Embedding formalisms: hypersequents and two-level systems of rules

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Abstract

A system of rules consists of (possibly labelled) sequent rules connected to each other by some variables and subject to the condition of appearing in a certain order in the derivation. The formalism of systems of rules is quite powerful and allows, e.g., the definition of analytic labelled sequent calculi for intermediate and modal logics characterised by frame conditions beyond the geometric fragment. Using propositional intermediate logics as a case study, we show how to use hypersequent calculus derivations to construct derivations using two-level systems of sequent rules and vice versa. Our transformations (embeddings) show that the hypersequent calculus and this proper restriction of systems of rules have the same expressive power.

Keywords: proof theory, hypersequent calculi, systems of rules, intermediate logics.

1 Introduction

Proof theory provides a constructive approach to investigating fundamental meta-logical and computational properties of a logic through the design and the study of analytic calculi. These calculi, whose proofs proceed by stepwise decomposition of the formulae to be proved, are also the base for developing computerised reasoning methods.

The difficulty in finding analytic sequent calculi for several logics of interest has lead to the introduction of many other formalisms and new ones emerge on a regular basis; prominent examples are the hypersequent calculus [1], which deals with sets of sequents rather than single sequents, and the labelled calculus [6,9], which manipulates sequents containing labelled formulae and relations on labels. The multitude and diversity of the introduced formalisms has made

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it increasingly important to identify their interrelationships and relative expressive power. *Embeddings* bewteen formalisms, i.e. functions that take any calculus in some formalism and yield a calculus for the same logic in another formalism, are useful tools to prove that a formalism subsumes another one in terms of expressiveness (or, when bi-directional, that two formalisms are equi-expressive). Such embeddings can also provide useful reformulations of known calculi and allow the transfer of certain proof-theoretic results, thus alleviating the need for independent proofs in each system and avoiding duplicating work; for example the highly technical proof of cut-elimination for modal provability logic GL for tree-hypersequents [11] could have been induced from the proof for labelled sequents [9], using the subsequently discovered embedding [8] of the former formalism into the latter. Various embeddings between formalisms have appeared in the literature, see, e.g., [12,8,7,13] (and the bibliography thereof).

In this paper we focus on the hypersequent calculus and on the formalism of systems of rules which was introduced in [10] to define analytic labelled calculi for logics semantically characterised by frame conditions beyond the geometric fragment. A system [2] of rules is a set of (possibly labelled) sequent rules linked together by some variables and by the requirement for the rules of appearing in a certain order in the derivation. Systems of rules are quite powerful and enable to define analytic labelled calculi, e.g., for *all* logics characterised by frame properties that correspond to formulae in the Sahlqvist fragment. The downside of this great expressivity is the non-locality of rule applications, which appears at two levels: horizontally, because of the dependency between rules occurring in disjoint branches; and vertically, because of rules that can only be applied above other rules. A possible connection between hypersequents and systems of rules is hinted in [10]. Our paper formalises in full this intuition. Focusing on propositional logics intermediate between intuitionistic and classical logic, we define a bi-directional *embedding* between hypersequents and a subclass of systems of rules in which the vertical non-locality is restricted to at most two (non labelled) sequent rules. We call *two-level systems of rules* this proper restriction of the full formalism. Beside showing that the two seemingly different formalisms are actually a notational variant of each other, our embeddings can be used

- to recover locality in two-level systems of rules;
- to transfer analyticity from hypersequents to two-level systems of rules;
- to define new cut-free two-level systems of rules; e.g. for substructural logics or intermediate logics characterised by Hilbert axioms within the class $\mathcal{P}_3$ in the classification of [5];
- to provide a reformulation of hypersequent calculi which may be of independent interest due to its close relation to natural deduction systems.

[2] The word "system" is used in the same sense as in linear algebra, where there are systems of equations with variables in common, and each equation is meaningful and can be solved only if considered together with the other equations of the system.

2 Preliminaries

A *hypersequent* [1,2] is a |-separated multiset of ordinary sequents, called *components*. The sequents we consider in this paper have the form $\Gamma \Rightarrow \Pi$ where Γ is a (possibly empty) multiset of formulae in the language of intuitionistic logic and Π contains at most one formula.

Notation. Unless stated otherwise we use upper-case Greek letters for multisets of formulae (where Π contains at most one element), lower-case Greek letters for formulae, and G, H for (possibly empty) hypersequents.

As with sequent calculi, the inference rules of hypersequent calculi consist of initial hypersequents (i.e., axioms), the cut-rule as well as logical and structural rules. The logical and structural rules are divided into *internal* and *external rules.* The internal rules deal with formulae within one component of the conclusion. Examples of external structural rules include external weakening (EW) and external contraction (EC), see Fig. 1.

Rules are usually presented as rule schemata. Concrete instances of a rule are obtained by substituting formulae for schematic variables. Following standard practice, we do not explicitly distinguish between a rule and a rule schema.

Fig. 1 displays the hypersequent version HLJ of the propositional sequent calculus LJ for intuitionistic logic. Note that the "hyperlevel" of HLJ is in fact

$$\varphi \Rightarrow \varphi \quad \bot \Rightarrow \Pi \qquad \frac{G \mid \Gamma, \varphi \Rightarrow \Pi \quad G \mid \Gamma, \psi \Rightarrow \Pi}{G \mid \Gamma, \varphi \vee \psi \Rightarrow \Pi}\,(\vee l) \qquad \frac{G \mid \Gamma \Rightarrow \varphi_i}{G \mid \Gamma \Rightarrow \varphi_1 \vee \varphi_2}\,(\vee r)$$

$$\frac{G \mid \Gamma, \varphi, \psi \Rightarrow \Pi}{G \mid \Gamma, \varphi \& \psi \Rightarrow \Pi}\,(\& l) \qquad \frac{G \mid \Gamma \Rightarrow \varphi \quad G \mid \Gamma \Rightarrow \psi}{G \mid \Gamma \Rightarrow \varphi \& \psi}\,(\& r) \qquad \frac{G \mid \Gamma \Rightarrow \Pi}{G \mid \varphi, \Gamma \Rightarrow \Pi}\,(w)$$

$$\frac{G \mid \Gamma \Rightarrow \varphi \quad G \mid \Gamma, \psi \Rightarrow \Pi}{G \mid \Gamma, \varphi \supset \psi \Rightarrow \Pi}\,(\supset l) \qquad \frac{G \mid \Gamma, \varphi \Rightarrow \psi}{G \mid \Gamma \Rightarrow \varphi \supset \psi}\,(\supset r) \qquad \frac{G \mid \varphi, \varphi, \Gamma \Rightarrow \Pi}{G \mid \varphi, \Gamma \Rightarrow \Pi}\,(c)$$

$$\frac{G \mid \Gamma \Rightarrow \varphi \quad G \mid \varphi, \Gamma' \Rightarrow \Pi}{G \mid \Gamma, \Gamma' \Rightarrow \Pi}\,(cut) \qquad \frac{G}{G \mid \Gamma \Rightarrow \Pi}\,(EW) \qquad \frac{G \mid \Gamma \Rightarrow \Pi \mid \Gamma \Rightarrow \Pi}{G \mid \Gamma \Rightarrow \Pi}\,(EC)$$

Fig. 1. Rules and axioms of HLJ.

redundant since a hypersequent $\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_k \Rightarrow \Pi_k$ is derivable in HLJ if and only if $\Gamma_i \Rightarrow \Pi_i$ is derivable in LJ for some $i \in \{1, \ldots, k\}$. Indeed, any sequent calculus can be trivially viewed as a hypersequent calculus. The added expressive power of the latter is due to the possibility of defining new rules which act simultaneously on several components of one or more hypersequents.

Example 2.1 By adding to HLJ the structural rule introduced in [2]

$$\frac{G \mid \Phi, \Gamma_1 \Rightarrow \Pi_1 \quad G \mid \Psi, \Gamma_2 \Rightarrow \Pi_2}{G \mid \Psi, \Gamma_1 \Rightarrow \Pi_1 \mid \Phi, \Gamma_2 \Rightarrow \Pi_2}\,(com)$$

we obtain a cut-free calculus for Gödel logic, which is (axiomatised by) intuitionistic logic with the linearity axiom $(\varphi \supset \psi) \vee (\psi \supset \varphi)$.

Since the usual interpretation of the symbol "|" is disjunctive, the hypersequent calculus is suitable to capture properties (Hilbert axioms, algebraic equations...) that can be expressed in a disjunctive form. More precisely, consider the following classification of intuitionistic formulae, which adapts that in [5] for substructural logics: $\mathcal{N}_0$ and $\mathcal{P}_0$ are the set of atomic formulae

$$\begin{array}{ll} \mathcal{P}_{n+1} & ::= \bot \mid \top \mid \mathcal{N}_n \mid \mathcal{P}_{n+1} \& \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \\ \mathcal{N}_{n+1} & ::= \bot \mid \top \mid \mathcal{P}_n \mid \mathcal{N}_{n+1} \& \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \supset \mathcal{N}_{n+1} \end{array}$$

As shown in [5] all axioms within the class $\mathcal{P}_3$ can be algorithmically transformed into equivalent *structural* hypersequent rules that preserve cut-elimination when added to the calculus HLJ. In particular the rule (com) in Example 2.1 can be automatedly extracted [3] from the linearity axiom.

Notation and Assumptions. Given a hypersequent rule (r) with premisses $G \mid H_1 \ldots G \mid H_n$ and conclusion $G \mid H$, we call *active* the components in the hypersequents $H_1, \ldots, H_n, H$. We call *context components* the components of G. In this paper we will only consider hypersequent rules that (i) are (external) context sharing, i.e., whose premisses all contain the same hypersequent context, and (ii) (except for (EC)) have one active component in each premiss, i.e., in which each H_i is a sequent. Note that (i) is not a restriction and, in absence of eigenvariables, neither is (ii), because we can always transform a rule into an equivalent one that satisfies these conditions.

System of rules were introduced in [10] to define analytic labelled calculi for logics semantically characterised by generalised geometric implications, a class of first-order formulae that includes the frame properties that correspond to formulae in the Sahlqvist fragment. In general, a *system of rules* consists of a set of (possibly labelled) sequent rules $\{(r_{1_1}), \ldots, (r_{1_n}), \ldots, (r_{k_1}), \ldots (r_{k_m}), (r_{end})\}$ connected to each other by (schematic) variables or labels and whose applicability conditions follow the schema

$$\dfrac{\begin{array}{c}\mathcal{D}_1 \\ \vdots \\ \Gamma \Rightarrow \Pi\end{array} \quad \ldots \quad \begin{array}{c}\mathcal{D}_k \\ \vdots \\ \Gamma \Rightarrow \Pi\end{array}}{\Gamma \Rightarrow \Pi}\,(r_{end}) \tag{1}$$

where each derivation $\mathcal{D}_i$, for $1 \le i \le k$, may contain applications of the (r_{i_j}) rules in a specific order. Analyticity of system of rules (when added to a sequent or a labelled sequent calculus for classical or intuitionistic logic) was proved in [10] for systems acting on atomic formulae or relational atoms. We define below a proper restriction of systems of rules which manipulate LJ sequents.

Definition 2.2 A *two-level system of rules* (2-*system* for short) is a set of LJ rules $\{(r_1), \ldots, (r_k), (r_{end})\}$ with applicability condition (1) and in which each derivation $\mathcal{D}_i$, for $1 \le i \le k$, contain (at most) *one* application of the rule (r_i). The rule (r_{end}) is called *ending rule* while $(r_1), \ldots, (r_k)$ *non-ending rules.*

[3] Program at http://www.logic.at/people/lara/axiomcalc.html.

Example 2.3 The 2-system $Sys_{(com^*)}$ in [10] for the linearity axiom (cf. Example 2.1) is the following (φ and ψ are metavariables for formulae):

$$\dfrac{\dfrac{\psi,\varphi,\Gamma_1 \Rightarrow \Pi_1}{\psi,\Gamma_1 \Rightarrow \Pi_1}\,(com'_1) \quad \cdots \quad \dfrac{\psi,\varphi,\Gamma_2 \Rightarrow \Pi_2}{\varphi,\Gamma_2 \Rightarrow \Pi_2}\,(com'_2)}{\dfrac{\Gamma \Rightarrow \Pi \qquad \Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi}\,(com'_{end})}$$

Given a calculus $\mathcal{C}$ and a set of rules $\mathbb{R}$, $\mathcal{C} + \mathbb{R}$ will denote the calculus obtained by adding the elements of $\mathbb{R}$ to $\mathcal{C}$, and $\vdash_{\mathcal{C}+\mathbb{R}}$ its derivability relation.

3 From 2-systems to hypersequent rules and back

Given a 2-system Sys we construct the corresponding hypersequent rule Hr_{Sys}; vice versa, from a hypersequent rule Hr we construct the corresponding 2-system Sys_{Hr}. The transformation of derivations from HLJ + Hr into LJ + Sys_{Hr} (and from LJ + Sys into HLJ + Hr_{Sys}) is shown in Section 4.

From 2-systems to hypersequent rules

Given a 2-system Sys of the form

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ \vdots \\ \Gamma \Rightarrow \Pi\end{matrix} \quad \ldots \quad \begin{matrix}\mathcal{D}_k \\ \vdots \\ \Gamma \Rightarrow \Pi\end{matrix}}{\Gamma \Rightarrow \Pi}\,(r_{end})$$

where each derivation $\mathcal{D}_i$, for $1 \le i \le k$, may contain an application of the rule

$$\dfrac{\varphi_i^1,\ldots,\varphi_i^{l_i},\Gamma_i \Rightarrow \Pi_i \quad \ldots \quad \psi_i^1,\ldots,\psi_i^{m_i},\Gamma_i \Rightarrow \Pi_i}{\theta_i^1,\,\ldots,\,\theta_i^{n_i},\Gamma_i \Rightarrow \Pi_i}\,(r_i)$$

the corresponding hypersequent rule Hr_{Sys} is as follows:

$$\dfrac{M_1 \qquad \ldots \qquad M_k}{G \mid \theta_1^1,\,\ldots,\,\theta_1^{n_1},\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \theta_k^1,\,\ldots,\,\theta_k^{n_k},\Gamma_k \Rightarrow \Pi_k}$$

where M_i, for $1 \le i \le k$, is the multiset of premisses

$$G \mid \varphi_i^1,\ldots,\varphi_i^{l_i},\Gamma_i \Rightarrow \Pi_i \quad \ldots \quad G \mid \psi_i^1,\ldots,\psi_i^{m_i},\Gamma_i \Rightarrow \Pi_i$$

Example 3.1 From Negri's 2-system in Example 2.3 we obtain

$$\dfrac{G \mid \varphi,\psi,\Gamma_1 \Rightarrow \Pi_1 \quad G \mid \varphi,\psi,\Gamma_2 \Rightarrow \Pi_2}{G \mid \psi,\Gamma_1 \Rightarrow \Pi_1 \mid \varphi,\Gamma_2 \Rightarrow \Pi_2}\,(com^*)$$

From hypersequent rules to 2-systems

Given any hypersequent rule Hr of the form

$$\dfrac{M_1 \qquad \ldots \qquad M_k}{G \mid \Theta_1^1,\,\ldots,\,\Theta_1^{n_1},\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Theta_k^1,\,\ldots,\,\Theta_k^{n_k},\Gamma_k \Rightarrow \Pi_k}\,(r)$$

where the sets M_i, for $1 \leq i \leq k$, constitute a partition of the set of premisses of (r) and each M_i contains the premisses

$$G \mid C_i^1 \quad \dots \quad G \mid C_i^{m_i}$$

where $C_i^1, \dots, C_i^{m_i}$ are sequents. The corresponding 2-system Sys_{Hr} is

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \Gamma \Rightarrow \Pi\end{array} \quad \dots \quad \begin{array}{c}\mathcal{D}_k\\ \vdots\\ \Gamma \Rightarrow \Pi\end{array}}{\Gamma \Rightarrow \Pi}\ (r_{end})$$

where the derivation $\mathcal{D}_i$, for $1 \leq i \leq k$, may contain an instance of the rule

$$\dfrac{C_i^1 \quad \dots \quad C_i^{m_i}}{\Theta_i^1, \dots, \Theta_i^{n_i}, \Gamma_i \Rightarrow \Pi_i}\ (r_i)$$

Example 3.2 The translation $Sys_{(com)}$ of the rule (com) in Example 2.1 is

$$\dfrac{\begin{array}{c}\dfrac{\Phi, \Gamma_1 \Rightarrow \Pi_1}{\Psi, \Gamma_1 \Rightarrow \Pi_1}\ (com_1)\\ \vdots\\ \Gamma \Rightarrow \Pi\end{array} \quad \begin{array}{c}\dfrac{\Psi, \Gamma_2 \Rightarrow \Pi_2}{\Phi, \Gamma_2 \Rightarrow \Pi_2}\ (com_2)\\ \vdots\\ \Gamma \Rightarrow \Pi\end{array}}{\Gamma \Rightarrow \Pi}\ (com_{end})$$

Definition 3.3 We say that the premisses of Hr contained in M_i, for $1 \leq i < k$, are *linked* to the component $\Theta_i^1, \dots, \Theta_i^{n_1}, \Gamma_i \Rightarrow \Pi_i$ of the conclusion.

Remark 3.4 A natural deduction calculus for Gödel logic has been introduced in [3]. Defined by reformulating (a variant of) the hypersequent calculus in Example 2.1, the calculus in [3] extends Gentzen NJ calculus for intuitionistic logic with non-local rules which simulate (com) and (EC). The rule $Sys_{(com)}$ in Example 3.2 turns out to be a suitable combination of these non-local rules.

4 Embedding the two formalisms

We introduce procedures to transform 2-system derivations into hypersequent derivations and vice versa.

4.1 From 2-systems to hypersequent derivations

Given any set $\mathbb{S}$ of 2-systems and set $\mathbb{H}$ of hypersequent rules s.t. if $Sys \in \mathbb{S}$ then $Hr_{Sys} \in \mathbb{H}$. Starting from a derivation $\mathcal{D}$ in LJ + $\mathbb{S}$ we construct a derivation $\mathcal{D}'$ in HLJ + $\mathbb{H}$ of the same end-sequent. The construction proceeds by a stepwise translation of the rules in $\mathcal{D}$: non-ending rules of the systems in $\mathbb{S}$ are translated into applications of the corresponding rules in $\mathbb{H}$ (and additional (EW), if needed), ending rules are translated into applications of (EC) and rules of LJ into rules of HLJ (possibly using (EW)). To keep track of the various translation steps, we mark the derivation $\mathcal{D}$. We start by marking and translating the leaves of $\mathcal{D}$. The rules with marked premisses are then translated one by one and the marks are moved to the conclusions of the rules. The process is repeated until we reach and translate the root of $\mathcal{D}$.

Definition 4.1 A *configuration* is a pair $(\mathcal{D}, \mathfrak{M})$ where $\mathcal{D}$ is an LJ + $\mathbb{S}$ derivation, for some set of 2-systems $\mathbb{S}$, and $\mathfrak{M}$ is a set of marks s.t. (i) each mark in $\mathfrak{M}$ refers to a sequent occurrence in $\mathcal{D}$ (*marked sequent*); (ii) in each path between a leaf of $\mathcal{D}$ and its root exactly one marked sequent occurs; and (iii) if a premiss of a non-ending rule of a 2-system $S \in \mathbb{S}$ is marked, then there are no marked sequents below the premisses of any non-ending rule of such instance.

The algorithm

Input: a derivation $\mathcal{D}$ in LJ + $\mathbb{S}$. Output: a derivation $\mathcal{D}'$ of the same sequent in HLJ + $\mathbb{H}$.

Translating axioms. The leaves of $\mathcal{D}$ are marked and copied as leaves of $\mathcal{D}'$.

Translating rules. Rules are translated one by one in the following order: first the one-premiss logical and structural rules applied to marked sequents, then the two-premiss logical rules and ending rules with all premisses marked, and finally (all) non-ending rules of one 2-system instance. After having translated each rule (or all non-ending rules of an occurrence of a 2-system simultaneously), we remove the marks from the premisses of the translated rules and mark their conclusions.

Since the LJ rules are particular instances of HLJ rules, we only show how to translate 2-systems.

$(*)$ For each configuration $(\mathcal{D}, \mathfrak{M})$ we have a set of hypersequent derivations s.t. each marked sequent in $\mathcal{D}$ is translated into one component of the root of exactly one HLJ + $\mathbb{H}$ derivation, and each component of the root of a HLJ + $\mathbb{H}$ derivation translates a marked sequent. This property holds for the leaves: we show that it is preserved by each translation step.

Consider a 2-system $Sys \in \mathbb{S}$ applied in $\mathcal{D}$ with the following instances of

(i) non-ending rules:

$$\dfrac{\begin{matrix}\vdots & & \vdots\\ C_1^1 & \dots & C_1^{m_1}\end{matrix}}{\Delta_1, \Gamma_1 \Rightarrow \Pi_1}\,(r_1) \qquad \dots \qquad \dfrac{\begin{matrix}\vdots & & \vdots\\ C_k^1 & \dots & C_k^{m_k}\end{matrix}}{\Delta_k, \Gamma_k \Rightarrow \Pi_k}\,(r_k)$$

where $C_1^1, \dots, C_1^{m_1}, \dots, C_k^1, \dots, C_k^{m_k}$ are marked sequents. By $(*)$ we have hypersequent derivations of

$$G_1^1 \mid C_1^1 \quad \dots \quad G_1^{m_1} \mid C_1^{m_1} \qquad \dots \qquad G_k^1 \mid C_k^1 \quad \dots \quad G_k^{m_k} \mid C_k^{m_k}$$

Lemma .1 in the Appendix ensures that each displayed instance C_i^j (corresponding to a marked occurrence) occurs exactly once in these hypersequents. We apply Hr_{Sys} as follows

$$\dfrac{M_1 \qquad \dots \qquad M_k}{G \mid \Delta_1, \Gamma_1 \Rightarrow \Pi_1 \mid \dots \mid \Delta_k, \Gamma_k \Rightarrow \Pi_k}$$

where $G = G_1^1 \mid \dots \mid G_1^{m_1} \mid \dots \mid G_k^1 \mid \dots \mid G_k^{m_k}$, and M_i, for $1 \leq i \leq k$, is the set of premisses $G \mid C_i^1 \quad \dots \quad G \mid C_i^{m_i}$ obtained from

$G_i^1 \mid C_i^1 \quad \ldots \quad G_i^{m_i} \mid C_i^{m_i}$ by repeatedly applying (EW). We move the marks to the conclusions of $(r_1), \ldots, (r_k)$.

(ii) ending rule:

$$\dfrac{\begin{matrix}\vdots & & \vdots \\ \Gamma \Rightarrow \Pi & & \Gamma \Rightarrow \Pi\end{matrix}}{\Gamma \Rightarrow \Pi}\ (r_{end})$$

Without loss of generality we can assume that the non-ending rules of the considered 2-system have been applied above the premisses of (r_{end}) (as otherwise the application of the 2-system is redundant). Hence we have a derivation in HLJ + $\mathbb{H}$ of $G \mid \Gamma \Rightarrow \Pi \mid \ldots \mid \Gamma \Rightarrow \Pi$. The desired derivation of $G \mid \Gamma \Rightarrow \Pi$ is obtained by repeatedly applying (EC). The mark is now moved to the conclusion of (r_{end}).

It is easy to see that property $(*)$ is preserved after each step.

Theorem 4.2 *For any set* $\mathbb{H}$ *of hypersequent rules and set* $\mathbb{S}$ *of* 2*-systems s.t. if* $Sys \in \mathbb{S}$ *then* $Hr_{Sys} \in \mathbb{H}$, *if* $\vdash_{\mathrm{LJ}+\mathbb{S}} \Gamma \Rightarrow \Pi$ *then* $\vdash_{\mathrm{HLJ}+\mathbb{H}} \Gamma \Rightarrow \Pi$.

Proof. Apply the above algorithm to the LJ + $\mathbb{S}$ derivation $\mathcal{D}$ of $\Gamma \Rightarrow \Pi$ to obtain $\mathcal{D}'$. The algorithm terminates because the number of rule applications in a derivation is finite. By induction on the number u of 2-system instances whose non-ending rules are still to be translated we prove that the algorithm does not get stuck before translating the root of $\mathcal{D}$. The claim then follows by property $(*)$. If $u = 0$ all remaining rules can be translated as soon as the premisses are marked. Assume $u = n + 1$. By Lemma 4.4 below there is a 2-system instance S that still has untranslated non-ending rules and is not blocked by any other 2-system. Given that all the non-ending rules above the non-ending rules of S must be translated, the only rule applications that have to be translated before S do not belong to any 2-system. These rule applications can be translated as soon as their premisses are marked, and we obtain $u = n$.□

Definition 4.3 Let S and S' be instances of possibly different 2-systems, (r) a non-ending rule occurrence of S and (r') a non-ending rule occurrence of S'. We say that S' *blocks* S (through (r')) and S *is blocked by* S' (on (r)) if (r') occurs some steps above (r).

Lemma 4.4 *Let* $\mathcal{D}$ *be a derivation in* LJ + $\mathbb{S}$ *and* $\overline{\mathbb{S}}$ *any set of instances of* 2*-systems occurring in* $\mathcal{D}$. *There is some* $S \in \overline{\mathbb{S}}$ *that is not blocked by any* $S' \in \overline{\mathbb{S}}$.

Proof. First we prove that $\mathcal{D}$ cannot contain a sequence of instances of 2-systems $S_1, \ldots, S_n, S_{n+1}$ s.t. S_i blocks S_{i+1} for $1 \leq i \leq n$ and $S_1 = S_{n+1}$. We call such a sequence $(S_1, \ldots, S_n, S_{n+1})$ a *loop* and show that the existence of a loop leads to a contradiction. Without loss of generality we assume that $(S_1, \ldots, S_n, S_{n+1})$ is a *distributed* loop, i.e. no S_i in the loop is blocked on a rule application (r) and blocks S_{i+1} through (r). Indeed, from any non-distributed loop we can always extract a subsequence that is a distributed loop by removing some elements from the sequence.

For any element k in the distributed loop either the ending rule of S_k occurs above a premiss of the ending rule of S_{k+1}, or the ending rule of S_{k+1} occur above a premiss of the ending rule of S_k. Otherwise the subtrees of the derivation rooted in S_k and in S_{k+1} are disjoint and S_k cannot block S_{k+1}. Consider now the occurrences $(r_1), \dots, (r_n)$ of the ending rules of $S_1, \dots, S_n$. Let (r_j) be the lowermost rule of the loop, i.e. no $(r_1), \dots, (r_n)$ occurs below (r_j). We distinguish two cases: either (i) all (r_k) $(k = 1, \dots, n;\ k \neq j)$ occur above (only) one premiss of (r_j); or (ii) the (r_k)'s occur above more than one premiss of (r_j). If (i) is the case the loop is not distributed, against the assumption, as S_j is a two-level system. If (ii) holds, $S_1, \dots, S_n$ is not a loop, as systems above different premisses of (r_j) cannot block each other.

We prove now the main statement. Assume that for each $S \in \overline{\mathbb{S}}$ there is $S' \in \overline{\mathbb{S}}$ such that S' blocks S. Either there is a loop among the elements of $\overline{\mathbb{S}}$, but we proved this is impossible; or the cardinality of $\overline{\mathbb{S}}$ is infinite, but this contradicts the fact that $\mathcal{D}$ contains a finite number of rule applications. □

Example 4.5 The following derivation in the calculus LJ+$Sys_{(com^*)}$ for Gödel logic (see Example 2.3)

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha \quad \dfrac{\bot \Rightarrow}{\bot, \alpha \Rightarrow}\,(w)}{\alpha \supset \bot, \alpha \Rightarrow}\,(\supset l)}{\dfrac{\alpha \Rightarrow}{\Rightarrow \alpha \supset \bot}\,(\supset r)}\,(com'_1)}{\Rightarrow (\alpha \supset \bot) \vee ((\alpha \supset \bot) \supset \bot)}\,(\vee r) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\bot \Rightarrow}{\alpha, \bot \Rightarrow}\,(w) \quad \alpha \Rightarrow \alpha}{\alpha, \alpha \supset \bot \Rightarrow}\,(\supset l)}{\alpha \supset \bot \Rightarrow}\,(com'_2)}{\Rightarrow (\alpha \supset \bot) \supset \bot}\,(\supset r)}{\Rightarrow (\alpha \supset \bot) \vee ((\alpha \supset \bot) \supset \bot)}\,(\vee r)}{\Rightarrow (\alpha \supset \bot) \vee ((\alpha \supset \bot) \supset \bot)}\,(com'_{end})$$

is translated into the HLJ + (com^*) derivation (see Example 3.1)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha \quad \dfrac{\bot \Rightarrow}{\bot, \alpha \Rightarrow}\,(w)}{\alpha \supset \bot, \alpha \Rightarrow}\,(\supset l) \quad \dfrac{\dfrac{\bot \Rightarrow}{\alpha, \bot \Rightarrow}\,(w) \quad \alpha \Rightarrow \alpha}{\alpha, \alpha \supset \bot \Rightarrow}\,(\supset l)}{\alpha \Rightarrow \mid \alpha \supset \bot \Rightarrow}\,(com^*)}{\alpha \Rightarrow \mid \Rightarrow (\alpha \supset \bot) \supset \bot}\,(\supset r)}{\Rightarrow \alpha \supset \bot \mid \Rightarrow (\alpha \supset \bot) \supset \bot}\,(\supset r)}{\Rightarrow \alpha \supset \bot \mid \Rightarrow (\alpha \supset \bot) \vee ((\alpha \supset \bot) \supset \bot)}\,(\vee r)}{\Rightarrow (\alpha \supset \bot) \vee ((\alpha \supset \bot) \supset \bot) \mid \Rightarrow (\alpha \supset \bot) \vee ((\alpha \supset \bot) \supset \bot)}\,(\vee r)}{\Rightarrow (\alpha \supset \bot) \vee ((\alpha \supset \bot) \supset \bot)}\,(EC)$$

4.2 From hypersequent to 2-system derivations

Given any set $\mathbb{H}$ of hypersequent rules and set $\mathbb{S}$ of 2-systems s.t. if $Hr \in \mathbb{H}$ then $Sys_{Hr} \in \mathbb{S}$. Starting from a derivation in HLJ + $\mathbb{H}$ we construct a derivation in LJ + $\mathbb{S}$ of the same end-sequent.

The algorithm

Input: a suitable derivation $\mathcal{D}$ of a sequent $\Gamma \Rightarrow \Pi$ in HLJ + $\mathbb{H}$. Output: a derivation $\mathcal{D}'$ of $\Gamma \Rightarrow \Pi$ in LJ + $\mathbb{S}$.

Intuitively, each application of a HLJ rule in $\mathcal{D}$ is rewritten as an application of an LJ rule in $\mathcal{D}'$. Some care is needed to handle the external structural rules in $\mathbb{H}$ as well as (EW) and (EC). To deal with the latter rules, which have no direct translation in LJ + $\mathbb{S}$, we consider derivations $\mathcal{D}$ in which (i) all applications of (EC) occur immediately above the root, and (ii) all applications of (EW) occur where immediately needed, that is they introduce components of the context of rules with more than one premiss. As shown in Section 4.2.1 each hypersequent derivation (of a sequent) can be transformed into an equivalent one having this shape.

The rules in $\mathbb{H}$ are translated in two steps. First for each component of the premiss of the uppermost application of (EC) in $\mathcal{D}$ we find a *partial derivation*, that is a derivation in LJ extended by the rules of the 2-systems in $\mathbb{S}$ without any applicability condition (Lemma 4.13). The desired derivation $\mathcal{D}'$ is then obtained by suitably applying to these partial derivations the corresponding ending rules (Theorem 4.14).

Definition 4.6 A partial derivation in LJ + $\mathbb{S}$ is a derivation in LJ extended with the non-ending rules of $\mathbb{S}$ (without their applicability conditions).

We show an example of translation to guide the reader's intuition through the proofs that follow.

Example 4.7 Consider the HLJ + (com) derivation (see Example 2.1)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha \Rightarrow \alpha \mid \beta \Rightarrow \alpha\&\beta}\,(EW) \qquad \dfrac{\dfrac{\beta \Rightarrow \beta \quad \alpha \Rightarrow \alpha}{\alpha \Rightarrow \beta \mid \beta \Rightarrow \alpha}\,(com) \qquad \dfrac{\beta \Rightarrow \beta}{\alpha \Rightarrow \beta \mid \beta \Rightarrow \beta}\,(EW)}{\alpha \Rightarrow \beta \mid \beta \Rightarrow \alpha\&\beta}\,(\&r)}{\alpha \Rightarrow \alpha\&\beta \mid \beta \Rightarrow \alpha\&\beta}\,(\&r)}{\alpha \Rightarrow \alpha\&\beta \mid\Rightarrow \beta \supset (\alpha\&\beta)}\,(\supset r)}{\Rightarrow \alpha \supset (\alpha\&\beta) \mid\Rightarrow \beta \supset (\alpha\&\beta)}\,(\supset r)}{\Rightarrow \alpha \supset (\alpha\&\beta) \mid\Rightarrow (\alpha \supset (\alpha\&\beta)) \vee (\beta \supset (\alpha\&\beta))}\,(\vee r)}{\Rightarrow (\alpha \supset (\alpha\&\beta)) \vee (\beta \supset (\alpha\&\beta)) \mid\Rightarrow (\alpha \supset (\alpha\&\beta)) \vee (\beta \supset (\alpha\&\beta))}\,(\vee r)}{\Rightarrow (\alpha \supset (\alpha\&\beta)) \vee (\beta \supset (\alpha\&\beta))}\,(EC)$$

First observe that this derivation satisfies property (i) and, as (EW) cannot be moved below the two-premiss rule $(\&r)$, also (ii). The partial derivations in LJ + $\{(com_1), (com_2)\}$ (see Example 3.2) for the components of the uppermost application of (EC) in the above proof are:

$$\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha \quad \dfrac{\beta \Rightarrow \beta}{\alpha \Rightarrow \beta}\,(com_1)}{\alpha \Rightarrow \alpha\&\beta}\,(\&r)}{\Rightarrow \alpha \supset (\alpha\&\beta)}\,(\supset r)}{\Rightarrow (\alpha \supset (\alpha\&\beta)) \vee (\beta \supset (\alpha\&\beta))}\,(\vee r) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\beta \Rightarrow \alpha}\,(com_2) \quad \beta \Rightarrow \beta}{\beta \Rightarrow \alpha\&\beta}\,(\&r)}{\Rightarrow \beta \supset (\alpha\&\beta)}\,(\supset r)}{\Rightarrow (\alpha \supset (\alpha\&\beta)) \vee (\beta \supset (\alpha\&\beta))}\,(\vee r)$$

These partial derivations have the same "structure" as the hypersequent derivations (see *ancestor tree* in Definition 4.11) of the corresponding components. The desired derivation of $\Rightarrow (\alpha \supset (\alpha\&\beta)) \vee (\beta \supset (\alpha\&\beta))$ in LJ+$Sys_{(com)}$ is simply obtained by connecting the partial derivations via (com_{end}).

We use Definition 4.8 and 4.9 to formalise and achieve properties (i) and (ii).

Definition 4.8 For any one-premiss rule (r) we call a *queue of* (r) a sequence of consecutive applications of (r) that is neither immediately preceded nor immediately followed by applications of (r).

Definition 4.9 We say that an HLJ+$\mathbb{H}$ derivation is in *structured form iff* all (EC) applications occur in a queue immediately above the root, and all (EW) applications occur in subderivations of the form

$$\dfrac{\dfrac{\dfrac{G_1 \mid C_1}{\vdots}\,(EW)}{G \mid C_1}\,(EW) \qquad \dots \qquad \dfrac{\dfrac{\dfrac{G_n \mid C_n}{\vdots}\,(EW)}{G \mid C_n}\,(EW)}{G \mid C_0}\,(r)$$

where (r) is any rule with more than one premiss and each component of G is contained in at least one of the hypersequents $G_1, \dots, G_n$.

A derivation in structured form can be divided into a part containing only (EC) applications and a part containing the applications of any other rule. We introduce a notation for the hypersequent separating the two parts.

Definition 4.10 If $\mathcal{D}$ is a derivation in structured form, we denote by $\widehat{H}_{\mathcal{D}}$ the premiss of the uppermost application of (EC) in $\mathcal{D}$.

We prove below that from any HLJ + $\mathbb{H}$ derivation $\mathcal{D}$ of a sequent we can construct a partial derivation for each component of $\widehat{H}_{\mathcal{D}}$ having the same structure as the ancestor tree of that component (Definition 4.11). By *having the same structure* we mean that the partial derivation of a hypersequent component contains the translation of the rules in the ancestor tree of that component in $\mathcal{D}$, with the exception of (EW).

Definition 4.11 Given a HLJ+$\mathbb{H}$ derivation. A sequent (hypersequent component) C' is a *parent* of a sequent C, denoted as $p(C, C')$, if one of the following conditions holds:

- C is active in the conclusion of an application of some $Hr \in \mathbb{H}$, and C' is the active component of a premiss linked to C (see Definition 3.3);
- C is active in the conclusion of an application of a rule of HLJ, and C' is the active component of a premiss of such application;
- C is a context component in the conclusion of any rule application, and C' is the corresponding context component in a premiss of such application.

We say that a sequent C' is an *ancestor* of a sequent C, and we write $a(C, C')$, if the pair (C, C') is in the transitive closure of the relation $p(\cdot,\cdot)$. The *ancestor tree* of a sequent C is the tree whose nodes are all sequents related to C by $a(\cdot,\cdot)$ and whose edges are defined by the relation $p(\cdot,\cdot)$ between such nodes.

Remark 4.12

- In an HLJ + $\mathbb{H}$ derivation that does not use (EC), the ancestor tree of each hypersequent is a sequent derivation.
- If C is the active component of an application of (EW), then there is no C' such that $p(C, C')$.

As usual, the *length* of a derivation is the the maximal number of applications of inference rules +1 occurring on any branch.

Lemma 4.13 *Let $\mathbb{H}$ be a set of hypersequent rules and $\mathbb{S}$ of 2-systems s.t. if $Hr \in \mathbb{H}$ then $Sys_{Hr} \in \mathbb{S}$. Given any* HLJ + $\mathbb{H}$ *derivation $\mathcal{D}$ in structured form, for each component C of $\widehat{H}_{\mathcal{D}}$ we can construct a partial derivation in* LJ + $\mathbb{S}$ *having the same structure as the ancestor tree of C in $\mathcal{D}$.*

Proof. Let H be a hypersequent in $\mathcal{D}$ derived without using (EC). We construct a partial derivation in LJ + $\mathbb{S}$ with the required property for each of its components. The proof proceeds by induction on the length l of the derivation of H by translating each rule of HLJ + $\mathbb{H}$, with the exception of (EW), into the corresponding sequent rule in LJ + $\mathbb{S}$.
Base case. If $l = 1$ (i.e. H is an axiom) the partial derivation in LJ + $\mathbb{S}$ simply contains H.
Inductive step. We consider the last rule (r) $(\neq (EW))$ applied in the (sub)derivation $\mathcal{D}'$ of H, and we distinguish the two cases: (i) (r) is a one-premiss rule and (ii) (r) has more premisses; for the latter case, since $\mathcal{D}'$ is in structured form, we deal also with possible queues of (EW) above its premisses.

(i) Assume that the derivation ending in a one-premiss rule $(r) \in$ HLJ is

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdots\\ G \mid C\end{array}}{G \mid C'}\ (r)$$

By induction hypothesis there is a partial derivation of C (and of each component of G) having the same structure as the ancestor tree of C. The partial derivation of C' is simply obtained by applying (r).

The case in which (r) is a one-premiss rule belonging to $\mathbb{H}$ is a special case of (ii), for which there is no need to consider queues of (EW).

(ii) Assume that $(r) = (Hr) \in \mathbb{H}$ has more than one premiss, the remaining cases ($(r) \in$ HLJ, and $(r) \in \mathbb{H}$ and has only one premiss) being simpler. Assume that the derivation $\mathcal{D}'$, of length n, ends as follows

$$\dfrac{\begin{array}{c}\mathcal{D}_1^1\\ \vdots\\ G \mid C_1^1\end{array}\quad \dots \quad \begin{array}{c}\mathcal{D}_1^{m_1}\\ \vdots\\ G \mid C_1^{m_1}\end{array}\quad \dots \quad \begin{array}{c}\mathcal{D}_k^1\\ \vdots\\ G \mid C_k^1\end{array}\quad \dots \quad \begin{array}{c}\mathcal{D}_k^{m_k}\\ \vdots\\ G \mid C_k^{m_k}\end{array}}{G \mid \Delta_1, \Gamma_1 \Rightarrow \Pi_1 \mid \dots \mid \Delta_k, \Gamma_k \Rightarrow \Pi_k}\ (Hr)$$

where the premisses $G \mid C^i_j$ of (Hr) are possibly inferred by a queue of (EW). When this is the case we consider the uppermost hypersequents in the queue. More precisely, we consider the following derivations (each of which has length strictly less than n)

$$\begin{array}{ccccccc} \mathcal{D}^1_1 & & \mathcal{D}^{m_1}_1 & & \mathcal{D}^1_k & & \mathcal{D}^{m_k}_k \\ \vdots & & \vdots & & \vdots & & \vdots \\ G^1_1 \mid C^1_1 & \dots & G^{m_1}_1 \mid C^{m_1}_1 & \dots & G^1_k \mid C^1_k & \dots & G^{m_k}_k \mid C^{m_k}_k \end{array}$$

where, for $1 \leq y \leq k$ and $1 \leq x \leq m_y$, the hypersequent G^x_y is G if there is no (EW) application immediately above $G \mid C^x_y$; otherwise, $G^x_y \mid C^x_y$ is the premiss of the uppermost (EW) application in the queue immediately above $G \mid C^x_y$.

Since $\mathcal{D}$ (and hence $\mathcal{D}'$) is in structured form each component of G must occur in at least one of the hypersequents $G^1_1, \dots, G^{m_1}_1, \dots, G^1_k, \dots, G^{m_k}_k$. Hence the induction hypothesis gives us partial derivations for each component of G [4]. We obtain partial derivations for $\Delta_1, \Gamma_1 \Rightarrow \Pi_1, \dots, \Delta_k, \Gamma_k \Rightarrow \Pi_k$ applying the non-ending rules of the 2-system Sys_{Hr} as follows

$$\frac{C^1_1 \quad \dots \quad C^{m_1}_1}{\Delta_1, \Gamma_1 \Rightarrow \Pi_1}\ (r_1) \qquad \dots \qquad \frac{C^1_k \quad \dots \quad C^{m_k}_k}{\Delta_k, \Gamma_k \Rightarrow \Pi_k}\ (r_k)$$

Indeed, by induction hypothesis we have a partial derivation for each C^x_y.

The obtained partial derivations clearly satisfy the following property: (with the exception of (EW) and of the dummy ending rules) a rule application occurs in the ancestor tree of a hypersequent component in $\mathcal{D}$ *iff* its translation occurs in the partial derivation of such component. □

Theorem 4.14 *For any set $\mathbb{H}$ of hypersequent rules and set $\mathbb{S}$ of 2-systems s.t. if $Hr \in \mathbb{H}$ then $Sys_{Hr} \in \mathbb{S}$, if $\vdash_{\mathrm{HLJ}+\mathbb{H}} \Gamma \Rightarrow \Pi$ then $\vdash_{\mathrm{LJ}+\mathbb{S}} \Gamma \Rightarrow \Pi$.*

Proof. Let $\mathcal{D}$ be a HLJ+$\mathbb{H}$ derivation of $\Gamma \Rightarrow \Pi$. By the results in Section 4.2.1 we can assume that $\mathcal{D}$ is in structured form. By applying the procedure of Lemma 4.13 to the premiss $\widehat{H}_{\mathcal{D}}$ of the uppermost application of (EC) in $\mathcal{D}$ we obtain a set of partial derivations $\{\mathcal{D}_i\}_{i\in I}$ whose rules translate those occurring in the ancestor trees of each component of $\widehat{H}_{\mathcal{D}}$. We show that we can suitably apply the ending rules of 2-systems in $\mathbb{S}$ to the roots of $\{\mathcal{D}_i\}_{i\in I}$ in order to obtain the required LJ + $\mathbb{S}$ derivation of $\Gamma \Rightarrow \Pi$.

To do that we first group all non-ending rule applications in $\{\mathcal{D}_i\}_{i\in I}$ according to the application of $Hr \in \mathbb{H}$ that these rules translate. For each such group we apply one ending rule below the partial derivations in which the non-ending

[4] In case we have different partial derivations for a component C of G we can always obtain one partial derivation by applying a "dummy" ending rule as

$$\frac{C \quad \dots \quad C}{C}$$

rules of the group occur. The needed ending rules are always applicable. This is not the case only when two non-ending rules belonging to the same 2-system instance occur in a same partial derivation $\mathcal{D}'_j$. Two cases can arise: either (i) $\mathcal{D}'_j \in \{\mathcal{D}_i\}_{i\in I}$, or (ii) $\mathcal{D}'_j$ is obtained by the application of an ending rule in LJ + $\mathbb{S}$ to two partial derivations. If (i), two nodes in the ancestor tree of a component of $\widehat{H}_{\mathcal{D}}$ must be active in the conclusion of a single Hr application, which contradicts Definition 4.11 and the fact that we deal with hypersequent rules having premisses with one active component only. If (ii), assume that there are two applications $(Hr_1), (Hr_2)$ of rules in $\mathbb{H}$ s.t. both have active components in the ancestor trees of the same two components C_1 and C_2 of $\widehat{H}_{\mathcal{D}}$. If (Hr_1) occurs below (Hr_2) two active components in (Hr_1) cannot have ancestors that are active in (Hr_2) (and vice versa), against the assumptions of (ii). Otherwise, (Hr_1) and (Hr_2) occur above different premisses of a rule application (r). If this holds, we show that we can modify the partial derivations in $\{\mathcal{D}_i\}_{i\in I}$ in order to apply the needed ending rule. Indeed, some nodes of the ancestor tree of one of C_1 and C_2 must be context components of (r) (otherwise the elements of the two ancestor trees never occur in the same hypersequent above (r)). Then the non-ending rules translating (Hr_1) and (Hr_2) occur in a partial derivation above different premisses of a dummy ending rule. To avoid this situation it is enough to split the two premisses in different partial derivations (removing some premisses of the dummy ending rule in each partial derivation). In general, there could be n applications of rules in $\mathbb{H}$ s.t. the 1^{st} and the n^{th} have active components in the ancestor tree of the same component of $\widehat{H}_{\mathcal{D}}$, and the i^{th} application (for $1 \leq i < n$) has active components in the ancestor tree of the same component of $\widehat{H}_{\mathcal{D}}$ as the $(i+1)^{\text{th}}$. We can reason in a similar way considering the whole sequence of applications instead of a pair.

Hence, we eventually obtain an LJ + $\mathbb{S}$ derivation of $\Gamma \Rightarrow \Pi$. □

4.2.1 Pre-processing of hypersequent derivations

In the previous algorithm we only considered hypersequent derivations in structured form, i.e. in which (EC) applications occur immediately above the root and (EW) applications occur where needed. Here we show how to transform each hypersequent derivation into a derivation in structured form.

Definition 4.15 The *external contraction rank (ec-rank)* of an application E of (EC) in a derivation is the number of applications of rules other than (EC) between E and the root of the derivation.

Lemma 4.16 *Each* HLJ + $\mathbb{H}$ *derivation* $\mathcal{D}$ *can be transformed into a derivation of the same end-hypersequent in which all* (EC) *applications have ec-rank* 0*.*

Proof. Proceed by double induction on the lexicographically ordered pair $\langle \mu, \nu \rangle$, where μ is the maximum ec-rank of any (EC) application in $\mathcal{D}$, and ν is the number of (EC) applications in $\mathcal{D}$ with maximum ec-rank.
Base case. If $\mu = 0$ the claim trivially holds.
Inductive step. Assume that $\mathcal{D}$ has maximum ec-rank μ and that there are ν applications of the rule (EC) with ec-rank μ. We show how to transform $\mathcal{D}$

into a derivation $\mathcal{D}'$ having either maximum ec-rank $\mu' < \mu$ or ec-rank μ and number of (EC) applications with maximum ec-rank $\nu' < \nu$.

Consider an (EC) application with ec-rank μ in $\mathcal{D}$ and the queue of (EC) containing it. There cannot be any applications of (EC) above this queue because the ec-rank of its elements is maximal. We distinguish cases according to the rule (r) applied to the conclusion of the last element of such queue.

Assume that (r) has one premiss. If $(r) = (EW)$, we apply (EW) (with the same active component) before the queue. If $(r) \neq (EW)$, we apply (r) immediately before the queue, possibly followed by applications of (EC).

Notation. Given a hypersequent H we denote by $(H)^u$ the hypersequent $H \mid \ldots \mid H$ containing u of copies of H $(u \geq 0)$.

Let (r) be a(ny external) context-sharing rule with more than one premiss and consider any subderivation of $\mathcal{D}$ of the form

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdots\\ G \mid G'_1 \mid (C_1)^{m_1}\end{matrix}}{\vdots}\,(EC)}{G \mid C_1}\,(EC) \quad \ldots \quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_n\\ \vdots\\ G \mid G'_n \mid (C_n)^{m_n}\end{matrix}}{\vdots}\,(EC)}{G \mid C_n}\,(EC)}{G \mid H}\,(r)$$

where G'_i, for $1 \leq i \leq n$, only contains components in G and the derivations $\mathcal{D}_1, \ldots, \mathcal{D}_n$ contain no application of (EC). We can transform $\mathcal{D}$ into a derivation $\mathcal{D}'$ in which all applications of (EC) occurring above the hypersequent $G \mid H$ are either immediately above it or immediately above another application of (EC); their ec-rank is thus reduced by 1.

We first prove that $(\star)$ the hypersequent $G \mid G'' \mid (H)^q$, where $G'' = G'_1 \mid \ldots \mid G'_n$ and $q = (\sum_{i=1}^n (m_i - 1)) + 1$ is derivable from

$$G \mid G'_1 \mid (C_1)^{m_1} \ , \ \ldots \ , \ G \mid G'_n \mid (C_n)^{m_n}$$

using only (EW) and (r). The hypersequent $G \mid H$ then follows from $G \mid G'' \mid (H)^q$ by (EC) as all the components of G'' occur also in G. The obtained derivation $\mathcal{D}'$ has maximum ec-rank $\mu' < \mu$, or the occurrences of (EC) with ec-rank μ occurring in it are $\nu' < \nu$.

It remains to prove claim $(\star)$. For each element of the set

$$\mathbb{Q} = \{G \mid G'' \mid (H)^0 \mid (C_1)^{x_1} \mid \ldots \mid (C_n)^{x_n} \ : \ \sum_{i=1}^n x_i \ = \ (\sum_{i=1}^n (m_i - 1)) + 1\}$$

there is a derivation from $G \mid G'_1 \mid (C_1)^{m_1}$, ... , $G \mid G'_n \mid (C_n)^{m_n}$ using only (EW). Indeed for any hypersequent in $\mathbb{Q}$ and for $1 \leq i \leq n$, there is at least one $x_i \geq m_i$, because otherwise $\sum_{i=1}^n x_i \ < \ (\sum_{i=1}^n (m_i - 1)) + 1$. The claim $(\star)$ therefore follows by Lemma 4.17 below being $G \mid G'' \mid (H)^q$ the only element of the set $(q = (\sum_{i=1}^n (m_i - 1)) + 1)$

$$\mathbb{Q}' = \{G \mid G'' \mid (H)^q \mid (C_1)^{x_1} \mid \ldots \mid (C_n)^{x_n} \ : \ \sum_{i=1}^n x_i \ = \ 0\}.$$

□

The following is the central lemma of the previous proof.

Lemma 4.17 *For any application of a hypersequent rule*

$$\frac{G \mid C_1 \qquad \dots \qquad G \mid C_n}{G \mid H}\ (r)$$

and natural number $d \geq 0$, consider the set of hypersequents

$$\mathbb{L}_d = \{G \mid (H)^c \mid (C_1)^{x_1} \mid \dots \mid (C_n)^{x_n} \ : \ \sum_{i=1}^{n} x_i = d\}$$

where G, H are hypersequents, $C_1, \dots, C_n$ sequents, and c is a natural number. For any natural number e, s.t. $0 \leq e \leq d$, each element of the set

$$\mathbb{L}_{(d-e)} = \{G \mid (H)^{c+e} \mid (C_1)^{x'_1} \mid \dots \mid (C_n)^{x'_n} \ : \ \sum_{i=1}^{n} x'_i = d - e\}$$

is derivable from hypersequents in $\mathbb{L}_d$ by repeatedly applying the rule (r).

Proof. By induction on e.
Base case: If $e = 0$, then $\mathbb{L}_d = \mathbb{L}_{d-e}$.
Inductive step: Assume that $e > 0$ and that the claim holds for all $e' < e$. By induction hypothesis there exists a derivation from the hypersequents in $\mathbb{L}_d$ for each element of the set

$$\mathbb{L}_{(d-(e-1))} = \{G \mid (H)^{c+(e-1)} \mid (C_1)^{x''_1} \mid \dots \mid (C_n)^{x''_n} \ : \ \sum_{i=1}^{n} x''_i = d - (e-1)\}$$

that only consists of applications of (r). Any hypersequent

$$G \mid (H)^{c+e} \mid (C_1)^{x'_1} \mid \dots \mid (C_n)^{x'_n}$$

in $\mathbb{L}_{(d-e)}$ can be derived from elements of $\mathbb{L}_{(d-(e-1))}$ as follows:

$$\frac{G \mid (H)^{c+(e-1)} \mid H'_1 \quad \dots \quad G \mid (H)^{c+(e-1)} \mid H'_n}{G \mid (H)^{c+e} \mid (C_1)^{x'_1} \mid \dots \mid (C_n)^{x'_n}}\ (r)$$

where, for $1 \leq i \leq n$, $H'_i = (C_1)^{y_1} \mid \dots \mid (C_n)^{y_n}$ is such that if $j \neq i$ then $y_j = x'_j$ and if $j = i$ then $x'_j + 1$; i.e., the components $C_1, \dots, C_n \notin G$ occur in the i^{th} premiss as many times as in the conclusion, except for C_i which occurs one more time.

All premisses of this rule application are hypersequents in $\mathbb{L}_{(d-(e-1))}$, indeed $(x'_1 + 1) + x'_2 + \dots + x'_n = \dots = x'_1 + \dots + x'_{n-1} + (x'_n + 1) = (\sum_{i=1}^{n} x'_i) + 1$ and $(\sum_{i=1}^{n} x'_i) + 1 = (d-e) + 1 = d-(e-1)$. Given that only the rule (r) is used to derive the elements of $\mathbb{L}_{d-(e-1)}$ from the elements of $\mathbb{L}_d$, also the elements of $\mathbb{L}_{(d-e)}$ can be derived from those of $\mathbb{L}_d$ by applying only (r). □

Lemma 4.18 *Any* HLJ + $\mathbb{H}$ *derivation of a sequent can be transformed into a derivation in structured form.*

Proof. Let $\mathcal{D}$ be a derivation of a sequent S in HLJ + $\mathbb{H}$. By Lemma 4.16 we can assume that all applications of (EC) in $\mathcal{D}$ occur in a queue immediately above S. Consider an application of (EW), with premiss G and conclusion $G \mid C$, which is not as in Definition 4.9. First notice that $G \mid C$ cannot be the root of $\mathcal{D}$. We show how to shift this application of (EW) below other rule applications until the statement is satisfied for such application. Three cases can arise:

(i) C is the active component in the premiss of an application of a rule (r). The conclusion of (r) is simply obtained by applying (EW) (possibly multiple times) to G.

(ii) C is a context component in the premiss of an application of a one-premiss rule (r). The (EW) is simply shifted below (r).

(iii) C occurs actively inside the queues of (EW) above all the premisses of an application of a rule (r). We remove all the applications of (EW) with active component C in the queues and apply (r) with one context component less, followed by (EW).

The termination of the procedure follows from the fact that $\mathcal{D}$ is finite and that (i)–(iii) always reduce the number of rules different from (EW) occurring below the (EW) applications. □

5 Applications and Future Work

We provided constructive transformations (embeddings) from hypersequent derivations into derivations in 2-systems of rules and back. Defined using intermediate logics as a case study, the embeddings do not depend on the considered calculus rules and can be naturally extended to other classes of (propositional) logics, e.g., substructural or modal logics. This shows that the two seemingly different proof frameworks have the same expressive power.

For 2-systems, the benefits of the embedding include: (i) analyticity proofs, (ii) new cut-free calculi and (iii) locality of derivations using the |-notation. Ad (i): the method in [10] transforms generalised geometric formulae in the class GA_1 into analytic 2-systems. The analiticity proof in [10] relies on the fact that the obtained 2-systems manipulate atomic formulae only; this is the case for labelled 2-systems arising from frame conditions, but it does not hold anymore when translating axiom schemata, e.g. $(\varphi \supset \psi) \vee (\psi \supset \varphi)$ for Gödel logic (cf. Example 2.1). In this case analyticity for the obtained 2-systems can be recovered by (a) first translating them into hypersequent rules, (b) applying the "completion" procedure [5] in [5] to the latter, and (c) translating them back. Ad (ii): for purely propositional formulae, the class of axioms that can be automatedly

[5] This amount in transforming each structural hypersequent rule into an equivalent one (w.r.t. intuitionistic logic) that preserves cut-elimination when added to the HLJ calculus.

transformed into analytic structural hypersequent rules (i.e. the class $\mathcal{P}_3$ in [5], see the program at http://www.logic.at/people/lara/axiomcalc.html) strictly contains GA_1. E.g. $\neg\alpha \vee \neg\neg\alpha$ belongs to $\mathcal{P}_3$ (and to GA_2) but *not* to GA_1; hence when applied to $\neg\alpha \vee \neg\neg\alpha$ the method in [10] does not lead to an analytic 2-system, which can instead be defined by translating the hypersequent rule equivalent to the axiom, i.e.

$$\frac{G \mid \Gamma, \Gamma' \Rightarrow}{G \mid \Gamma \Rightarrow \mid \Gamma' \Rightarrow}\ (lq)$$

The transformation from hypersequent derivations into 2-systems allows us to reformulate the former without using |-separated components and without the need of (EC), which is "internalised" within the (ending rules of the) 2-systems. The resulting calculi can be rewritten as natural deduction systems. As future work we plan to explore their potential for extracting the computational content of the formalised logics (see, e.g. [3] for an attempt of translating the hypersequent calculus for Gödel logic – cf. Example 2.1 – into a natural deduction system to be used for establishing a Curry-Howard correspondence).

Finally, as shown in [4], all propositional axiomatisable intermediate logics are definable by adding to intuitionistic logic suitable formulae (*canonical formulae*) belonging to the class in the hierarchy of [5] immediately above the one that can be handled by hypersequents. The established connection between hypersequents and two-level systems of rules suggests the use of *three-level* systems of rules for dealing with the only class still escaping uniform analytic hypersequent calculi.

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Appendix

Lemma .1 *Let C'_1 and C'_2 be two occurrences of marked sequents that are translated by the algorithm of Section 4.1 as components of the same hypersequent. Then C'_1 and C'_2 are neither premisses of a rule which is not the ending rule of any 2-system, nor premisses of two non-ending rules belonging to the same 2-system instance.*

Proof. Let C_1 and C_2 be the components translating C'_1 and C'_2, respectively. Assume that C_1 and C_2 occur in the same hypersequent.

We first specify when the algorithm of Section 4.1 translates two marked sequents into two components of the same hypersequent. In order to do this we introduce the notion of *inner path* of a 2-system instance, i.e. a path between a premiss of the ending rule of a 2-system instance and the conclusion of one of its non-ending rules. The components that occur in the same hypersequent are those that translate sequents that occur either (i) in inner paths of a single instance S of 2-system, or (ii) in inner paths of two distinct instances S_1 and S_2 of possibly different 2-systems that are related by the transitive closure of the relation of sharing part of a path. If the latter holds, we say that S_1 and S_2 are *chained*. To see that these are the only possible cases consider when the algorithm translates two conclusions of different rule applications into components of the same hypersequent. This only happens when we translate the conclusions of the non-ending rules of a 2-system instance (or the sequents occurring below these conclusions but above the ending rule of the 2-system instance), and when a single marked sequent occurs in the intersection of the inner paths of two different 2-systems instances. In this case, due to the handling of hypersequent contexts in the algorithm, all marked sequents in the inner paths of the 2-systems instances are translated into components of a single hypersequent.

Suppose that (i) holds, i.e. C'_1 and C'_2 occur in two different inner paths of a single instance S of 2-system. Then the sequents C'_1 and C'_2 cannot be the premisses of a rule which is not an ending rule because otherwise two inner paths of S would occur above the same premiss of the ending rule of S, against the definition of inner path. Moreover, the sequents C'_1 and C'_2 cannot be the premisses of two non-ending rules belonging to a single instance S_3 of 2-system. Otherwise, two non-ending rules of S_3 would occur along two inner paths of S, against the definition of 2-system.

Suppose now that (ii) is the case, i.e. C'_1 and C'_2 occur in the inner paths of two chained instances S_1 and S_2 of possibly different 2-systems. We distinguish two sub-cases: (ii.a) S_1 and S_2 share part of an inner path, (ii.b) S_1 and S_2 do not share part of an inner path.

Assume that (ii.a) holds. The shared inner path does not contain C'_1 and C'_2 (as otherwise the two occurrences C_1 and C_2 would coincide). If C'_1 and C'_2

are the premisses of some rule (which is not an ending rule), then S_1 and S_2 also share part of the path in which C_1' and C_2' occur, but then the ending rule applications of S_1 and S_2 would coincide, against the definition of 2-system. If C_1' and C_2' are the premisses of two non-ending rules belonging to a single 2-system instance S_3, then two non-ending rules of S_3 occur above different premisses of the lower of the ending rules of S_1 and S_2, which contradicts the definition of 2-system.

If (ii.b) holds then, firstly, some instances of 2-systems are chained to both S_1 and S_2, and the ending rule of one of these instances must occur below the ending rules of S_1 and S_2. Otherwise, no 2-system instance chained to S_1 can be chained also to S_2. We call S_0 the 2-system instance with the uppermost ending rule among the instances that are chained to both S_1 and S_2. Secondly, S_1 and S_2 occur above different premisses of the ending rule of S_0. Otherwise, either C_1 and C_2 are not in the same hypersequent (because C_1' and C_2' occur above the rule application that joins the inner paths of the 2-systems chained to S_1 and S_2) or C_1' and C_2' cannot be marked (because all the inner paths of one among S_1 and S_2 occur above an inner path of the other). Finally, the non-ending rules of S_0 must have already been translated by the algorithm, otherwise C_1 and C_2 cannot occur in the same hypersequent. Assume, by contradition, that C_1' and C_2' are the premisses of the same rule application which is not an ending rule. This contradicts the fact that C_1' and C_2' occur above different premisses of the ending rule of S_0. Suppose now that C_1' and C_2' are the premisses of two non-ending rules belonging to a single 2-system instance S_3. Then, two non-ending rules of S_3 occur above different premisses of S_0, against the definition of 2-system (S_3 and S_0 cannot coincide as the non-ending rules of S_0 have already been translated). □