Locales, Nuclei, and Dragalin Frames

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Abstract

It is a classic result in lattice theory that a poset is a complete lattice iff it can be realized as fixpoints of a closure operator on a powerset. Dragalin [9,10] observed that a poset is a locale (complete Heyting algebra) iff it can be realized as fixpoints of a nucleus on the locale of upsets of a poset. He also showed how to generate a nucleus on upsets by adding a structure of "paths" to a poset, forming what we call a Dragalin frame. This allowed Dragalin to introduce a semantics for intuitionistic logic that generalizes Beth and Kripke semantics. He proved that every spatial locale (locale of open sets of a topological space) can be realized as fixpoints of the nucleus generated by a Dragalin frame. In this paper, we strengthen Dragalin's result and prove that every locale—not only spatial locales—can be realized as fixpoints of the nucleus generated by a Dragalin frame. In fact, we prove the stronger result that for every nucleus on the upsets of a poset, there is a Dragalin frame based on that poset that generates the given nucleus. We then compare Dragalin's approach to generating nuclei with the relational approach of Fairtlough and Mendler [11], based on what we call FM-frames. Surprisingly, every Dragalin frame can be turned into an equivalent FM-frame, albeit on a different poset. Thus, every locale can be realized as fixpoints of the nucleus generated by an FM-frame. Finally, we consider the relational approach of Goldblatt [13] and characterize the locales that can be realized using Goldblatt frames.

Keywords: nucleus, locale, Heyting algebra, intuitionistic logic, lax logic

1 Introduction

A well-known result of Shehtman [30] (cf. [23]) shows that there are intermediate logics that cannot be characterized by Kripke frames [21]. This incompleteness result renewed interest in the earlier topological semantics for intuitionistic

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logic due to Tarski [33]. It remains a famous open problem of Kuznetsov [22] whether every intermediate logic is topologically complete. Dragalin [9,10] made the important but somewhat neglected observation that by generalizing Kripke frames in a way inspired by Beth semantics [4], one obtains a semantics for intuitionistic logic that is at least as general as topological semantics.

Dragalin frames are triples $(S, \leq, D)$ where $(S, \leq)$ is a poset and $D\colon S \to \wp(\wp(S))$ satisfies natural conditions stated below.[3] Dragalin called each $X \in D(x)$ a *path starting from* x. In the literature on Beth semantics, 'path' suggests a linearly ordered subset of $(S, \leq)$, so we will instead call X a *development* starting from x and elements of X *stages* of the development. For the poset $(S, \leq)$, we use the following standard notation for $Y \subseteq S$ and $y \in S$:

- ${\uparrow}Y = \{z \in S \mid \exists y \in Y : y \leq z\}$ and ${\uparrow}y = {\uparrow}\{y\}$;
- ${\downarrow}Y = \{z \in S \mid \exists y \in Y : z \leq y\}$ and ${\downarrow}y = {\downarrow}\{y\}$.

A subset U of S is an *upset* (upward closed set) if $x \in U$ implies ${\uparrow}x \subseteq U$. A *downset* (downward closed set) is defined dually. If $X \subseteq {\downarrow}Y$, so $\forall x \in X$ $\exists y \in Y$: $x \leq y$ (every stage of development in X is extended by a stage of development in Y), then we say that X is *bounded by* Y.

Definition 1.1 A Dragalin frame is a triple $\mathfrak{F} = (S, \leq, D)$ where $(S, \leq)$ is a poset and $D\colon S \to \wp(\wp(S))$ (a *Dragalin function*) is such that for all $x, y \in S$:

(1°) $\varnothing \notin D(x)$;

(2°) if $y \in X \in D(x)$, then $\exists z \in X$: $x \leq z$ and $y \leq z$;

(3°) if $x \leq y$, then $\forall Y \in D(y)$ $\exists X \in D(x)$: $X \subseteq {\downarrow}Y$;

(4°) if $y \in X \in D(x)$, then $\exists Y \in D(y)$: $Y \subseteq {\downarrow}X$.

A Dragalin frame is *normal* if $D(x) \neq \varnothing$ for all $x \in S$.

Conditions (1°)–(4°) admit intuitive intepretations. (1°) says that the empty set is not a development of anything. (2°) says that every stage y in a development starting from x is at least *compatible* with x, in that x and y have a common extension z. Dragalin also mentions the stronger condition:

(2°°) if $X \in D(x)$, then $X \subseteq {\uparrow}x$,

so the stages in a development starting from x are extensions of x. Next, (3°) says that if at some "future" stage y a development Y will become available, then it is already possible to follow a development that is bounded by Y. Dragalin also mentions the stronger condition:

(3°°) if $x \leq y$, then $D(y) \subseteq D(x)$,

[3] Dragalin used the term 'Beth-Kripke frame'. To give due credit to Dragalin, we introduce the term 'Dragalin frame' instead. Note that Dragalin started with a preordered set $(S, \leq)$, but there is no loss of generality in starting with a poset (see Remark 2.6 and Theorem 3.5). Also note that Dragalin worked with downsets in $(S, \leq)$. Others, including Goldblatt [13,14] and Fairtlough and Mendler [11], work instead with upsets. It proves to be more convenient to flip Dragalin's approach to use upsets than to flip the other approaches to use downsets. Thus, we will work with upsets, at the expense of another flip of perspective in Theorem 2.8.

so developments available at future stages are already available. Finally, (4°) says that we "can always stay inside" a development, in the sense that for every stage y in X, we can follow a development Y from y that is bounded by X. A stronger notion of "staying inside" comes from replacing $Y \subseteq \downarrow X$ with $Y \subseteq X$:

(4°°) if $y \in X \in D(x)$, then $\exists Y \in D(y)$: $Y \subseteq X$.

In §3 we will see that (2°°), (3°°), and (4°°) can be assumed without loss of generality. This motivates the following definition.

Definition 1.2 A Dragalin frame is *standard* if it satisfies (2°°)–(4°°).

To use Dragalin frames to give semantics for the language of propositional logic, we use models $\mathcal{M} = (S, \leq, D, V)$ where $(S, \leq, D)$ is a Dragalin frame and V assigns to each propositional variable an upset $V(p)$ of $(S, \leq)$ with the property that $x \in V(p)$ iff $\forall X \in D(x)\ X \cap V(p) \neq \varnothing$, i.e., each development starting from x hits the interpretation of p. The forcing clauses for the connectives are the same as in intuitionistic Kripke semantics except for $\bot$ and $\vee$:

- $\mathcal{M}, x \Vdash \bot$ iff $D(x) = \varnothing$;
- $\mathcal{M}, x \Vdash \varphi \vee \psi$ iff $\forall X \in D(x)\ \exists y \in X$: $\mathcal{M}, y \Vdash \varphi$ or $\mathcal{M}, y \Vdash \psi$.

What is going on here is that we are evaluating formulas not in the full Heyting algebra $\mathsf{Up}(S, \leq)$ of upsets of $(S, \leq)$, as in Kripke semantics, but rather in the Heyting algebra of just those upsets U such that $x \in U$ iff $\forall X \in D(x)$ $X \cap U \neq \varnothing$. As Dragalin explained, and as we review in §2, the function D gives rise to a *nucleus* on the Heyting algebra $\mathsf{Up}(S, \leq)$, and we are evaluating formulas in the Heyting algebra of fixpoints of this nucleus.[4] This idea of evaluating formulas as fixpoints of a closure operator, of which a nucleus is a special case, and interpreting disjunction by taking the closure of the union appears in semantics for substructural logic [27, §12.2] and in recent philosophical discussions of the relation between intuitionistic and classical logic [28,29].

Nuclei play an important role in pointfree topology [18]. The pointfree generalization of a topological space is a complete Heyting algebra, also known as a *locale*,[5] and nuclei on a locale describe sublocales of the locale. The locale $\mathsf{Up}(S, \leq)$ is a very special locale (see Remark 2.7). Shehtman's result shows that not all intermediate logics are complete with respect to such locales. On the other hand, Dragalin proved that every locale can be represented as the algebra of fixpoints of a nucleus on some $\mathsf{Up}(S, \leq)$. This representation theorem is related to the classic result in lattice theory that every complete lattice can be represented as the lattice of fixpoints of a closure operator on a powerset.

Dragalin's representation theorem motivates the notion of a *nuclear frame* that we introduce in §2. The question is then how a nucleus on $\mathsf{Up}(S, \leq)$ can be realized more concretely. Dragalin frames do so with the function D, and

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[4] Dragalin [9,10] used the term 'completion operator' instead of 'nucleus'. In this paper, we follow the now standard terminology and notation from pointfree topology.

[5] Locales are also known as *frames* in the pointfree topology literature, but in this paper we use the term 'frame' as it is used in the modal logic literature.

Dragalin proved that every *spatial* locale (locale of open sets of a topological space) can be represented as fixpoints of the nucleus generated by D for some Dragalin frame. In §3, we prove that every locale—not only spatial locales—can be represented as fixpoints of the nucleus generated by a Dragalin frame. In fact, we prove the stronger result that for every nucleus on the upsets of a poset, there is a Dragalin frame based on that poset that realizes the given nucleus. In §4, we compare Dragalin's approach to generating nuclei with the relational approach of Fairtlough and Mendler [11], based on what we call FM-frames. Surprisingly, every Dragalin frame can be turned into an equivalent FM-frame, albeit on a different poset. Thus, every locale can be represented as fixpoints of the nucleus generated by an FM-frame. Finally, in §5, we consider the relational approach of Goldblatt [13] and characterize the locales representable using Goldblatt frames. We conclude in §6 with directions for further research.

2 Nuclear Frames

Our basic objects of study will be nuclei on Heyting algebras and locales [24,25,12,18].

Definition 2.1 A *nucleus* on a Heyting algebra H is a function $j\colon H \to H$ such that for all $a, b \in H$: $a \leq ja$ (inflationary); $jja = ja$ (idempotent); $j(a \wedge b) = ja \wedge jb$ (multiplicative). A nucleus is *dense* if $j0 = 0$.

A *nuclear algebra* is a pair $\mathbb{H} = (H, j)$ where H is a Heyting algebra and j is a nucleus on H. It is a *localic nuclear algebra* if H is a locale.

The following result is well known (see, e.g., [10, p. 71]).

Theorem 2.2 *If* $\mathbb{H} = (H, \wedge, \vee, \to, 0, j)$ *is a nuclear algebra, then the* algebra of fixpoints $\mathbb{H}_j = (H_j, \wedge_j, \vee_j, \to_j, 0_j)$ *is a Heyting algebra where* $H_j = \{a \in H \mid a = ja\}$ *is the set of fixpoints of* j *in* H *and for* $a, b \in H_j$*:*

(i) $a \wedge_j b = a \wedge b$; (ii) $a \vee_j b = j(a \vee b)$;

(iii) $a \to_j b = a \to b$; (iv) $0_j = j0$.

If $\mathbb{H}$ *is a localic nuclear algebra, then* $\mathbb{H}_j$ *is a locale, where for* $X \subseteq H_j$*:*

(v) $\bigwedge_j X = \bigwedge X$; (vi) $\bigvee_j X = j \bigvee X$.

We will often abuse notation and conflate H_j and $\mathbb{H}_j$.

Remark 2.3 An important example of a nucleus on a Heyting algebra is the nucleus of double negation $\neg\neg$, where $\neg x = x \to 0$. It is well known that the algebra of fixpoints of double negation as in Theorem 2.2 forms a Boolean algebra, which is complete if the original Heyting algebra is complete. For example, since the Heyting algebra $\mathsf{Up}(S, \leq)$ of upsets in a poset is a locale, the algebra of fixpoints in $(\mathsf{Up}(S, \leq), \neg\neg)$ is a complete Boolean algebra. Going from a poset to a complete Boolean algebra in this way is a standard technique in set theory for relating forcing posets to Boolean-valued models [32].

The nuclei on a Heyting algebra H are naturally ordered by $j \leq k$ iff $ja \leq ka$ for all $a \in H$. It is well known that in the case of a locale L, the collection

$N(L)$ of all nuclei on L with the natural ordering is itself a locale (see, e.g., [12, Th. 2.20], [18, Prop. II.2.5]). Meets in $N(L)$ are computed pointwise, whereas joins are more difficult to describe.

There are several families of nuclei that can generate all nuclei in $N(L)$. Here we focus on the following: given $a \in L$, define the function w_a on L by

$$w_a b = (b \to a) \to a. \tag{1}$$

It can be verified that w_a is a nucleus. Note that when $a = 0$, w_0 is the nucleus of double negation. [6] One of the special roles of these nuclei is shown by the following observation of Simmons [31, p. 243], which we will utilize in Theorem 3.5. To keep the paper self-contained, we include a proof.

Lemma 2.4 (Simmons) *Given a locale L and a nucleus j on L,*

$$j = \bigwedge\{w_{ja} \mid a \in L\}.$$

Proof. First, observe that for any nucleus k on L,

$$k \leq w_a \text{ iff } ka \leq a. \tag{2}$$

From left to right, if $k \leq w_a$, then $ka \leq w_a a = (a \to a) \to a = a$. From right to left, the multiplicativity of k implies $k(b \to a) \leq kb \to ka$, and the inflationarity of k yields $(b \to a) \leq k(b \to a)$. It follows that $kb \wedge (b \to a) \leq kb \wedge k(b \to a) \leq ka$ and hence $kb \leq (b \to a) \to ka$. Thus, if $ka \leq a$, then we have $kb \leq (b \to a) \to ka \leq (b \to a) \to a = w_a b$ for every $b \in L$, so $k \leq w_a$.

It follows from (2) that j is a lower bound of $\{w_{ja} \mid a \in L\}$. To see that it is the greatest, suppose k is also a lower bound of $\{w_{ja} \mid a \in L\}$, so for every $a \in L$, we have $k \leq w_{ja}$. Then (2) implies $ka \leq ja$ for every $a \in L$, so $k \leq j$. Therefore, j is the greatest lower bound. □

In this paper, we are interested in nuclear algebras in which the underlying Heyting algebra is the locale $\mathsf{Up}(S, \leq)$ of upsets of some poset $(S, \leq)$, in which implication is defined by $U \to V = \{x \in S \mid {\uparrow}x \cap U \subseteq V\}$. The locale $\mathsf{Down}(S, \leq)$ of downsets of $(S, \leq)$ is defined dually.

Definition 2.5 A *nuclear frame* is a triple $\mathfrak{F} = (S, \leq, j)$ where $(S, \leq)$ is a poset and j is a nucleus on $\mathsf{Up}(\mathfrak{F}) := \mathsf{Up}(S, \leq)$. We say that $\mathfrak{F}$ is *dense* if j is dense. The *nuclear algebra of* $\mathfrak{F}$ is the nuclear algebra $(\mathsf{Up}(\mathfrak{F}), j)$.

Remark 2.6 One could also allow the relation $\leq$ in Definition 2.5 to be a preorder. However, as is well known, for any preordered set $(S, \leq)$, taking its quotient with respect to the equivalence relation defined by $x \sim y$ iff $x \leq y$ and $y \leq x$ produces a poset $(S', \leq')$, called the *skeleton* of $(S, \leq)$, such that $\mathsf{Up}(S', \leq')$ is isomorphic to $\mathsf{Up}(S, \leq)$. Thus, any preordered nuclear frame $(S, \leq, j)$ can be turned into a partially ordered nuclear frame $(S', \leq', j')$ such that their nuclear algebras are isomorphic.

[6] As in the case of double negation, for any nuclear algebra of the form (H, w_a), its algebra of fixpoints is a Boolean algebra (see [12, p. 330], [18, p. 51]).

Remark 2.7 Nuclear frames generate only a special class of localic nuclear algebras. Recall that for a locale L, an element $a \in L$ is *completely join-prime* if from $a \leq \bigvee X$ it follows that $a \leq x$ for some $x \in X$. Let $J^\infty(L)$ be the set of completely join-prime elements of L. We call a locale L *Alexandroff* if $J^\infty(L)$ is join-dense in L, i.e., each element of L is the join of completely join-prime elements below it. Then L is Alexandroff iff L is isomorphic to the locale of upsets of a poset (see, e.g., [8,5]) and hence to the locale of open sets in an Alexandroff space. Thus, nuclear frames generate exactly the localic nuclear algebras based on Alexandroff locales, which we call *Alexandroff nuclear algebras*.

Although not every localic nuclear algebra can be represented as the nuclear algebra of a nuclear frame, nonetheless every locale can be represented as the algebra of fixpoints in the nuclear algebra of a nuclear frame. To keep the paper self-contained, we include a proof of this important result from [10, p. 75].

Theorem 2.8 (Dragalin) *A poset P is a locale iff there is a dense nuclear frame $\mathfrak{F}$ such that P is isomorphic to the algebra of fixpoints in the nuclear algebra of $\mathfrak{F}$.*

Proof. From right to left, since $\mathsf{Up}(\mathfrak{F})$ is a locale, the algebra of fixpoints in the nuclear algebra of $\mathfrak{F}$ is a locale by Theorem 2.2.

From left to right, suppose P is a locale. Let $S = P \setminus \{0\}$ and $\leq$ be the restricted order. We will build a nuclear frame $\mathfrak{F}$ whose poset is $(S, \geq)$. Since $\mathsf{Up}(S, \geq) = \mathsf{Down}(S, \leq)$, we can work with the locale of *downsets* in the poset $(S, \leq)$. Define a unary function j on this locale by

$$jX = \downarrow \bigvee X, \tag{3}$$

where $\bigvee X$ is the join of X in P, which exists since P is complete, and $\downarrow$ indicates the downset in $(S, \leq)$. It is easy to see that j is inflationary, idempotent, and that $\downarrow \bigvee (X \cap Y) \subseteq (\downarrow \bigvee X) \cap (\downarrow \bigvee Y)$ for $X, Y \in \mathsf{Down}(S, \leq)$. To see that $\downarrow \bigvee (X \cap Y) \supseteq (\downarrow \bigvee X) \cap (\downarrow \bigvee Y)$, suppose that $a \in S$ is in the right hand side, so $a \leq \bigvee X$ and $a \leq \bigvee Y$, whence $a \leq (\bigvee X) \wedge (\bigvee Y)$. By the join-infinite distributive law for locales,

$$(\bigvee X) \wedge (\bigvee Y) = \bigvee \{x \wedge y \mid x \in X, y \in Y\},$$

so $a \leq \bigvee \{x \wedge y \mid x \in X, y \in Y\}$. Since X and Y are downsets, we have $\{x \wedge y \mid x \in X, y \in Y\} \subseteq (X \cap Y) \cup \{0\}$, so $\bigvee \{x \wedge y \mid x \in X, y \in Y\} \leq \bigvee ((X \cap Y) \cup \{0\}) = \bigvee (X \cap Y)$. Thus, $a \leq \bigvee (X \cap Y)$ and hence $a \in \downarrow \bigvee (X \cap Y)$. Therefore, j is a nucleus. To see that j is dense, observe that $j\varnothing = \downarrow \bigvee \varnothing = \downarrow 0 = \varnothing$ since $0 \notin S$.

Finally, we must check that our original locale P is isomorphic to the algebra of fixpoints in the nuclear algebra of $\mathfrak{F}$. Observe that the fixpoints of j in the nuclear algebra of $\mathfrak{F}$ are exactly the principal downsets in $(S, \leq)$ plus $\varnothing$. Thus, the map sending each x to $\downarrow x$ is the desired isomorphism. $\square$

Remark 2.9 The proof technique of Theorem 2.8 is related to a standard technique in set theory, whereby one goes from a complete Boolean algebra to a poset by deleting the bottom element [32]. The poset thereby obtained is a *separative* poset (if $y \not\leq x$, then $\exists y' \leq y\ \forall y'' \leq y'\colon\ y'' \not\leq x$). Moreover, the nucleus j defined in (3) above is the nucleus of double negation [6].

Theorem 2.8 shows that nuclear frames suffice to represent arbitrary locales. However, since nuclear frames are a mix of the concrete $(S, \leq)$ and the algebraic j, it is natural to ask if we can replace the nucleus j with more concrete data from which j can be recovered. We will answer this question in the next section.

3 Dragalin Frames

The Dragalin frames of Definition 1.1 replace the nucleus j in a nuclear frame $(S, \leq, j)$ with the function $D\colon S \to \wp(\wp(S))$. As shown by Dragalin [10, pp. 72-73], this D indeed gives rise to a nucleus, as in Proposition 3.1. Given its importance in our story, we include a proof of this result.

Proposition 3.1 (Dragalin) *Given a Dragalin frame $\mathfrak{F} = (S, \leq, D)$, define a function $[D\rangle$ on $\mathsf{Up}(\mathfrak{F})$ by*

$$[D\rangle U = \{x \in S \mid \forall X \in D(x)\colon X \cap U \neq \varnothing\}. \tag{4}$$

(i) *$[D\rangle$ is a nucleus on $\mathsf{Up}(\mathfrak{F})$;*

(ii) *$[D\rangle$ is a dense nucleus iff $\mathfrak{F}$ is normal.*

We call $(\mathsf{Up}(\mathfrak{F}), [D\rangle)$ the nuclear algebra *of $\mathfrak{F}$.*

Proof. For part (i), to see that $U \in \mathsf{Up}(\mathfrak{F})$ implies $[D\rangle U \in \mathsf{Up}(\mathfrak{F})$, suppose $x \in [D\rangle U$ and $x \leq y$. For each $Y \in D(y)$, by (3°) there is an $X \in D(x)$ with $X \subseteq {\downarrow}Y$. Since $x \in [D\rangle U$, $X \cap U \neq \varnothing$, which with $U \in \mathsf{Up}(\mathfrak{F})$ and $X \subseteq {\downarrow}Y$ implies $Y \cap U \neq \varnothing$. Since this holds for each $Y \in D(y)$, we have $y \in [D\rangle U$.

For inflationarity, for any $X \in D(x)$, there is a $z \in X$ with $x \leq z$ by (1°)–(2°). So if $x \in U \in \mathsf{Up}(\mathfrak{F})$, then $z \in U$, so $X \cap U \neq \varnothing$. Hence $x \in [D\rangle U$.

For idempotence, suppose $x \notin [D\rangle U$, so there is an $X \in D(x)$ such that $X \cap U = \varnothing$. We claim that $X \cap [D\rangle U = \varnothing$. For any $y \in X$, by (4°) we have a $Y \in D(y)$ such that $Y \subseteq {\downarrow}X$. For reductio, suppose there is a $z \in Y \cap U$. Then since $Y \subseteq {\downarrow}X$, there is a $z' \in X$ such that $z \leq z'$, which with $z \in U \in \mathsf{Up}(\mathfrak{F})$ implies $z' \in U$. But then $z' \in X \cap U$, contradicting $X \cap U = \varnothing$ from above. Hence $Y \cap U = \varnothing$, so $y \notin [D\rangle U$. Since this holds for all $y \in X$, $X \cap [D\rangle U = \varnothing$, which with $X \in D(x)$ implies $x \notin [D\rangle[D\rangle U$.

Finally, $[D\rangle$ is monotonic ($A \subseteq B$ implies $[D\rangle A \subseteq [D\rangle B$) by its definition, so $[D\rangle(U \cap U') \subseteq [D\rangle U \cap [D\rangle U'$. Conversely, if we can show that $A \cap [D\rangle A' \subseteq [D\rangle(A \cap A')$, then by two applications of this fact, plus monotonicity and idempotence, $[D\rangle U \cap [D\rangle U' \subseteq [D\rangle([D\rangle U \cap U') \subseteq [D\rangle[D\rangle(U \cap U') \subseteq [D\rangle(U \cap U')$. So suppose $x \in A \cap [D\rangle A'$. Then for $X \in D(x)$, there is a $y \in X \cap A'$. Therefore, by (2°), there is a $z \in X$ with $x \leq z$ and $y \leq z$. Since $x \in A$, $y \in A'$, and $A, A' \in \mathsf{Up}(\mathfrak{F})$, we have $z \in A \cap A'$, so $X \cap A \cap A' \neq \varnothing$. Thus, $x \in [D\rangle(A \cap A')$.

For part (ii), by (4) we have that $[D\rangle\varnothing = \{x \in S \mid D(x) = \varnothing\}$, so $[D\rangle\varnothing = \varnothing$ ($[D\rangle$ is dense) iff $D(x) \neq \varnothing$ for all $x \in S$ ($\mathfrak{F}$ is normal). $\square$

Example 3.2 For any poset $(S, \leq)$ and $x \in S$, define $D(x) = \{\uparrow y \mid x \leq y\}$. One can easily check that $(S, \leq, D)$ is a standard Dragalin frame as in Definition 1.2. Observe that $x \in [D\rangle U$ iff $\forall y \geq x\ \exists z \geq y$: $z \in U$, so $[D\rangle$ is the nucleus of double negation (recall Remark 2.3).

The obvious next question is: which nuclear frames can be generated by Dragalin frames as in Proposition 3.1? In addition, in light of Theorem 2.8, another obvious question is: which locales can be generated as the algebra of fixpoints in the nuclear algebra of a Dragalin frame?

Dragalin gave a partial answer to the second question. We provide a sketch of his proof [10, pp. 75-76] of Theorem 3.3 to convey the main idea, but we omit the details since we will prove a more general result below. Recall that a *spatial* locale is a locale isomorphic to the locale of open sets of a topological space, and a *normal* Dragalin frame $(S, \leq, D)$ is one in which $D(x) \neq \varnothing$ for all $x \in S$.

Theorem 3.3 (Dragalin) *If L is a spatial locale, then there is a normal Dragalin frame $\mathfrak{F}$ such that L is isomorphic to the algebra of fixpoints in the nuclear algebra of $\mathfrak{F}$.*

Proof. [Sketch] Given a topological space (X, Ω) and $x \in X$, a $\mathcal{B} \subseteq \Omega$ is a *local basis of x* if (i) $x \in \bigcap \mathcal{B}$ and (ii) whenever $x \in U \in \Omega$, there is a $V \in \mathcal{B}$ with $V \subseteq U$. From (X, Ω), we define $(S, \leq, D)$ where $S = \Omega \setminus \{\varnothing\}$, $U \leq V$ iff $U \supseteq V$, and $D(U) = \{\mathcal{B} \mid \exists x \in U : \mathcal{B}$ is a local basis of x and $\bigcup \mathcal{B} \subseteq U\}$. Dragalin showed that $(S, \leq, D)$ is a normal Dragalin frame. In addition, for the locale $\Omega(X)$ of opens of (X, Ω), he showed that $f : \Omega(X) \to \mathsf{Up}(S, \leq)_{[D\rangle}$ defined by $f(U) = \{V \in S \mid U \leq V\}$ is an isomorphism. $\Box$

In the other direction, it is not the case that for every Dragalin frame $\mathfrak{F}$, the algebra of fixpoints in the nuclear algebra of $\mathfrak{F}$ is spatial.

Example 3.4 Consider the Dragalin frame $(S, \leq, D)$ where $(S, \leq)$ is the poset associated with the complete infinite binary tree and D gives the nucleus of double negation as in Example 3.2. Then the algebra of fixpoints in the nuclear algebra $(\mathsf{Up}(S, \leq), [D\rangle)$ is a complete atomless Boolean algebra [16, Ex. 2.40]. But a complete Boolean algebra is spatial iff it is atomic.

Indeed, *every* locale can be realized as the algebra of fixpoints in the nuclear algebra of a Dragalin frame. To prove this, it would suffice to show that the nuclear frame used in the proof of Theorem 2.8 can be generated by a Dragalin frame. We will prove the following stronger result.

Theorem 3.5 *Given any nuclear frame $(S, \leq, j)$, there is a standard Dragalin frame $(S, \leq, D)$ such that $j = [D\rangle$.*

To prove Theorem 3.5, we will utilize Simmons's result (Lemma 2.4) that every nucleus on a locale L is the meet, in $N(L)$, of nuclei of the w_a type. We will apply this to the locale $L = \mathsf{Up}(S, \leq)$. First, we show how nuclei of the w_a type on $\mathsf{Up}(S, \leq)$ can be realized by a Dragalin frame $(S, \leq, D)$.

Lemma 3.6 *Given a poset $(S, \leq)$ and $A \in \mathsf{Up}(S, \leq)$, let w_A be the nucleus on $\mathsf{Up}(S, \leq)$ defined as in* (1) *and define $D_A(x) = \{\uparrow x' \setminus A \mid x' \in \uparrow x \setminus A\}$. Then:*

(i) $(S, \leq, D_A)$ *is a standard Dragalin frame;*

(ii) $[D_A\rangle = w_A$.

Proof. For (i), D_A satisfies (1°) because if $x' \in {\uparrow}x \setminus A$, then $x' \in {\uparrow}x' \setminus A$, which shows that $\varnothing \notin D(x)$. Clearly D_A also satisfies (2°°), as well as (3°°), for if $x \leq y$ and $y' \in {\uparrow}y \setminus A$, then $y' \in {\uparrow}x \setminus A$. Finally, for (4°°), suppose $y \in X \in D(x)$, so $X = {\uparrow}x' \setminus A$ for some $x' \in {\uparrow}x \setminus A$. Let $Y = {\uparrow}y \setminus A$. Since $y \in X$ and hence $y \notin A$, we have $y \in Y$, which implies $Y \in D(y)$. Moreover, since $y \in X$ and hence $y \in {\uparrow}x'$, we have ${\uparrow}y \setminus A \subseteq {\uparrow}x' \setminus A$, so $Y \subseteq X$.

For (ii), observe that $x \in w_A U = (U \to A) \to A$ iff for all $x' \geq x$, if $x' \notin A$, then there is an $x'' \geq x'$ such that $x'' \in U \setminus A$. This is equivalent to the condition that for all $x' \in {\uparrow}x \setminus A$, we have $({\uparrow}x' \setminus A) \cap U \neq \varnothing$. That is in turn equivalent to the condition that for all $X \in D_A(x)$, $X \cap U \neq \varnothing$, which is finally equivalent to $x \in [D_A\rangle U$. □

Next we show that we can build up meets of nuclei from Dragalin frames.

Lemma 3.7 *Given a family* $\{j_\alpha\}_{\alpha \in I}$ *of nuclei on* $\mathsf{Up}(S, \leq)$ *and a family* $\{D_\alpha\}_{\alpha \in I}$ *of Dragalin functions on* $(S, \leq)$ *such that*

$$x \in j_\alpha U \text{ iff } x \in [D_\alpha\rangle U, \tag{5}$$

the function D defined by

$$D(x) = \bigcup_{\alpha \in I} D_\alpha(x) \tag{6}$$

is a Dragalin function such that

$$x \in \Big(\bigwedge_{\alpha \in I} j_\alpha\Big)U \text{ iff } x \in [D\rangle U. \tag{7}$$

Moreover, if each D_α is standard, then so is D.

Proof. We first prove (7). Since meets of nuclei are computed pointwise and meets in $\mathsf{Up}(S, \leq)$ are intersections, we have

$$\Big(\bigwedge_{\alpha \in I} j_\alpha\Big)U = \bigwedge_{\alpha \in I} j_\alpha U = \bigcap_{\alpha \in I} j_\alpha U.$$

Now suppose $x \notin \bigcap_{\alpha \in I} j_\alpha U$, so there is some $\alpha \in I$ such that $x \notin j_\alpha U$. Then by (5), we have $x \notin [D_\alpha\rangle U$, so there is some $X \in D_\alpha(x)$ with $X \cap U = \varnothing$. By (6), $X \in D(x)$, which with $X \cap U = \varnothing$ implies $x \notin [D\rangle U$. Conversely, suppose $x \notin [D\rangle U$, so there is some $X \in D(x)$ with $X \cap U = \varnothing$. Then by (6), there is some $\alpha \in I$ such that $X \in D_\alpha(x)$, which with $X \cap U = \varnothing$ implies $x \notin [D_\alpha\rangle U$, which implies $x \notin j_\alpha U$ by (5), so $x \notin \bigcap_{\alpha \in I} j_\alpha U$.

Next, we show that D satisfies (1°)–(4°), assuming that each D_α does. Clearly if (1°) holds for each D_α, then it holds for D. The same is true of (2°).

For (3°), suppose $x \leq y$ and $Y \in D(y)$. Then $Y \in D_\alpha(x)$ for some $\alpha \in I$. Applying (3°) for D_α, there is an $X \subseteq {\downarrow}Y$ such that $X \in D_\alpha(x)$ and hence $X \in D(x)$, so (3°) also holds for D. Similarly, for (4°), if $y \in X \in D(x)$, then $X \in D_\alpha(x)$ for some $\alpha \in I$, in which case (4°) for D_α gives us a $Y \subseteq {\downarrow}X$ such that $Y \in D_\alpha(y)$ and hence $Y \in D(y)$, so (4°) also holds for D. It is also easy to see that if each D_α satisfies (2°°)–(4°°), then so does D. ☐

We can now put the pieces together to prove Theorem 3.5.

Proof. By Lemmas 2.4, 3.6, and 3.7, we have:

$$j = \bigwedge\{w_{jA} \mid A \in \mathsf{Up}(S, \leq)\} = \bigwedge\{[D_{jA}\rangle \mid A \in \mathsf{Up}(S, \leq)\} = [D\rangle,$$

where D is defined from the D_{jA}'s as in Lemma 3.7. By Lemma 3.6, each D_{jA} is standard, so D is as well by Lemma 3.7. ☐

Putting together Theorems 2.8 and 3.5 and Proposition 3.1(ii), we obtain the following.

Corollary 3.8 *A poset P is a locale iff there is a standard normal Dragalin frame $\mathfrak{F}$ such that P is isomorphic to the algebra of fixpoints in the nuclear algebra of $\mathfrak{F}$.*

In §4, we shall see that an analogue of Corollary 3.8 holds for frames that replace Dragalin's function D with a partial order $\preceq$. For the purposes of comparing these frames, we will use the fact that every Dragalin frame can be turned into one satisfying a property stronger than (2°°).

Definition 3.9 A Dragalin frame $\mathfrak{F} = (S, \leq, D)$ is *convex* if for all $x \in S$ and $X \in D(x)$, we have $X = {\uparrow}x \cap {\downarrow}X$.

Remark 3.10

(i) If a Dragalin frame $\mathfrak{F} = (S, \leq, D)$ is convex, then $X \in D(x)$ implies $x \in X$. For if $X \in D(x)$, then by (1°), there is a $y \in X = {\uparrow}x \cap {\downarrow}X$, so $x \leq y$ and $y \in {\downarrow}X$, which implies $x \in {\downarrow}X$ and hence $x \in {\uparrow}x \cap {\downarrow}X = X$.

(ii) A convex $\mathfrak{F}$ typically does not satisfy (3°°). Consider $x, y \in S$ such that $D(y) \neq \varnothing$. By convexity each $Y \in D(y)$ is such that $Y \subseteq {\uparrow}y$, and by (i) each $X \in D(x)$ is such that $x \in X$, so if $x < y$, then $D(x) \cap D(y) = \varnothing$.

(iii) By contrast, every convex $\mathfrak{F}$ satisfies (4°°). Suppose $y \in X \in D(x)$, so by (4°) there is a $Y \in D(y)$ such that $Y \subseteq {\downarrow}X$. By convexity, $Y \subseteq {\uparrow}y$ and $X = {\uparrow}x \cap {\downarrow}X$. Since $y \in X \in D(x)$ implies $x \leq y$ by convexity, we have ${\uparrow}y \subseteq {\uparrow}x$. Thus, $Y \subseteq {\uparrow}x$, which with $Y \subseteq {\downarrow}X$ implies $Y \subseteq {\uparrow}x \cap {\downarrow}X = X$.

Proposition 3.11 *For each Dragalin frame $\mathfrak{F}$, there is a convex Dragalin frame $\mathfrak{G}$ such that the nuclear algebras of $\mathfrak{F}$ and $\mathfrak{G}$ are isomorphic. Moreover, if $\mathfrak{F}$ is normal, then so is $\mathfrak{G}$.*

Proof. Given a Dragalin frame $\mathfrak{F} = (S, \leq, D)$, define $\mathfrak{G} = (S, \leq, D')$ by:

$$D'(x) = \{{\uparrow}x \cap {\downarrow}X \mid X \in D(x)\}.$$

We claim that for all $U \in \mathsf{Up}(S, \leq)$, $[D\rangle U = [D'\rangle U$. Suppose $x \in [D\rangle U$ and consider some $\uparrow x \cap \downarrow X \in D'(x)$, so $X \in D(x)$. Since $x \in [D\rangle U$ and $X \in D(x)$, there is a $y \in X \cap U$. Then by (2°), there is a $z \in X$ with $x \leq z$ and $y \leq z$. From $z \in X$ and $x \leq z$, we have $z \in \uparrow x \cap X \subseteq \uparrow x \cap \downarrow X$. From $y \in U$ and $y \leq z$, we have $z \in U$. Therefore, $(\uparrow x \cap \downarrow X) \cap U \neq \varnothing$ and hence $x \in [D'\rangle U$. Conversely, if $x \notin [D\rangle U$, so there is an $X \in D(x)$ such that $X \cap U = \varnothing$, then since U is an upset, $(\uparrow x \cap \downarrow X) \cap U = \varnothing$ and $\uparrow x \cap \downarrow X \in D'(x)$, so $x \notin [D'\rangle U$. It follows that the nuclear algebras of $\mathfrak{F}$ and $\mathfrak{G}$ are isomorphic.

Next we show that $\mathfrak{G}$ satisfies (1°)–(4°), so it is a Dragalin frame. Since $\mathfrak{G}$ is convex, (2°) is immediate. For (1°) for D', if $X \in D(x)$, then by (1°)–(2°) for D, $\uparrow x \cap X \neq \varnothing$, so $\uparrow x \cap \downarrow X \neq \varnothing$ and hence $\varnothing \notin D'(x)$.

For (3°) for D', suppose $x \leq y$ and $Y' \in D'(y)$, so $Y' = \uparrow y \cap \downarrow Y$ for some $Y \in D(y)$. Then by (3°) for D, there is an $X \in D(x)$ such that $X \subseteq \downarrow Y$. Setting $X' = \uparrow x \cap \downarrow X$, we have $X' \in D'(x)$, and we claim that $X' \subseteq \downarrow Y'$, which will establish (3°) for D'. If $a \in X'$, then $a \in \downarrow X$, so there is a b with $a \leq b \in X$. Then since $X \subseteq \downarrow Y$, there is a c with $b \leq c \in Y$. Given $Y \in D(y)$, it follows by (2°) for D that there is a $z \in Y$ such that $y \leq z$ and $c \leq z$. Thus, $a \leq b \leq c \leq z \in \uparrow y \cap Y \subseteq \uparrow y \cap \downarrow Y$, so $a \in \downarrow(\uparrow y \cap \downarrow Y) = \downarrow Y'$.

Next we prove (4°°). Suppose $y \in X' \in D'(x)$, so $X' = \uparrow x \cap \downarrow X$ for some $X \in D(x)$. We need a $Y' \in D'(y)$ with $Y' \subseteq X'$, so we need a $Y \in D(y)$ with $\uparrow y \cap \downarrow Y \subseteq \uparrow x \cap \downarrow X$. Since $y \in X'$, $y \in \downarrow X$, so there is a z with $y \leq z \in X$. Given $X \in D(x)$, it follows by (4°) for D that there is a $Z \in D(z)$ such that $Z \subseteq \downarrow X$. Then given $y \leq z$, it follows by (3°) for D that there is a $Y \in D(y)$ such that $Y \subseteq \downarrow Z$, which with $Z \subseteq \downarrow X$ implies $Y \subseteq \downarrow X$, which in turn implies $\downarrow Y \subseteq \downarrow X$. Since $y \in X'$, $y \in \uparrow x$, so we also have $\uparrow y \subseteq \uparrow x$. Therefore, $\uparrow y \cap \downarrow Y \subseteq \uparrow x \cap \downarrow X$.

Finally, it is obvious that if $\mathfrak{F}$ is normal, then $\mathfrak{G}$ is normal too. □

4 Fairtlough-Mendler Frames

In this section, we consider another way of replacing the nucleus j in a nuclear frame with more concrete data. Fairtlough and Mendler [11] (also see [1]) give a semantics for an intuitionistic modal logic called *propositional lax logic* with a modality $\bigcirc$ obeying the axioms of a nucleus. The frames used in their semantics therefore provide another method for representing nuclear algebras.

Definition 4.1 An *FM-frame* (Fairtlough-Mendler frame) is a tuple $\mathfrak{F} = (S, \leq, \preceq, F)$, where $\leq$ and $\preceq$ are preorders on the set S, $\preceq$ is a subrelation of $\leq$, and $F \in \mathsf{Up}(\mathfrak{F}) := \mathsf{Up}(S, \leq)$. We say that $\mathfrak{F}$ is *normal* if $F = \varnothing$, and $\mathfrak{F}$ is *partially ordered* if $\leq$ (and hence $\preceq$) is a partial order.

Each FM-frame gives rise to a nuclear algebra. To see this, for each FM-frame $\mathfrak{F} = (S, \leq, \preceq, F)$, let $\mathsf{Up}(\mathfrak{F})_F = \{U \in \mathsf{Up}(\mathfrak{F}) \mid F \subseteq U\}$ be the relativization of $\mathsf{Up}(\mathfrak{F})$ to F. Then $\mathsf{Up}(\mathfrak{F})_F$ is a locale, where the operations $\wedge, \vee, \to$ on $\mathsf{Up}(\mathfrak{F})_F$ are the restrictions of the corresponding operations on $\mathsf{Up}(\mathfrak{F})$, and F is the bottom element of $\mathsf{Up}(\mathfrak{F})_F$. When $F = \varnothing$, we obviously have $\mathsf{Up}(\mathfrak{F})_F = \mathsf{Up}(\mathfrak{F})$. In order to define a nucleus on $\mathsf{Up}(\mathfrak{F})_F$, recall that for a binary relation R on S, $x \in S$, and $U \subseteq S$, it is customary to define

$R(x) = \{y \in S \mid xRy\}$ and let

$$\Box_R(U) = \{x \in S \mid R(x) \subseteq U\} \text{ and } \Diamond_R(U) = \{x \in S \mid R(x) \cap U \neq \varnothing\}.$$

Now consider the following operator on $\mathsf{Up}(\mathfrak{F})_F$:

$$\Box_{\leq}\Diamond_{\preceq}U = \{x \in S \mid \forall y(x \leq y \Rightarrow \exists z(y \preceq z \;\&\; z \in U))\}.$$

Fairtlough and Mendler use this operator to interpret the modality $\bigcirc$ of lax logic in FM-frames. The following proposition is essentially their soundness result for lax logic [11, p. 9], which we prove for the reader's convenience.

Proposition 4.2 (Fairtlough-Mendler) *If $\mathfrak{F} = (S, \leq, \preceq, F)$ is an FM-frame, then $(\mathsf{Up}(\mathfrak{F})_F, \Box_{\leq}\Diamond_{\preceq})$ is a nuclear algebra, which we call the* nuclear algebra *of $\mathfrak{F}$. Moreover, if $F = \varnothing$, then $\Box_{\leq}\Diamond_{\preceq}$ is a dense nucleus.*

Proof. For any $U \in \mathsf{Up}(\mathfrak{F})_F$, since $\preceq$ is a reflexive subrelation of $\leq$, we have $U \subseteq \Box_{\leq}\Diamond_{\preceq}U$. Since $\leq$ is reflexive and $\preceq$ is transitive, we also have $\Box_{\leq}\Diamond_{\preceq}\Box_{\leq}\Diamond_{\preceq}U \subseteq \Box_{\leq}\Diamond_{\preceq}U$. Clearly $\Box_{\leq}\Diamond_{\preceq}(U \cap V) \subseteq \Box_{\leq}\Diamond_{\preceq}U \cap \Box_{\leq}\Diamond_{\preceq}V$. Conversely, for any $U, V \in \mathsf{Up}(\mathfrak{F})_F$, if $x \in \Box_{\leq}\Diamond_{\preceq}U \cap \Box_{\leq}\Diamond_{\preceq}V$ and $x \leq y$, then there is a z with $y \preceq z \in U$ and hence $y \leq z$. So $x \leq z$, which with $x \in \Box_{\leq}\Diamond_{\preceq}V$ implies that there is a w with $z \preceq w \in V$ and hence $z \leq w$. Then since $z \in U \in \mathsf{Up}(\mathfrak{F})_F$, we have $w \in U \cap V$. Thus, $x \in \Box_{\leq}\Diamond_{\preceq}(U \cap V)$. Finally, since $\leq$ is reflexive, $\Box_{\leq}\Diamond_{\preceq}\varnothing = \varnothing$, so $\Box_{\leq}\Diamond_{\preceq}$ is dense if $F = \varnothing$. $\Box$

Example 4.3 If $\leq = \preceq$ in an FM-frame $(S, \leq, \preceq, F)$, then $\Box_{\leq}\Diamond_{\preceq}$ is the nucleus of double negation (cf. [11, p. 23] and [3]).

Remark 4.4 Let us now consider extracting a nuclear *frame* rather than a nuclear algebra from an FM-frame. If the FM-frame $\mathfrak{F} = (S, \leq, \preceq, F)$ is normal, then $(S, \leq, \Box_{\leq}\Diamond_{\preceq})$ is a preordered nuclear frame as in Remark 2.6, which we can turn into a partially ordered nuclear frame by taking the skeleton, and the nuclear algebra of this nuclear frame is isomorphic to that of $\mathfrak{F}$. Now suppose $\mathfrak{F}$ is not normal. Define the preordered nuclear frame $(S^-, \leq^-, j)$ where $S^- = S \setminus F$, $\leq^-$ is the restriction of $\leq$ to S^-, and j is defined for $U \in \mathsf{Up}(S^-, \leq^-)$ by $jU = \Box_{\leq}\Diamond_{\preceq}(U \cup F) \setminus F$. The nuclear algebra of $(S^-, \leq^-, j)$ is then isomorphic to that of $\mathfrak{F}$, and once again we can turn $(S^-, \leq^-, j)$ into a partially ordered nuclear frame as in Remark 2.6.

Turning FM-frames into partially ordered FM-frames is more difficult. It is not clear how to define the skeleton $\mathfrak{F}' = (S', \leq', \preceq', F')$ of an FM-frame $\mathfrak{F} = (S, \leq, \preceq, F)$ such that $(\mathsf{Up}(\mathfrak{F})_F, \Box_{\leq}\Diamond_{\preceq})$ and $(\mathsf{Up}(\mathfrak{F}')_{F'}, \Box_{\leq'}\Diamond_{\preceq'})$ are isomorphic nuclear algebras. The difficulty is in defining $\preceq'$, as standard ways of defining a new binary relation on a quotient do not work in this case. Below we take a different approach by unwinding instead of collapsing clusters to produce a partially ordered FM-frame $\mathfrak{F}^\dagger$. While the nuclear algebra of $\mathfrak{F}^\dagger$ will be "larger" than that of $\mathfrak{F}$, their algebras of fixpoints will be isomorphic.

Proposition 4.5 *For any FM-frame $\mathfrak{F}$, there is a partially ordered FM-frame $\mathfrak{F}^\dagger$ such that the algebra of fixpoints in the nuclear algebra of $\mathfrak{F}$ is isomorphic to that of $\mathfrak{F}^\dagger$. Moreover, $\mathfrak{F}$ is normal iff $\mathfrak{F}^\dagger$ is normal.*

Proof. Given an FM-frame $\mathfrak{F} = (S, \sqsubseteq, \preceq, F)$, define $\mathfrak{F}^\dagger = (S^\dagger, \sqsubseteq^\dagger, \preceq^\dagger, F^\dagger)$ as follows (in this proof, $\leq$ is the usual ordering on $\mathbb{N}$):

- $S^\dagger = \{\langle x, t\rangle \mid x \in S,\, t \in \mathbb{N}\}$;
- $\langle x, t\rangle \sqsubseteq^\dagger \langle x', t'\rangle$ iff either $[x = x'$ and $t \leq t']$ or $[x \sqsubseteq x'$ and $t < t']$;
- $\langle x, t\rangle \preceq^\dagger \langle x', t'\rangle$ iff either $[x = x'$ and $t \leq t']$ or $[x \preceq x'$ and $t < t']$;
- $F^\dagger = \{\langle x, t\rangle \in S^\dagger \mid x \in F\}$.

Observe that $\sqsubseteq^\dagger$ and $\preceq^\dagger$ are partial orders. Moreover, $F = \varnothing$ iff $F^\dagger = \varnothing$, so $\mathfrak{F}$ is normal iff $\mathfrak{F}^\dagger$ is normal.

Let j be the nucleus associated with $\mathfrak{F}$ and $j^\dagger$ the nucleus associated with $\mathfrak{F}^\dagger$. Let $g\colon S^\dagger \to S$ be defined by $g(x, t) = x$. We claim that the function G that maps each fixpoint U of $j^\dagger$ to $G(U) = g[U]$ is an isomorphism between the algebras of fixpoints in the nuclear algebras of $\mathfrak{F}^\dagger$ and $\mathfrak{F}$.

First, toward showing that G sends fixpoints of $j^\dagger$ to fixpoints of j, we show that for every $\langle x, t\rangle \in S^\dagger$ and $U \in \mathsf{Up}(\mathfrak{F}^\dagger)_{F^\dagger}$:

$$\langle x, t\rangle \in j^\dagger U \text{ iff } x \in jg[U]. \tag{8}$$

From left to right, if $x' \sqsupseteq x$, then $\langle x', t+1\rangle \sqsupseteq^\dagger \langle x, t\rangle$. Since $\langle x, t\rangle \in j^\dagger U$, there is $\langle x'', t''\rangle \succeq^\dagger \langle x', t+1\rangle$ such that $\langle x'', t''\rangle \in U$. This implies $x'' \succeq x'$ and $x'' \in g[U]$. Hence we have shown that $x \in jg[U]$. From right to left, if $\langle x', t'\rangle \sqsupseteq^\dagger \langle x, t\rangle$, then $x' \sqsupseteq x$. Since $x \in jg[U]$, there is an $x'' \succeq x'$ with $x'' \in g[U]$. It follows that there is some $s \in \mathbb{N}$ such that $\langle x'', s\rangle \in U$. Since $U \in \mathsf{Up}(\mathfrak{F}^\dagger)_{F^\dagger}$, it follows that $\langle x'', t'+s+1\rangle \in U$. Given $x'' \succeq x'$ and $t'+s+1 > t'$, we have $\langle x'', t'+s+1\rangle \succeq^\dagger \langle x', t'\rangle$. Therefore, $\langle x, t\rangle \in j^\dagger U$.

Now we can see that if U is a fixpoint of $j^\dagger$, then $g[U]$ is a fixpoint of j. To see that $jg[U] \subseteq g[U]$, observe that if $x \in jg[U]$, then by (8) and the assumption that U is a fixpoint of $j^\dagger$, we have $\langle x, t\rangle \in j^\dagger U \subseteq U$, whence $x \in g[U]$.

Second, we claim that G is surjective. Suppose V is a fixpoint of j. Then we claim that $g^{-1}[V]$ is a fixpoint of $j^\dagger$, which with $g[g^{-1}[V]] = V$, given by the surjectivity of g, will show that G is surjective. We begin by showing that $g^{-1}[V] \in \mathsf{Up}(\mathfrak{F}^\dagger)_{F^\dagger}$. To see that $g^{-1}[V] \in \mathsf{Up}(\mathfrak{F}^\dagger)$, observe that if $\langle x, t\rangle \in g^{-1}[V]$ and $\langle x', t'\rangle \sqsupseteq^\dagger \langle x, t\rangle$, then $x \in V$ and $x' \sqsupseteq x$, which with $V \in \mathsf{Up}(\mathfrak{F})$ implies $x' \in V$ and hence $\langle x', t'\rangle \in g^{-1}[V]$. To see that $g^{-1}[V] \in \mathsf{Up}(\mathfrak{F}^\dagger)_{F^\dagger}$, observe that since $g[F^\dagger] = F \subseteq V$, we have $F^\dagger \subseteq g^{-1}[g[F^\dagger]] \subseteq g^{-1}[V]$. Next, we must show that $j^\dagger g^{-1}[V] \subseteq g^{-1}[V]$. If $\langle x, t\rangle \in j^\dagger g^{-1}[V]$, then by (8), $x \in jg[g^{-1}[V]] = jV = V$, using that V is a fixpoint of j. Hence $\langle x, t\rangle \in g^{-1}[V]$.

Third, we show that G preserves and reflects order: for any fixpoints U and V of $j^\dagger$, we have $U \subseteq V$ iff $g[U] \subseteq g[V]$. If $U \subseteq V$ and $x \in g[U]$, then for some $t \in \mathbb{N}$, $\langle x, t\rangle \in U \subseteq V$, so $x \in g[V]$. Conversely, suppose $U \not\subseteq V$, so there is an $\langle x, t\rangle \in U \setminus V$. Then $x \in g[U]$, and since $V = j^\dagger V$, $\langle x, t\rangle \notin j^\dagger V$, which with (8) implies $x \notin jg[V]$ and hence $x \notin g[V]$. So $g[U] \not\subseteq g[V]$.

Thus, G is an isomorphism between the algebras of fixpoints of $\mathfrak{F}^\dagger$ and $\mathfrak{F}$. $\Box$

Turning preordered FM-frames into partially ordered Dragalin frames does not require the unwinding in the previous proof. In this case we may collapse clusters without difficulty, by first turning a preordered FM-frame into a partially ordered nuclear frame as in Remark 4.4 and then turning that nuclear frame into a Dragalin frame as in Theorem 3.5.

Remark 4.6 In the case of a normal FM-frame $\mathfrak{F} = (S, \leq, \preceq, F)$, there is an even more direct approach: we can define a standard Dragalin frame $\mathfrak{D} = (S', \leq', D)$ where $(S', \leq')$ is the skeleton of $(S, \leq)$ and $D([x]) = \{\Uparrow y \mid x \leq y\}$, where $[x]$ is the equivalence class of x and $\Uparrow y = \{[z] \in S' \mid y \preceq z\}$. For lack of space, we omit the proof that the nuclear algebras of $\mathfrak{F}$ and $\mathfrak{D}$ are isomorphic.

It is rather surprising that we can also go in the other direction, from Dragalin to FM-frames. This we do by "enlarging" the underlying poset.

Theorem 4.7 *For any Dragalin frame $\mathfrak{D}$, there is an FM-frame $\mathfrak{F}$ such that the nuclear algebras of $\mathfrak{D}$ and $\mathfrak{F}$ are isomorphic. Moreover, $\mathfrak{D}$ is normal iff $\mathfrak{F}$ is normal.*

Proof. By Proposition 3.11, we may assume that the Dragalin frame $\mathfrak{D}$ is convex and therefore satisfies ($4^{\circ\circ}$) by Remark 3.10(iii). First we give the proof assuming that $\mathfrak{D}$ is normal and then show how to modify the proof to lift this assumption. The construction is similar to the construction of intuitionistic relational frames from intuitionistic neighborhood frames in [19] (cf. the construction of birelational frames from monotonic neighborhood frames in [20]). Given a normal Dragalin frame $\mathfrak{D} = (S, \leq, D)$, we define an FM-frame $\mathfrak{F} = (S', \leq', \preceq', F')$ with $F' = \varnothing$ as follows:

- $S' = \{(x, X) \mid x \in S, X \in D(x)\}$;
- $(x, X) \leq' (y, Y)$ iff $x \leq y$;
- $(x, X) \preceq' (y, Y)$ iff $y \in X$ and $Y \subseteq X$.

Clearly $\leq'$ is a preorder. To see that $\preceq'$ is a preorder, $X \in D(x)$ implies $x \in X$ by convexity and Remark 3.10(i), so $\preceq'$ is reflexive. For transitivity, if $(x, X) \preceq' (y, Y) \preceq' (z, Z)$, then $Z \subseteq Y \subseteq X$, so $Z \subseteq X$, and $z \in Y \subseteq X$, so $z \in X$. Hence $(x, X) \preceq' (z, Z)$. Finally, $\preceq'$ is a subrelation of $\leq'$: if $(x, X) \preceq' (y, Y)$, then by ($2^{\circ\circ}$), $y \in X \in D(x)$ implies $x \leq y$, so $(x, X) \leq' (y, Y)$.

Define $f \colon \mathsf{Up}(\mathfrak{D}) \to \mathsf{Up}(\mathfrak{F})$ by

$$f(U) = \{(x, X) \mid x \in U, X \in D(x)\}.$$

It is routine to check that f is an isomorphism between $\mathsf{Up}(\mathfrak{D})$ and $\mathsf{Up}(\mathfrak{F})$. To show it is an isomorphism between the nuclear algebras of $\mathfrak{D}$ and $\mathfrak{F}$, we show:

$$f([D\rangle U) = \Box_{\leq'} \Diamond_{\preceq'} f(U). \tag{9}$$

Suppose $(x, X) \in f([D\rangle U)$, so $x \in [D\rangle U$ and $X \in D(x)$. Consider any (y, Y) such that $(x, X) \leq' (y, Y)$, so $x \leq y$. Then from $x \in [D\rangle U$ we have $y \in [D\rangle U$, since $[D\rangle U$ is an upset whenever U is. Given $y \in [D\rangle U$ and $Y \in D(y)$, there is a $z \in Y$ such that $z \in U$. Given $z \in Y \in D(y)$, by ($4^{\circ\circ}$) there is a $Z \in D(z)$ such that $Z \subseteq Y$. Thus, $(y, Y) \preceq' (z, Z)$. Then since $z \in U$, we have $(z, Z) \in f(U)$. Hence we have shown that $(x, X) \in \Box_{\leq'} \Diamond_{\preceq'} f(U)$.

Conversely, suppose $(x, X) \notin f([D\rangle U)$, so $x \notin [D\rangle U$. Then there is a $Y \in D(x)$ with $Y \cap U = \varnothing$ (but $Y \neq \varnothing$ by (1°)). Therefore, $(x, X) \leq' (x, Y)$, and for any (z, Z) such that $(x, Y) \preceq' (z, Z)$, we have $z \in Y$ and hence $z \notin U$, so $(z, Z) \notin f(U)$. Thus, $x \notin \Box_{\leq'}\Diamond_{\preceq'} f(U)$. This completes the proof of (9).

If $\mathfrak{D} = (S, \leq, D)$ is not normal, define the FM-frame $\mathfrak{F} = (S', \leq', \preceq', F')$ by:

- $S' = \{(x, X) \mid x \in S, X \in D(x)\} \cup \{(x, \varnothing) \mid D(x) = \varnothing\} \cup \{m\}$; $F' = \{m\}$;
- $(x, X) \leq' (y, Y)$ iff $x \leq y$; and m is the maximum of $\leq'$;
- $(x, X) \preceq' (y, Y)$ iff $y \in X$ and $\varnothing \neq Y \subseteq X$;
- for all $(x, \varnothing) \in S'$, $(x, \varnothing) \preceq' (x, \varnothing)$, $(x, \varnothing) \preceq' m$, and $m \preceq' m$.

By (1°), for all $x \in S$, $\varnothing \notin D(x)$, so adding $(x, \varnothing)$ to S' when $D(x) = \varnothing$ does not cause any ambiguity. Note that $\leq'$ and $\preceq'$ are still preorders, and $\preceq'$ is a subrelation of $\leq'$. Since $\mathfrak{F}$ is an FM-frame, its nuclear algebra is based on the locale $\mathsf{Up}(\mathfrak{F})_{F'}$. Define $g\colon \mathsf{Up}(\mathfrak{D}) \to \mathsf{Up}(\mathfrak{F})_{F'}$ by

$$g(U) = \{(x, X) \mid x \in U, X \in D(x)\} \cup \{(x, \varnothing) \mid x \in U, D(x) = \varnothing\} \cup \{m\}.$$

As in the case of f above, it is routine to check that g is an isomorphism between $\mathsf{Up}(\mathfrak{D})$ and $\mathsf{Up}(\mathfrak{F})_{F'}$. To see that it is an isomorphism between the nuclear algebras of $\mathfrak{D}$ and $\mathfrak{F}$, the proof of (9) above works for g in place of f with only small additions. In particular, we must show that $(x, \varnothing) \in g([D\rangle U)$ iff $(x, \varnothing) \in \Box_{\leq'}\Diamond_{\preceq'} g(U)$. In fact, for any $(x, \varnothing) \in S'$, both $(x, \varnothing) \in g([D\rangle U)$ and $(x, \varnothing) \in \Box_{\leq'}\Diamond_{\preceq'} g(U)$. To see the first, since $D(x) = \varnothing$, for any $U \in \mathsf{Up}(\mathfrak{D})$ we have $x \in [D\rangle U$ and hence $(x, \varnothing) \in g([D\rangle U)$. To see the second, consider any (y, Y) such that $(x, \varnothing) \leq' (y, Y)$. Then $x \leq y$, which with $D(x) = \varnothing$ and (3°) implies $D(y) = \varnothing$, so $(y, Y) = (y, \varnothing)$. By construction, $(y, \varnothing) \preceq' m$ and $m \in g(U)$. Thus, $(x, \varnothing) \in \Box_{\leq'}\Diamond_{\preceq'} g(U)$. $\Box$

From Theorems 3.5 and 4.7, we obtain the following.

Corollary 4.8 *For any nuclear frame $\mathfrak{F}$, there is an FM-frame $\mathfrak{G}$ such that the nuclear algebras of $\mathfrak{F}$ and $\mathfrak{G}$ are isomorphic.*

This is weaker than what we had for Dragalin frames in Theorem 3.5, which showed that we can always go from a nuclear frame to a Dragalin frame *based on the same poset*. The following example shows that when going from a nuclear frame to an FM-frame, changing the underlying poset may be unavoidable.

Example 4.9 If $\leq$ is identity, so $\mathsf{Up}(S, \leq) = \wp(S)$, then $\preceq = \leq$, so $\Box_{\leq}\Diamond_{\preceq}$ is the identity nucleus on $\wp(S)$ and any relativization thereof. Yet for any non-trivial Boolean algebra, there is a nucleus distinct from the identity nucleus.

Putting together Corollary 3.8, Theorem 4.7, and Proposition 4.5, we obtain the following analogue of Corollary 3.8.

Corollary 4.10 *A poset P is a locale iff there is a partially ordered normal FM-frame $\mathfrak{F}$ such that P is isomorphic to the algebra of fixpoints in the nuclear algebra of $\mathfrak{F}$.*

5 Goldblatt Frames

A nucleus on a Heyting algebra is a special case of a *dual operator* on a Heyting algebra, a unary function that preserves all finite meets (including 1). Following the tradition in modal logic, we denote such a function by $\Box$.

Definition 5.1 A *modal Heyting algebra* is a pair $(H, \Box)$ where H is a Heyting algebra and $\Box$ is a dual operator. It is a *modal locale* if H is a locale.

Typical examples of modal locales come from intuitionistic modal frames [34].

Definition 5.2 An *IM-frame* (intuitionistic modal frame) is a triple $\mathfrak{F} = (S, \leq, R)$ where $(S, \leq)$ is a poset and $R \subseteq S^2$ is such that $\leq \circ R \circ \leq = R$.

The condition that $\leq \circ R \circ \leq = R$ guarantees that for $U \in \mathsf{Up}(\mathfrak{F})$, we also have $\Box_R U \in \mathsf{Up}(\mathfrak{F})$. It is straightforward to check that $\mathfrak{F}^+ := (\mathsf{Up}(\mathfrak{F}), \Box_R)$ is a modal locale. A natural question, then, is whether there are conditions on an IM-frame $\mathfrak{F}$ that are equivalent to $\mathfrak{F}^+$ being a nuclear algebra. These conditions were identified by Goldblatt [13, pp. 500-01]. Recall that a relation $R \subseteq S^2$ is *dense* if whenever xRy, there is a $z \in S$ such that $xRzRy$.

Lemma 5.3 (Goldblatt) *Let $\mathfrak{F} = (S, \leq, R)$ be an IM-frame.*

(i) *R is a subrelation of $\leq$ iff $U \subseteq \Box_R U$ for each $U \in \mathsf{Up}(\mathfrak{F})$.*

(ii) *R is dense iff $\Box_R \Box_R U \subseteq \Box_R U$ for each $U \in \mathsf{Up}(\mathfrak{F})$.*

Remark 5.4 Goldblatt [13] did not assume the full IM-frame condition that $\leq \circ R \circ \leq = R$, but only the weaker condition that $R \circ \leq = R$, which is still sufficient for $\Box_R$ to be a function on $\mathsf{Up}(S, \leq)$.[7] Relative to frames satisfying the weaker condition, the property that $\Box_R \Box_R U \subseteq \Box_R U$ for each $U \in \mathsf{Up}(\mathfrak{F})$ corresponds to a "pseudo-density" condition on R [13, p. 501], rather than density. But any frame satisfying the weaker condition can be turned into an IM-frame such that their associated modal locales are isomorphic (by simply defining a new relation $R' = \leq \circ R \circ \leq$).

Following Goldblatt's notation, we will use $\prec$ for a dense subrelation of $\leq$.

Definition 5.5 A *Goldblatt frame* is an IM-frame $\mathfrak{F} = (S, \leq, \prec)$ such that $\prec$ is a dense subrelation of $\leq$.

For a Goldblatt frame $\mathfrak{F} = (S, \leq, \prec)$, the modal operator $\Box_\prec$ is a nucleus by Lemma 5.3, which gives us the following.

Proposition 5.6 (Goldblatt) *If $\mathfrak{F} = (S, \leq, \prec)$ is a Goldblatt frame, then $(\mathsf{Up}(\mathfrak{F}), \Box_\prec)$ is a nuclear algebra, which we call the* nuclear algebra *of $\mathfrak{F}$.*

Remark 5.7 A special case of the frames considered by Goldblatt [13]—with $\prec$ not only dense, but also serial—appears in the semantics for *intuitionistic epistemic logic* in [2,26], which treats $\Box_\prec$ as an intuitionistic knowledge modality. Note that in a Goldblatt frame, $\Box_\prec$ is a dense nucleus iff $\prec$ is serial. The logic IEL^+ of [2,26] is exactly the logic of a dense nucleus. This is the extension

[7] A still weaker sufficient condition is that $R \circ \leq \subseteq \leq \circ R$ [7].

of propositional lax logic with the axiom $\neg\bigcirc\bot$, which Fairtlough and Mendler [11, Thm. 4.5] prove is the propositional lax logic of normal FM-frames.

We have seen that for every nuclear frame $\mathfrak{F}$, there is a Dragalin frame and an FM-frame whose nuclear algebras are isomorphic to that of $\mathfrak{F}$ (Theorem 3.5 and Corollary 4.8). Let us now consider for which nuclear frames there is a Goldblatt frame with an isomorphic nuclear algebra.

First, we define a necessary and sufficient condition for a modal locale $(L, \Box)$ to be isomorphic to $\mathfrak{F}^+$ for some IM-frame $\mathfrak{F}$ (recall Remark 2.7).

Definition 5.8 A modal locale $(L, \Box)$ is *perfect* if L is Alexandroff and $\Box$ is completely multiplicative: for every $X \subseteq L$, $\Box \bigwedge X = \bigwedge\{\Box x \mid x \in X\}$.

As is well known, if a function on a complete lattice is completely multiplicative, then it admits an adjoint.

Lemma 5.9 *Given a modal locale $(L, \Box)$ with $\Box$ completely multiplicative, the function $\Diamond^*\colon L \to L$ defined by $\Diamond^* a = \bigwedge\{x \in L \mid a \leq \Box x\}$ is a left adjoint of $\Box$: for all $a, b \in L$, $\Diamond^* a \leq b$ iff $a \leq \Box b$.*

Lemma 5.9 is used in the proof of the following characterization.

Theorem 5.10 *Let $(L, \Box)$ be a modal locale.*

(i) *$(L, \Box)$ is isomorphic to $\mathfrak{F}^+$ for an IM-frame $\mathfrak{F}$ iff $(L, \Box)$ is perfect.*

(ii) *$(L, \Box)$ is isomorphic to $\mathfrak{F}^+$ for a Goldblatt frame $\mathfrak{F}$ iff $(L, \Box)$ is a perfect nuclear algebra.*

Proof. For part (i), for any IM-frame $\mathfrak{F}$, it is easy to see that $\mathfrak{F}^+$ is perfect. Conversely, suppose $(L, \Box)$ is perfect, so L is Alexandroff. Let $S = J^\infty(L)$ (see Remark 2.7) and $\sqsubseteq$ be the dual of the restriction of the order $\leq$ on L to S. It is well known that the function $f\colon L \to \mathsf{Up}(S, \sqsubseteq)$ defined by $f(a) = \{x \in S \mid a \sqsubseteq x\}$ is an isomorphism between L and $\mathsf{Up}(S, \sqsubseteq)$ (see, e.g., [8,5]).

If $\Box$ is a completely multiplicative operator on L, then we define R on S by xRy iff $y \leq \Diamond^* x$. To see that $\mathfrak{F} := (S, \sqsubseteq, R)$ is an IM-frame, suppose $x \sqsubseteq yRz \sqsubseteq u$. Then $u \leq z$, $z \leq \Diamond^* y$, and $y \leq x$. Therefore, $u \leq \Diamond^* y$ and $\Diamond^* y \leq \Diamond^* x$, yielding $u \leq \Diamond^* x$, so xRu. To see that f is an isomorphism between $(L, \Box)$ and $\mathfrak{F}^+$, it only remains to show that $f(\Box a) = \Box_R f(a)$. Suppose $x \in f(\Box a)$ and xRy. Then $\Box a \sqsubseteq x$, so $x \leq \Box a$, and $y \leq \Diamond^* x$, so by Lemma 5.9, $y \leq \Diamond^* x \leq a$ and hence $a \sqsubseteq y$. Thus, $y \in f(a)$, which shows $x \in \Box_R f(a)$. Conversely, if $x \notin f(\Box a)$, then $\Box a \not\sqsubseteq x$, so $x \not\leq \Box a$ and hence $\Diamond^* x \not\leq a$ by Lemma 5.9. Since L is Alexandroff, there is a $y \in S$ such that $y \leq \Diamond^* x$ and $y \not\leq a$. Hence xRy and $a \not\sqsubseteq y$, so $y \notin f(a)$, whence $x \notin \Box_R f(a)$.

For part (ii), apply part (i) and Lemma 5.3. $\Box$

Note that when L is the locale of upsets in a poset P, the poset $(J^\infty(L), \sqsubseteq)$ constructed in the proof of Theorem 5.10 is isomorphic to P. Thus, the following is an immediate corollary of Theorem 5.10.

Corollary 5.11 *Given any nuclear frame $\mathfrak{F} = (S, \leq, j)$ with j completely multiplicative, there is a Goldblatt frame $\mathfrak{G} = (S, \leq, \prec)$ such that $j = \Box_\prec$.*

Using the above results, we can also characterize the locales that can be realized as the algebra of fixpoints in the nuclear algebra of a Goldblatt frame.

Theorem 5.12 *A locale L is isomorphic to the algebra of fixpoints in the nuclear algebra of a Goldblatt frame iff L is completely distributive.*

Proof. From right to left, it suffices to observe that in the proof of Theorem 2.8, if the poset P is a completely distributive locale, then the nucleus j defined in (3) is completely multiplicative, whence Corollary 5.11 gives us the desired Goldblatt frame. The proof that j is completely multiplicative is just like the original proof that j is multiplicative, which used the join-infinite distributive law for locales, but now we use the completely distributive law.

From left to right, by Theorem 5.10, the nucleus in the nuclear algebra of any Goldblatt frame is completely multiplicative, and the underlying locale of the nuclear algebra of any Goldblatt frame is Alexandroff and hence completely distributive. Thus, it suffices to show that for any localic nuclear algebra $\mathbb{L} = (L, j)$ with L completely distributive and j completely multiplicative, its algebra of fixpoints $\mathbb{L}_j$ is completely distributive. Let $\{x_{\varphi,\psi} \mid \varphi \in \Phi,\, \psi \in \Psi_\varphi\}$ be a doubly indexed family of elements from $\mathbb{L}_j$, F the set of functions f assigning to each $\varphi \in \Phi$ some $f(\varphi) \in \Psi_\varphi$, $\bigwedge$ and $\bigvee$ the operations in L, and $\bigsqcap$ and $\bigsqcup$ the operations in $\mathbb{L}_j$. Then we have:

$$\begin{aligned}
\bigsqcap_{\varphi\in\Phi} \bigsqcup_{\psi\in\Psi_\varphi} x_{\varphi,\psi} &= \bigwedge_{\varphi\in\Phi} j \bigvee_{\psi\in\Psi_\varphi} x_{\varphi,\psi} \text{ by definition of } \mathbb{L}_j \\
&= j \bigwedge_{\varphi\in\Phi} \bigvee_{\psi\in\Psi_\varphi} x_{\varphi,\psi} \text{ by complete multiplicativity of } j \\
&= j \bigvee_{f\in F} \bigwedge_{\varphi\in\Phi} x_{\varphi,f(\varphi)} \text{ by complete distributivity of } L \\
&= \bigsqcup_{f\in F} \bigsqcap_{\varphi\in\Phi} x_{\varphi,f(\varphi)} \text{ by definition of } \mathbb{L}_j,
\end{aligned}$$

so $\mathbb{L}_j$ is completely distributive. □

Remark 5.13 While every Alexandroff locale is completely distributive, there are completely distributive non-Alexandroff locales, such as the interval $[0, 1]$.

6 Conclusion

We now have a complete picture of the ability of Dragalin frames, FM-frames, and Goldblatt frames to represent nuclear algebras and, via their algebras of fixpoints, to represent locales. This is summarized in the following table: [8]

[8] We put the dash next to FM because we must typically change the underlying poset of a nuclear frame in order to find an FM-frame with an isomorphic nuclear algebra (see Theorem 4.7 and Example 4.9). Recall that for a nuclear frame $(S, \leq, j)$, its associated locale is $\mathsf{Up}(S, \leq)$, whereas for an FM frame $(S, \leq, \preceq, F)$, its associated locale is $\mathsf{Up}(S, \leq)_F$.

Frames	**Nuclear Frames**	**Nuclear Algebras**	**Locales**
Dragalin	all	Alexandroff	all
FM	–	Alexandroff	all
Goldblatt	completely multiplicative j	perfect	completely distributive

The frames studied in this paper are not the only frames in the literature for representing nuclear algebras. Goldblatt [14] introduces *localic cover systems* and proves that every locale can be realized as the algebra of fixpoints in the nuclear algebra of a localic cover system, by showing that the nucleus j as in the proof of Theorem 2.8 can be generated by one of his cover systems. In future work [6], we will present a detailed comparison of this "cover" perspective and the "development" perspective of Dragalin, thereby relating Scott-Montague-style neighborhood semantics with Beth-style path semantics.

We will also explain in future work how Dragalin frames $(S, \leq, D)$ for intuitionistic propositional logic extend to *modal Dragalin frames* $(S, \leq, D, R)$ for intuitionistic modal logic, where R is a binary relation on S that interacts with $\leq$ and D in a natural way. This provides an intuitionistic generalization of the recently studied "possibility semantics" for classical modal logic [17,15,16,3].

Returning to the logical angle with which we began, it is an open problem whether every intermediate logic is the logic of some class of locales. Given the results of this paper, we can equivalently rephrase the problem as follows: is every intermediate logic the logic of some class of Dragalin frames?

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