we pay for this property is that we use a proper class of auxiliary operators, $\{\Box_\alpha \mid \alpha \in \mathrm{Ord}\}$.

This suggests a few directions for further research. Firstly, we could try to use similar techniques to design semantics for other circular properties. Examples of such circular properties include "agent a knows at least as much as agent b"

$$\mathcal{M}, w \models a \succeq b \Leftrightarrow \forall\psi : \mathcal{M}, w \models K_b\psi \to K_a\psi$$

and "everything agent a believes is true" [8]

$$\mathcal{M}, w \models T(a) \Leftrightarrow \forall\psi : \mathcal{M}, w \models K_a\psi \to \psi.$$

Or consider *knowledge based programs* [12,13] (or similarly: *epistemic protocols* [2,3]). Such programs contain instructions for multiple agents to perform actions. But, importantly, every action has to come with an epistemic precondition for the agent that is supposed to carry out the action. So a knowledge based program can only contain clauses of the form "if $K_a\varphi$, then a should do x." These kinds of programs are useful when modeling distributed systems. Suppose we use $\blacksquare\psi$ to denote "there is a knowledge based program π that, if executed, guarantees outcome ψ." Then $\blacksquare\psi$ is circular, since π could contain a clause "if $K_a\blacksquare\psi$, then a should do x."

Secondly, we could attempt to reduce the conceptual cost of F-APAL by using fewer auxiliary operators. Ideally we would use no auxiliary operators at all, but barring that it would be nice to have a *set* of auxiliary operators instead of a proper class. We conjecture that it is possible to have fully arbitrary public announcements with a set of auxiliary operators, but we think it may be

[8] If we work in S5, $K_a\psi \to \psi$ is always true for every ψ. So in order to make this property interesting we would need to use a different class of models.

impossible to have fully arbitrary public announcements without any auxiliary operators.

Acknowledgments

We would like to thank the AiML reviewers for their insightful remarks. Their questions about fixed points were especially useful, and prompted the addition of Section 6. Hans van Ditmarsch and Louwe B. Kuijer acknowledge support from ERC project EPS 313360. Hans van Ditmarsch is also affiliated to IMSc, Chennai, India, and Zhejiang University, China.

References

[1] Ågotnes, T., P. Balbiani, H. van Ditmarsch and P. Seban, *Group announcement logic*, Journal of Applied Logic **8** (2010), pp. 62–81.

[2] Attamah, M., H. van Ditmarsch, D. Grossi and W. van der Hoek, *Knowledge and gossip*, in: T. Schaub, G. Friedrich and B. O'Sullivan, editors, *Proceedings of the 21st European Conference on Artificial Intelligence (ECAI 2014)*, 2014, pp. 21–26.

[3] Attamah, M., H. van Ditmarsch, D. Grossi and W. van der Hoek, *The pleasure of gossip*, in: C. Başkent, L. Moss and R. Ramanujam, editors, *Rohit Parikh on Logic, Language and Society*, To Appear.

[4] Balbiani, P., A. Baltag, H. van Ditmarsch, A. Herzig, T. Hoshi and T. de Lima, *What can we achieve by arbitrary announcements? A dynamic take on Fitch's knowability*, in: D. Samet, editor, *Proceedings of the 11th conference on Theoretical aspects of rationality and knowledge*, 2007, pp. 42–51.

[5] Balbiani, P., A. Baltag, H. van Ditmarsch, A. Herzig, T. Hoshi and T. de Lima, *'Knowable' as 'Known after an Announcement'*, The Review of Symbolic Logic **1** (2008), pp. 305–334.

[6] Baltag, A., L. Moss and S. Solecki, *The logic of public announcements, common knowledge, and private suspicions*, in: I. Gilboa, editor, *Proceedings of the 7th conference on Theoretical aspects of rationality and knowledge* (1998), pp. 43–56.

[7] Barwise, J. and L. Moss, "Vicious Circles," CSLI Publications, 1996.

[8] Blackburn, P., M. de Rijke and Y. Venema, "Modal Logic," Cambridge University Press, 2001.

[9] Bozzelli, L., H. van Ditmarsch, T. French, J. Hales and S. Pinchinat, *Refinement modal logic*, Information and Computation **239** (2014), pp. 303–339.

[10] van Ditmarsch, H., W. van der Hoek and P. Iliev, *Everything is knowable - how to get to know whether a proposition is true*, Theoria **78** (2012), pp. 93–114.

[11] van Ditmarsch, H., W. van der Hoek and B. Kooi, *Arbitrary arrow update logic*, presented at Advances in Modal Logic 2014.

[12] Fagin, R., J. Y. Halpern, Y. Moses and M. Vardi, "Reasoning About Knowledge," MIT Press, Cambridge, MA, USA, 1995.

[13] Fagin, R., J. Y. Halpern, Y. Moses and M. Y. Vardi, *Knowledge based programs*, Distributed Computing **10** (1997), pp. 199–225.

[14] Hales, J., *Arbitrary action model logic and action model synthesis*, in: *28th Annual ACM/IEEE Symposium on Logic in Computer Science*, 2013, pp. 253–262.

[15] Jech, T., "Set Theory," Springer, Berlin, 1997.

[16] Kuijer, L. B., *How arbitrary are arbitrary public announcements?*, in: M. Colinet, S. Katrenko and R. K. Rendsvig, editors, *Pristine Perspectives on Logic, Language and Computation*, 2014, pp. 109–123.