The Logic of Where and While
in the 13th and 14th Centuries

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Abstract

Medieval analyses of molecular propositions include many non-truthfunctional connectives in addition to the standard modern binary connectives (conjunction, disjunction, and conditional). Two types of non-truthfunctional molecular propositions considered by a number of 13th- and 14th-century authors are temporal and local propositions, which combine atomic propositions with 'while' and 'where'. Despite modern interest in the historical roots of temporal and tense logic, medieval analyses of 'while' propositions are rarely discussed in modern literature, and analyses of 'where' propositions are almost completely overlooked. In this paper we introduce 13th- and 14th-century views on temporal and local propositions, and connect the medieval theories with modern temporal and spatial counterparts.

Keywords: Jean Buridan, Lambert of Auxerre, local propositions, Roger Bacon, temporal propositions, Walter Burley, William of Ockham

1 Introduction

Modern propositional logicians are familiar with three kinds of compound propositions: Conjunctions, disjunctions, and conditionals. Thirteenth-century logicians were more liberal, admitting variously five, six, seven, or more types of compound propositions. In the middle of the 13th century, Lambert of Auxerre [1] in his *Logic* identified six types of 'hypothetical' (i.e., compound, as opposed to atomic 'categorical', i.e., subject-predicate, propositions) propositions: the three familiar ones, plus causal, local, and temporal propositions [17, ¶99]. Another mid-13th century treatise, Roger Bacon's *Art and Science of Logic*, lists these six and adds expletive propositions (those which use the connective 'however'), and other ones not explicitly classified such as "Socrates

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[1] The identity of the author of this text is not known for certain. Only one manuscript identifies him further than simply *Lambertus*, and there his place of origin is given as Ligny-le-Châtel. Two candidates for who Lambert is have been advanced in the literature: Lambert of Auxerre, a Dominican and canon of the cathedral of Auxerre in the late 1230s or early 1240s, and Lambert of Lagny, a secular cleric and teacher of Theobald II who later became a Dominican and a papal penitentiary. The introduction of [17] lays out the arguments for and against both positions, and comes down in the favor of an attribution of Auxerre. We follow this status quo here with the caveat that nothing we say turns on his precise identification.

is such as Plato is", "Socrates runs as often as Plato argues", etc. [2, ¶170]). Numerous other anonymous treatises from the first half of the 13th century include similar divisions. The *Ars Burana* (AB) [9, pp. 175–213], dating around 1200 [8, pp. 42, 397], and the *Ars Emmerana* (AE) [9, pp. 143–174], from the first half of the 13th century [8, p. 43] both identify the same six types of hypothetical sentences as Lambert, and add a seventh type, the 'adjunctive' [9, pp. 158, 190]. The *Dialectica Monacensis* (DM) [9, pp. 453–638], composed between the 1160s and the first decade of the 13th century [8, pp. 410–414] gives the modern logician's three types as the primary types, but subdivides conditionals into a further four categories: temporal, local, causal, and subconditional [9, p. 484].[2]

From this we can see that the other two of the four main textbooks from the middle of the century, Peter of Spain's *Summaries of Logic* and William of Sherwood's *Introduction to Logic*, are unusual, as neither of them mention these types of sentences [6, p. 115]; [19, p. 34]. In his early 14th century *Summary of Dialectic*, Jean Buridan explains that "some texts do not provide the species 'temporal' and 'local', because they can be reduced to conjunctive propositions, for saying 'Socrates is where Plato is' amounts to the same as saying 'Socrates is somewhere and Plato is there', and in the same way, saying 'Socrates lectured when Plato disputed' is equivalent to saying 'Socrates lectured sometime and then Plato disputed'" [3, p. 60]. (Note that this 'then' must be interpreted logically, not temporally.) Despite giving this reduction from temporal or local sentences to conjunctions, Buridan himself discusses these two types separately from conjunctive propositions, as do Walter Burley (c. 1275–1344) and William of Ockham (c. 1287–1347) in their textbooks, ensuring that temporal propositions remained entrenched in logical discourse throughout the rest of the 14th century and beyond.[3]

Much attention has been devoted to medieval temporal logic since the birth of modern temporal logic in the works of Arthur Prior, whose debt to medieval logicians is explicit [22, Ch. 1, §7]. However, the focus has tended to be on the semantics of tensed categorical propositions (cf. [26,27]), as opposed to the use of temporal connectives. (Two exception are [20] and [21, ch. 1.8].) When it comes to local propositions, involving the connective 'where', the case is even worse: We know of no modern investigation of medieval theories of local propositions (perhaps because so few of the medieval authors discussed these themselves!).

Thus, our first goal in this paper is to collect and present 13th- and early 14th-century views on the logic of temporal (§2) and local propositions (§4). Our second aim is to identify and model the formal properties of the medieval doctrines, so as to compare them to modern approaches. In §3 we consider two modern approaches to the logic of 'while', and show that both of them fail to

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[2] References to AB, AE, and DM are to the Latin editions; all translations are my own.
[3] For example, Richard Lavenham (fl. 1380), Paul of Venice (c. 1369–1429), and John Dorp (fl. 1499) also discuss temporal propositions [20, pp. 167, 169].

capture the medieval ideas. We introduce a new definition of 'while' that does. Next, in §5 we provide a spatial analogue of the medieval 'while', as well as look at modern approaches to logics of space and spatial reasoning, including the logic of elsewhere. We conclude in §6.

2 Temporal propositions

2.1 Temporal propositions in the 13th century

Lambert defines temporal propositions as follows [17, ¶105]:

Definition 2.1 *A temporal* proposition is one whose parts are joined by the adverb 'while', as in 'Socrates runs while Plato argues'.

Similar definitions are given in AE [9, pp. 158–159], AB [9, pp. 190–191], and DM [9, pp. 484–485]. All four definitions crucially include reference to the use of an adverb; as Bacon notes, local and temporal propositions differ from the other type of compound propositions because they are complex 'in virtue of a relation' rather than a connective [2, ¶170]. Bacon himself doesn't mention the presence of an adverb, but that is because he defines local and temporal propositions by ostension. His example of a temporal proposition is the sentence 'Socrates hauls [the boat] in when Plato runs' [2, ¶170], similar to the examples given in the other texts, such as AE's 'While Socrates runs, Plato moves' and AB's 'Socrates reads while Plato disputes'.

None of Bacon, AE, or DM give truth conditions for temporal propositions. AB gives the same truth conditions for both temporal and causal propositions:

Definition 2.2 *If the antecedent is false and the consequent true, the proposition is worthless (*nugatoria*) [9, p. 191].*

The most explicit condition is given by Lambert:

Definition 2.3 *A temporal proposition is true if the two actions stated in the temporal proposition are carried out at the same time; it is false otherwise [17, ¶105].* [4]

The same basic idea is expressed in late-12th-century *Tractatus Anagnini*, when its author notes that "Generally, every temporal proposition is true of which each part is true" [9, p. 252]—a definition which supports Buridan's contention that some authors treated temporal propositions as if they were conjunctions.

2.2 Temporal propositions in the 14th century

The 14th-century views are distinguished from the 13th-century accounts by their sophistication; they are more nuanced, and have greater scope.

Ockham, Buridan, and Burley define the syntax of temporal propositions and their truth conditions almost identically to Lambert; however, all three include a constraint, namely that a temporal proposition be composed "out of categoricals mediated by a temporal adverb" [4, p. 127], [3, p. 65], [18, p. 191]. This has a consequence of disallowing embedded temporal propositions.

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[4] More accurately, it is "in the same time"; the Latin phrase is *in eodem tempore* [7, p. 17].

Additionally, Ockham allows temporal propositions to be composed out of more than two propositions [18, p. 191] (though he gives no explanation of how this would be done, and all his examples involve only two).

Of all the accounts we consider, Ockham's and Burley's are the most detailed. They both allow for temporal adverbs beyond those signifying simultaneity, such as those indicating priority or posteriority in the temporal order. Examples of temporal adverbs of the first type include *dum* 'while, as long as, until' and *quando* 'when, at which time', while examples of the second type include *ante* 'before', *post* and *postquam* 'after', and *priusquam* 'before, until' [4, p. 128]. For temporal adverbs indicating simultaneity, Burley gives the following truth conditions:

Definition 2.4 *For the truth of a temporal [proposition], in which categorical propositions are conjoined by means of an adverb conveying simultaneity of time, it is required that both parts be true for the same time [4, p. 128].*

This condition is further specified depending on whether the time signified is present, past, or future:

> For if the parts of such a temporal [proposition] are propositions of the present, then it is required that both parts be now true for this present time, and if it is of the past, it is required that both parts were true for some past time, this is, because they themselves were true in the present tense for some past time. And if they are propositions of the future, then it is required that both parts be true for some future time, that is, because they themselves will be true in the present tense for some future time [4, p. 128].

Thus, the truth conditions of temporal propositions will depend on the tenses of the sentences being conjoined with the temporal adverb.

There is an important point in which the 14th-century truth conditions offered by Ockham and Burley differ from the 13th-century ones as typified by Lambert, a point which Buridan makes explicit. Lambert requires that the two actions indicated by the temporal propositions are carried out "*at* the same time", and his view is typical of the 13th century. However, the 14th-century conditions change *at* to *for*. Buridan makes this point explicitly, grounded in his token-based semantics which allows for a distinction between a proposition being 'true of' vs. being 'true at' something [23,25]:

> It does not suffice for its categoricals to be true at the same time; for the propositions 'Aristotle existed' and 'The Antichrist will exist' are true at the same time, namely now, but it is required and sufficient that the copulas of the categoricals consignify the same time and that they be true *for* the same time, although not at that time [3, p. 65].

That is, given that the propositions which go into a temporal proposition can themselves be tensed, there is a difference between 'while' statements where the time of reference is the same and 'while' statements where the time of evaluation is the same. Which is taken as primary is one of the ways in which the 14th-century views differ from the 13th-century views.

For adverbs indicating something other than simultaneity, Ockham notes that it is necessary that "the propositions [be] true for different times" [18, p. 191].[5] He also notes that these two conditions do not exclude each other: 'The apostles preached while Christ preached' and 'The apostles preached after Christ preached' are consistent with each other [18, p. 191]. Both Ockham and Burley are also the only ones to consider when a temporal proposition is necessary, impossible, or contingent: "In order for a temporal proposition to be necessary it is required that each part be necessary" [18, p. 191], [4, p. 129]. Given the definitions of impossibility and contingency in terms of necessity and negation, the conditions under which a temporal proposition is impossible or contingent follow naturally [18, p. 192]. A consequence of this definition is that many statements which seem to be expressing necessities, such as 'Wood becomes warm when fire is brought near it', or 'A donkey is risible when it is a man', are not necessary, because neither of their parts is necessary (and in the case of the second example, both parts are necessarily false) [18, p. 191]. These only look like they are necessary because if we interpreted them as conditionals rather than temporal statements, they would be always true, and hence necessary [18, p. 192].

From these definitions, Burley offers a few corollaries concerning inferences involving temporal propositions:

Corollary 2.5 *The negation (*oppositum*) of a temporal [proposition] is a disjunction composed from the opposites of those which were required for the truth of the temporal [4, p. 131].*

However, this is merely a sufficient condition for the falsity of the temporal proposition; it is not a necessary condition.

Corollary 2.6 *A temporal [proposition] implies both of its parts, and not conversely [4, p. 131].*

Corollary 2.7 *A temporal [proposition] implies a conjunction made of the temporal parts, but not conversely [4, p. 131].*

The second two can also be found in Ockham [18, p. 192].

3   The logic of while

Let P and F be the usual backward- and forward-facing unary temporal possibility operators and $\Box$ be universal necessity, both past and future. Our temporal models $\mathfrak{T} = \langle W, \leq, V \rangle$ are linear and continuous, reflecting the fact that for the medievals, 'time' and 'place' are two of the five continuous quantities. These two are differentiated from the other three—line, surface, and body—in that time and place both have corresponding categories, the categories of 'when' and 'where' (cf. [17, ¶561]). We define the truth conditions of these operators in the usual fashion:

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[5] The same condition is given for both 'after' and 'before', which is clearly incomplete.

Definition 3.1 (Unary temporal operators) *For $w \in W$:*

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\begin{aligned}
w &\vDash Pp \quad \textit{iff} \quad \textit{there is } w' \leq w, w' \vDash p \\
w &\vDash Fp \quad \textit{iff} \quad \textit{there is } w' \geq w, w' \vDash p \\
w &\vDash \Box p \quad \textit{iff} \quad \textit{for all } w', w' \vDash p
\end{aligned}
$$

That some medieval authors argue that temporal propositions are reducible to conjunctions, while others argue that they are a type of conditional is grounded in the intuition that, in principle, there are two ways in which 'the same time' can be construed, either existentially or universally. In the first, temporal statements are equivalent to conjunctions, while in the second, temporal statements are equivalent to strict implications. We introduce Q (*quando* 'while', 'at every time') as a binary connective 'while' [6]. The two cases then are:

(i) $w \vDash pQq$ iff there exists w', $w' \vDash p$ and $w' \vDash q$.

(ii) $w \vDash pQq$ iff for all w', if $w' \vDash q$ then $w' \vDash p$.

In order to see which of these is correct, we must make more precise the way in which temporal compounds interact with tensed propositions.

Case (i) can be divided into three possibilities: (a) $w = w'$, (b) $w < w'$, and (c) $w' < w$. Case (a) corresponds to the case when p and q are both present-tensed statements referring to now, in which case, if time is not extended (that is, propositions are true or false at single points, or instants, of time), then for any two present-tensed propositions p and q, pQq is equivalent to $p \wedge q$: For the only way in which two present-tensed propositions can be true at the same time is if they are true now, and if now is a single instant, this means they must both be true at this instant, which is equivalent to their conjunction being true. If two present-tensed propositions are both true now, then they are true at the same instant, and hence each is true while the other is true. Cases (b) and (c) correspond to when the point of reference is either in the past or in the future, that is, statements such as 'Socrates lectured while Plato disputed' or—forgive the somewhat awkward English grammar—'Socrates will lecture while Plato will dispute'. Let us focus on the past-tensed case: Given that the components of 'Socrates lectured while Plato disputed' are 'Socrates lectured' and 'Plato disputed', themselves past-tensed, it would be natural to formalize this as a 'while' connective between two past-tensed statements, e.g., $PpQPq$, in analogy to when the atomic statements are present-tensed. However, it is not clear that this gets the truth conditions right. On an Ockham-Buridan-Burley account, 'Socrates lectured while Plato disputed' being true implies that there was a time for which both propositions were true, i.e., $P(p \wedge q)$. By ordinary temporal reasoning this implies $Pp \wedge Pq$, but from $Pp \wedge Pq$, one cannot make the reverse inference, because there is no guarantee that the time for which p is true and the time for which q is true is the same, a fact which both Ockham and Burley point out. Ockham says:

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[6] We use Q rather than W to avoid confusion with both W a set of worlds and with 'where'.

$$w \xrightarrow{\hspace{2cm}} w'' \xrightarrow{\hspace{2cm}} w'$$
$$p \qquad\qquad\qquad \neg p \qquad\qquad\qquad p$$
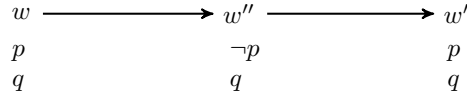$$q \qquad\qquad\qquad\ q \qquad\qquad\qquad\ q$$

Fig. 1. $w \vDash pQq$

...a conjunctive proposition follows from a temporal proposition—but not conversely. For this does not follow: 'Adam existed and Noah existed, therefore Adam existed when Noah existed'. Nor does this follow: 'Jacob existed and Esau existed, therefore Jacob existed when Esau existed' [18, p. 192].[7]

Whether $PpQPq$ implies $P(p \wedge q)$ depends on how, precisely, we interpret Q.

In modern temporal logic, the 'while' operator is most commonly found in the context of dynamic temporal logic, where 'while φ, do α' constructions are commonly used. It is clear that this imperative, dynamic conception of 'while' is *not* what the medieval logicians had in mind. Instead, their notion is static. Modern static temporal logic tends to omit discussion of 'while', taking as basic instead the forward-looking $\mathcal{U}$ 'until' and the backward-looking $\mathcal{S}$ 'since':

Definition 3.2 (Weak until) *For $w \in W$:*

$$w \vDash p\mathcal{U}q \ \textit{iff if there is a } w' \geq w \textit{ s.t. } w' \vDash q$$
$$\textit{then for every } w'', w \leq w'' < w', w'' \vDash p$$

$\mathcal{S}$ *is defined symmetrically.*

Malachi and Owicki use this weak notion of 'until' to define a correspondingly weak notion of 'while' [14, p. 206]:[8]

Definition 3.3 (Malachi & Owicki 'while') *For $w \in W$:*

$$w \vDash pQq \quad \textit{iff} \quad w \vDash p\mathcal{U}(\neg q)$$
$$\textit{iff} \quad \textit{if there is a } w' \geq w \textit{ s.t. } w' \vDash \neg q$$
$$\textit{then for every } w'', w \leq w'' < w', w'' \vDash p$$

Intuitively, defining 'while' in terms of 'until' makes sense, for "p is true while q is true" seems to mean nothing more than "p is true until q is false". However, the 'until' used in the English sentence here is not the same as the weak until defined in Definition 3.2, since if q is always true, then the antecedent will be false and p can be either true or false, as illustrated in Figure 1. As a result, we must reject Malachi and Owicki's definition as unsuitable. An alternative analysis of 'while' is given in [15, p. 260]:

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[7] Burley's example is similar: "Adam was when Noah was, therefore Adam was and Noah was" follows, but "Adam was and Noah was, therefore Adam was when Noah was" does not [4, p. 131].

[8] They read pQq as "q is true as long as p is true", but this cannot be a correct interpretation of $p\mathcal{U}(\neg q)$.

$$t_1 \longrightarrow t_2 \longrightarrow t_3 \longrightarrow t_4 \longrightarrow t_5$$

p	$\neg p$	Pp	Pp	Pp
$\neg q$	q	Pq	Pq	Pq
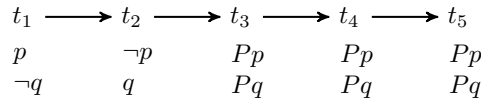
Fig. 2. $t_3 \vDash PpQPq \wedge \neg P(p \wedge q)$

Definition 3.4 (Manna & Pnueli 'while') *For $w \in W$:*

$$w \vDash pQq \text{ iff } w' \vDash p \text{ for every } w' \geq w \text{ such that}$$
$$w'' \vDash q \text{ for all } w'', w \leq w'' \leq w'$$

But this also fails to capture the medieval account, in two ways. First, on this definition, $PpQPq$ does not imply $P(p \wedge q)$. When p and q are past-tensed statements, it is possible for them to both be true at the same time without there being any time for which the present-tense conjunction is true (see Figure 2), which is contrary to what Ockham and Burley argue for above. Thus, if we were to adopt these truth conditions for Q, 'Socrates lectured while Plato disputed' could not be formalized as a temporal compound of two past-tensed sentences. This raises two questions: (1) How should it be formalized?, and (2) What, if anything, does $PpQPq$ represent, given that it appears to be well-formed?

We answer the second question first, by returning to Buridan's distinction of a proposition being *true for* a time vs. being *true at* a time; this is a distinction between the time of evaluation and the time of reference. In Figure 2, Pp and Pq are *true at* the same time, namely t_3, but they are not *true for* the same time; Pp is true for t_1, because t_1 is the witness for the truth of Pp at t_3, while Pq is true for t_2, because t_2 is the witness for the truth of Pq at t_3. Thus, Buridan would reject $PpQPq$ as an appropriate analysis of temporal compounds of past-tensed sentences. However, one needn't deny the acceptability of the 'true at' analysis, and indeed, inspection of the 13th-century views show that this is precisely how they differ from the 14th-century ones: Recall that Lambert says that a temporal proposition is true "if the two actions stated in the temporal proposition are carried out at the same time", rather than that the two actions described are *true for* the same time. In taking this account, he shows similarities with Peter Abelard's views in the middle of the 12th century. As Martin describes Abelard's views:

> The compound temporal proposition formed from the two propositions 'Socrates was a youth' and 'Socrates was an infant' is true at a given time, [Abelard] tells us, just in case each component is true at that time, and so true now of the old Socrates [16, p. 165].

Martin doesn't say how the compound proposition is formed from these two statements, and if it is merely 'Socrates was an infant while Socrates was a youth', then this statement should never be true, since being an infant excludes being a youth. However, if the compound proposition expresses that these two past-tensed statements can be true *at* (as opposed to *for*) the same time, e.g.,
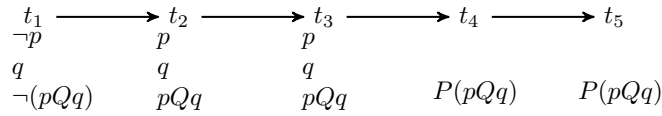
$$t_1 \longrightarrow t_2 \longrightarrow t_3 \longrightarrow t_4 \longrightarrow t_5$$

$\neg p$	p	p		
q	q	q		
$\neg(pQq)$	pQq	pQq	$P(pQq)$	$P(pQq)$

Fig. 3. $t_4 \vDash P(pQq)$

'Socrates was a youth is true at the same time as Socrates was an infant is true', then this is true any time Socrates is alive and was both previously an infant and previously a youth.

Ideally, we would like an account of 'while' which doesn't require us to exclude either the *true for* or the *true at* analysis, and which treats both of these in a uniform fashion. If we adopt $PpQPq$, i.e., applying the tenses directly to the atomic propositions, to indicate that the two tensed propositions are true at the same time, then the natural alternative to represent the 'true for' case is to make the entire temporal compound past-tensed, e.g., $P(pQq)$. Figure 3 gives a model where $P(pQq)$ is true. From this figure, it should be clear that $P(pQq)$ will always imply $PpQPq$, but not conversely. It should also be clear that $\varphi Q\psi$ implies $\varphi \wedge \psi$, regardless of the tenses of the individual components φ and ψ, and regardless of the tense of the entire compound, from which it follows that $P(pQq)$ implies $Pp \wedge Pq$, but also not conversely.

A problem still remains with analysing Q via the conditions given in Definition 3.4. When p and q are both present-tensed, if q is always false, pQq will always be true. [9] For Definition 3.4 can be rewritten, informally, as "for every $w' \geq w$, if w'''s being between w and w' implies that $w'' \vDash q$, then $w' \vDash p$. When q is always false, w'''s being between w and w' does not imply that $w'' \vDash q$, and hence the antecedent of the conditional is falsified, making the entire condition satisfied. But this goes against the medieval requirement that pQq imply $p \wedge q$. If we add this requirement to Definition 3.4, we obtain a characterization of Q that captures the medieval account, wherein pQq implies $p \wedge q$ but is not implied by it:

Definition 3.5 (Medieval 'while') *For $w \in W$:*

$$w \vDash pQq \quad iff \quad w \vDash p \wedge q \text{ and for all } w' \geq w$$
$$if \text{ for all } w'', w \leq w'' < w', w'' \vDash q \text{ then } w' \vDash p$$

An advantage of this account is that it helps understand why some medieval authors try to reduce temporal propositions to conjunctions and others to implications, because the truth conditions have both conjunctive and implicative

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[9] A similar objection can be levied against Kröger's definition of Q [13, p. 22]: For $w \in W$:

$$w \vDash pQq \text{ iff } (1) \text{ there is } w' > w \text{ s.t. } w' \vDash \neg q \text{ and}$$
$$\text{for all } w'', w < w'' < w', w'' \vDash p \text{ or}$$
$$(2) \text{ for all } w' > w, w' \vDash p$$

Because the relations are strict, there is no requirement for $p \wedge q$ to be true at w.

conditions. A further advantage is that on this definition we are able to prove that Ockham's account of the necessity of temporal propositions is correct:

Lemma 3.6 (Ockham) $\Box(pQq)$ *iff* $\Box p \wedge \Box q$.

Proof. Fix $w \in W$.

($\Rightarrow$) If $w \vDash \Box(pQq)$, then for all w', $w' \vDash pQq$. By Definition 3.5, $w' \vDash p \wedge q$, so $w \vDash \Box(p \wedge q)$. Since necessity distributes, $w \vDash \Box p \wedge \Box q$.

($\Leftarrow$) If $w \vDash \Box p \wedge \Box q$, then $w \vDash \Box(p \wedge q)$. By Definition 3.1, for all $w' \in W$, $w' \vDash p \wedge q$, and in particular, $w' \vDash p$. From this it follows that for all $w'' < w' \in W$, $w'' \vDash q$ implies $w' \vDash p$, and for any $w \leq w''$, $w \vDash pQq$. But since w' was arbitrary, this holds of all $w \in W$, so $w \vDash \Box(pQq)$. $\Box$

4 Local propositions

The 13th-century discussions of local propositions are not as extensive, and resemble each other quite a bit. AE [9, p. 159], AB [9, p. 191], and DM [9, p. 485] all give similar accounts which closely resemble Lambert's definition:

Definition 4.1 (Lambert) *A* local *proposition is one whose parts are joined by the adverb 'where', as in 'Socrates runs where Plato argues' [17, ¶104].*

Bacon says that 'Socrates is where Plato is' is called local, and this is a complex proposition in virtue of a relation (namely 'being in the same place as') instead of a connective [2, ¶170], treating the syntactic construction of local propositions analogously to temporal ones. Of the 13th-century authors, only Lambert provides any truth conditions:

Definition 4.2 (Lambert) *A local proposition is true if the two actions stated in the local proposition are carried out in the same place; it is false otherwise [2, ¶104].*

The 14th-century discussions of local propositions are more detailed, but still relatively circumscribed compared to the temporal analyses. Buridan's definition of local propositions mirrors his definition of temporal ones, with the difference that 'temporal adverb' is replaced with 'local adverb' and 'when' with 'where' (though he notes it is possible to replace the local adverb with an equivalent phrase, such as 'Socrates is in the place in which Plato is' [3, p. 66]). For their truth, he provides a necessary but not sufficient condition:

Definition 4.3 (Buridan) *A local proposition is true if "both categoricals [are] true for the place designated by the word 'where', and it is not sufficient that they are true for the same place" [3, p. 66].*

Thus, local propositions will entail conjunctions but not vice versa, as with temporal propositions.

Ockham allows local propositions to be composed of more than two categoricals [18, p. 192], as he does with temporal propositions and with the same caveat noted earlier. An interesting point of textual interpretation arises in the statement of the truth conditions. Some manuscripts say that "for the truth of [a local] proposition, it is required that each part be true for the same place

or for different places" while others give the last clause as "*and not for* different places" (emphasis added) [18, pp. 193, 205]. There is clear evidence that the latter reading should be preferred: After giving this condition, Ockham points it out as the very characteristic by which local propositions differ from temporal ones:

> In this regard it [a local proposition] differs from a temporal one. For in order for a temporal proposition to be true it is required that both parts be true for the same place or for different places, while in order for a local proposition to be true it is required that both parts be true for the same place *and not for* different places [18, p. 193]. [10]

For local propositions to be distinguished from temporal ones on precisely these grounds, it must be the case that one of the two causes of truth is excluded. Burley does not discuss local propositions as a separate type. Instead, he says that local propositions such 'Socrates is moved where he runs' are reducible to one of the five basic types of hypothetical propositions: conditional, causal, temporal, conjunctive, and disjunctive [4, p. 107]. He does not say which, but from other accounts, it is clear that local propositions are analogous to temporal ones.

5   The logic of where

It is important to note that we have the same "true at (or in)" vs. "true for" distinction for local propositions that we had for temporal ones. Therefore, we should expect our analysis of the logic of where to account for this distinction, as we required of our logic of while.

The first, and simplest, modern attempt to capture a local notion in modal logic is the logic of elsewhere introduced by von Wright [29] and completely axiomatized by Segerberg [24]. In this system, $\Box$ is interpreted as 'everywhere else' and $\Diamond$ is read as 'somewhere else'; the frames for this logic are ones where xRy iff $x \neq y$, that is, R is the relation of nonidentity. The logic of elsewhere is the smallest normal modal logic containing all instances of the schemata: [11]

A: $p \to (\Box p \to \Box\Box p)$

B: $p \to \Box\Diamond p$

This logic, of course, describes exactly the opposite of what we need to capture the medieval analyses, because what we want is not the logic of elsewhere but the logic of, so to say, 'here'. However, the natural approach, taking R as identity, is clearly not correct: Such a system would collapse into the trivial system Triv. Adopting this would make local propositions equivalent to conjunctions and destroy the analogy with temporal propositions, something which all medieval authors we consider maintain.

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[10] The italicized portion is our correction of the reading in the translation "or for", following the other manuscript tradition.

[11] In [10,11], axiom A is replaced by the axiom $\Diamond\Diamond p \to (p \lor \Diamond p)$.

Since von Wright, substantially more complex spatial logics, such as ones characterizing topological notions such as 'nearness' and 'distance', have been introduced [1,28]. These logics are aimed at capturing what Aiello and van Bentham call "the ontological structure [of space]: What are the primitive objects and their relations?" [1, p. 320]. This approach can be contrasted with one that looks at "some existing human practice, e.g., a language with spatial expressions (say locative propositions) or a diagrammatic way of visualizing things" [1, p. 320], and identifies and classifies the types of spatial structures that 'fit' these linguistic expressions. Many modern modal logicians prefer the former approach, whereas clearly here it is the latter that will be most beneficial for understanding the medieval practices: It is how we use local propositions, not the nature of the underlying structure of space, that guides the correct analysis of 'where' compounds.

All of the authors we have looked at agree that temporal propositions and local propositions should be treated analogously (though not necessarily identically: As noted in the previous section Ockham requires that the component propositions not be true for different places): 'where' is substituted for 'while', 'same place' is substituted for 'same time'. We begin with looking at how far the analogy between local propositions and temporal propositions can be pushed, specifically at our assumptions concerning the nature of time and whether they are appropriate to transfer to space. We first point out a clear disanalogy between local propositions and temporal ones. The truth of 'p where q' does not depend on the place of evaluation, whereas the truth of pQq depends on the time of evaluation. This reflects the fact that if no explicit time is specified, a temporal proposition must have both of its temporal parts true now; but if no explicit place is specified, a local proposition can be true without either of its parts being true here. A consequence of this is that we can ignore location when considering temporal propositions, but we cannot ignore time or tense when considering local propositions, for it is perfectly natural to say such things as "I will walk where Socrates disputed" or "That church stands where a Roman temple used to stand". In order for sentences like these to be sensible, we need to evaluate propositions not merely at places, but at place/time pairs. This will be reflected below when we evaluate local propositions in $\mathbb{R}^3$.

One assumption about time that we did not address explicitly in the previous section is whether time is point-based or extended (i.e., interval-based). We adopted a point-based approach without comment, for two reasons: First, it is within the bounds of medieval approaches to temporal reasoning to consider at least some statements as being true or false at instants [12], even though many sentences express propositions that are extended in duration. [13] Second,

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[12] 13th- and 14th-century literature on *incipit* 'begins' and *desinit* 'ceases' define these in terms of instants.

[13] For example, DM's example is "When the sun is above the earth, it is day" [9, p. 485] (rewritable into the equivalent "it is day while the sun is above the earth"), or the other examples involving activities of extended duration such as walking, reading, and disputing, all of which are more naturally suited to evaluation at intervals than at points.

actions with extended duration are taken into account by the second conjunct
of the truth conditions in Definition 3.5; while intervals are not used in the
definition under that name, it is clear that the truth of every temporal propo-
sition depends upon intervals bounded below by some point where $p \wedge q$ is true
and extending until q is false.[14]   If we adopt the same approach for space,
we are committed to the view that an infinite number of actions can occur
at a single spatial point; while this is unproblematic for temporal points, it
may be less palatable for spatial points. As an alternative, we could adopt an
approach where regions rather than points are taken as basic, such that truth
in a region does not necessarily propagate down to subregions. For example,
taking an entire football pitch as a region, if Socrates is disputing on the left
half while Plato is running on the right half, "Socrates is disputing where Plato
is running" is true for the entire pitch; but it is false for either of the individual
halves. However, on an alternative view of events, they "unlike material ob-
jects, do not *occupy* the space at which they are located" [5, p. 17], meaning we
can, in principle, allow for an infinite number of events happening at a single
point, even if the material objects acting in those events do not co-exist at that
point.

A further assumption that we made about the nature of time—that it is
linear—is not plausible for space. Dropping linearity means that we are no
longer looking at points and intervals on a line but rather points in $\mathbb{R}^2$ and 2D
regions around these points, regions which will fill the same role as intervals
play in the temporal case. These regions need to be 'well-behaved' or 'normal'
in some intuitive sense: They shouldn't be disjoint, they shouldn't contain
holes,[15] they should be extended in the spatial dimensions only, etc. How
precisely this notion of 'region' is to be defined is ultimately immaterial—
many different possibilities are acceptable, and we do not wish to discriminately
unduly—but since we need a definition, we take the following:

Definition 5.1 (Neighborhood) *A set $A \subset \mathbb{R}^2$ is a* neighborhood *of a point*
$(x, y) \in \mathbb{R}^2$ *if A contains an open set containing (x, y).*[16]

Definition 5.2 (Region) *Let $(x, y) \in \mathbb{R}^2$. We say that $\mathcal{R}(x, y) = A$ is a*
region *of (x, y) if A is a simply connected neighborhood of (x, y).*[17]

This captures the definition of 'where' given by Lambert, quoting the anony-

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[14] It is for this reason that hybrid approaches are not immediately adaptable to our present
needs, because in standard hybrid logics, nominals range over time points, not time intervals.

[15] The requirement that regions not have holes may be too strong. For suppose that Socrates
is a moderately good hammer thrower, such that no hammer he throws ever lands less than
1 meter from him. However, he's not *very* good, so they could land in any direction. Then,
one might want the region referred to in the sentence "Plato walks where Socrates's hammers
fall" to be a torus, rather than a circle. If this example, suggested to me by Charles Walker, is
motivating enough, the requirement of simple connectedness in Definition 5.1 can be dropped.

[16] An alternative definition of 'neighborhood' requires that A itself be the open set containing
(x, y), rather than merely containing an open set, but we do not need this constraint.

[17] Note that this differs from the definition of 'regions' as 'regular closed sets' in [12, p. 514].

mous author of the 12th-century *Book of Six Principles*: [18]

> "[W]here is the circumscription of a body proceeding from the circumscription of a place." For example, water collected in a container adopts the figure of the container and is transfigured in accord with the figure of the interior surface of the container, and the configuration that it has from the interior surface of the container is named 'where' [17, ¶567]. [19]

We are now in a position to define the binary local connective 'U' (*ubi* 'where'), analogous to 'Q but replacing intervals with regions, representing 'wheres':

Definition 5.3 (Medieval 'where') *For* $(t, x, y) \in (\mathbb{R}^3, \leq)$:

$$(t, x, y) \vDash pUq \quad \textit{iff} \quad \textit{there is } x', y' \textit{ s.t. } (t, x', y') \vDash p \wedge q \textit{ and for all } \mathcal{R}(x', y'),$$
$$\textit{if for all } (x'', y'') \in \mathcal{R}(x', y'), (t, x'', y'') \vDash q$$
$$\textit{then for all } (x'', y'') \in \mathcal{R}(x', y'), (t, x'', y'') \vDash p$$

We then redefine the truth conditions for the relevant unary temporal operators with a specification of place:

Definition 5.4 *For* $(t, x, y) \in (\mathbb{R}^3, \leq)$:

$$(t, x, y) \vDash Pp \quad \textit{iff} \quad \textit{there is } t' \leq t, (t', x, y) \vDash p$$
$$(t, x, y) \vDash Fp \quad \textit{iff} \quad \textit{there is } t' \geq t, (t', x, y) \vDash p$$
$$(t, x, y) \vDash \Box p \quad \textit{iff} \quad \textit{for all } t', (t', x, y) \vDash p$$

Because the place is kept fixed, this is equivalent to Definition 3.1 when the place is unspecified. Updating Definition 3.5 to include reference to place is also straightforward.

6   Conclusion

One of the most interesting scientific results of the 20th century was the production of a model under which time and place can be considered in symmetric fashion: One can simply treat the dimension of time as another dimension on a par with the three dimensions of space, length, breadth, and width. This parity is arguably reflected in the medieval theories we've discussed: We have seen a variety of 13th- and 14th-century views of temporal and local propositions, on which the analyses of being true at the same time and being true at the same place are (almost) symmetric, and both arise from natural uses of everyday language. We gave truth conditions for the operators Q and U which both capture the medieval views concerning the analogous structure and truth conditions of these operators as well as provide interesting alternatives to modern approaches. Our definition of Q differs from two modern definitions by requiring the implication from pQq to $p \wedge q$, while our definition of

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[18] And that author in turn takes the definition from Aristotle's *Physics*.

[19] Note that this example is three-dimensional. In our discussion, we are restricting ourselves to the two-dimensional case, but only for simplicity's sake. Our definitions should be generalizable.

U is unique. This paper is only a first step to a complete analysis: The next step after defining the semantics would be to identify the characteristic axioms governing these new operators. We will pursue this in future work.

Acknowledgments

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