Canonical Filtrations and Local Tabularity

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Abstract

The paper deals with a special type of filtration in modal logic called "canonical". This filtration has been known since the 1970s, but was used only occasionally. Applying it in a systematic way allows us to prove new results on finite model property (and in some cases — local tabularity) for different polymodal logics. In particular, we consider products of logics of finite depth with **S5** and **DL**, and also temporal logics of finite depth.

Keywords: modal logic, temporal logic, the finite model property, local tabularity, filtration,canonical model, modal product.

1 Introduction

The filtration method is a standard and powerful instrument in modal logic. Filtrations for Kripke models were first introduced and studied by John Lemmon [9] and (in a general form) by Krister Segerberg [11], [12]

A filtration of a Kripke model M through a set of formulas Ψ is given by a truth-preserving map $h : M \longrightarrow M'$ onto another Kripke model M'. This map is monotonic for all relations in M. So if we can prove that a certain modal logic L is complete w.r.t a class of frames $\mathcal{C}$ and for any model over a $\mathcal{C}$-frame there exists a finite filtration M', also over a $\mathcal{C}$-frame, then L has the finite model property.

This definitely holds if we can construct finite filtrations, for which the filtration map h is a p-morphism. For example, such an argument works in model-theoretic proofs of the well-known Bull's theorem, cf. [4]. In general h need not be a p-morphism, but in some cases p-morphic filtrations can be obtained in a regular way. The corresponding procedure was discovered also by Segerberg [13] [2] . Viz., consider a Kripke model $M = (W, R, \theta)$ and a modal logic L such that $M \vDash L$. Let Ψ be the set of all modal m-formulas (i.e., formulas in proposition letters $p_1, \ldots, p_m$), and let M' be the greatest filtration

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[2] Segerberg used this filtration to show that every Kripke model is equivalent to a distinguished one. Applications to the fmp proofs were not realized at that time.

of M through Ψ. We call such a filtration *canonical*. In this case M' can be identified with a submodel of the weak canonical model $M_{L\lceil m}$, and one can easily show that h is p-morphic whenever M' is finite.

So to prove the fmp of a certain logic L, we can try to find a class of models characterizing L and prove that their canonical filtrations are finite. So far this method has been used only occasionally [7], [8], but in this paper we will show different situations when it is applicable. For the proofs of finiteness of canonical filtrations we shall use the following strategy. Let $\equiv$ be the equivalence relation modulo Ψ. We construct its *stratification*, i.e., we present $\equiv$ as the intersection of a decreasing sequence of equivalence relations $\equiv_0 \supseteq \equiv_1 \supseteq \ldots$, for which the quotient sets $W/\equiv_n$ are finite. Now if the sequence $(\equiv_n)$ stabilizes, this readily implies the finiteness of M'.

Actually for many logics this argument proves not only the fmp, but *local tabularity*, i.e., finiteness of all weak canonical models. Traditional proofs of local tabularity were just by examining points in the canonical model; cf. for example, the proof of Segerberg's theorem on local tabularity of transitive logics of finite depth [13]. However, canonical filtrations may simplify the job. To this end, we first unravel a canonical model into a tree, then go back by the canonical filtration and prove finiteness by stratification.

The plan of the paper is as follows. Section 2 contains basic material on modal logic. The main result of section 3 is the local tabularity of products of (finitely many) modal logics of finite depth with **S5**; hence we deduce the fmp for $\mathbf{S5} \times \mathbf{K}^r$. A similar argument is applied to $\mathbf{DL} \times \mathbf{K}^r$. In section 4 we consider temporal logics of finite depth and show their local tabularity.

2 Preliminaries

We begin with recalling some standard notions and facts.

In this paper we consider normal polymodal propositional logics understood as usual, as sets of polymodal formulas. *r-modal formulas* are built from a countable set $PL = \{p_1, p_2, \dots\}$ of proposition letters, the classical connectives $\to, \bot$, and the modal connectives $\Box_1, \dots, \Box_n$. A *k-formula* is a formula using only proposition letters from the set $PL\lceil k := \{p_1, p_2, \dots p_k\}$. A formula without proposition letters is called *closed*.

$\mathcal{L}_r$ (respectively, $\mathcal{L}_r\lceil k$) denotes the set of all r-modal formulas (respectively, r-modal k-formulas).

An *r-modal logic* is a set of r-modal formulas containing the classical tautologies, the axioms $\Box_i(p_1 \to p_2) \to (\Box_i p_1 \to \Box_i p_2)$, closed under Substitution, Modus Ponens, and Necessitation. The *k-restriction* of a modal logic L is $L\lceil k := L \cap \mathcal{L}_n\lceil k$. These sets $L\lceil k$ are called *k-weak modal logics*.

$\mathbf{K}_r$ denotes the minimal r-modal logic; $\mathbf{K} = \mathbf{K}_1$.

An *r-temporal logic* is a $2r$-modal logic (with the modal connectives $\Box_1, \dots, \Box_r, \Box_{-1}, \dots, \Box_{-r}$) containing the axioms $\Diamond_i \Box_{-i} p \to p,\ \Diamond_{-i}\Box_i p \to p$. $\mathbf{K.t}_r$ denotes the minimal r-temporal logic; $\mathbf{K.t} = \mathbf{K.t}_1$.

For a modal formula A, $md(A)$ denotes its *modal depth* defined by induction:

$$md(\bot) = md(p_i) = 0,\ md(A \to B) = \max(md(A), md(B)),$$
$$md(\Box_j A) = md(A) + 1.$$

Recall that the *fusion* of two logics, r-modal L_1 and m-modal L_2 is $L_1 * L_2 := \mathbf{K}_{r+m} + L_1 + L_2^{+r}$, where L_2^{+r} is obtained from L_2 by replacing every occurrence of any $\Box_j$ with $\Box_{j+r}$.

An *(r-modal) Kripke frame* is a tuple $F = (W, R_1, \dots, R_r)$, where $W \neq \varnothing$, $R_i \subseteq W \times W$.

We use the standard notation $R_i(x) := \{y \in W \mid xR_iy\}$.

A *Kripke model* over F is a pair $M = (F, \theta)$, where $\theta : PL \longrightarrow 2^W$ is a *valuation*.

$M, x \vDash A$ denotes that a formula A is true at a point x in a Kripke model M; the definition is standard.

A *submodel* of a Kripke model is its restriction to some subset of F. A submodel M' of M is called *reliable* if $M, x \vDash A \Leftrightarrow M', x \vDash A$ for any modal formula A and x in M'.

A *k-weak Kripke model* is $M = (F, \theta)$, where $\theta : PL\lceil k \longrightarrow 2^W$ is a *k-valuation*; in this case we can find truth values only for k-formulas.

A formula A is *valid in a frame* F (notation: $F \vDash A$) if it is true at every world of every Kripke model over F. A set of formulas Γ is *valid* in F (notation: $F \vDash \Gamma$) if every $A \in \Gamma$ is valid. In the latter case we also say that F is a Γ-*frame*. The logic *determined* by a class of frames $\mathcal{C}$ is the set of all formulas valid in all frames from $\mathcal{C}$; it is denoted by $\mathbf{L}(\mathcal{C})$.

In particular, $\mathbf{K}_r$ is determined by all r-modal frames; $\mathbf{K.t}_r$ by all *n-temporal frames* of the form $(W, R_1, \dots, R_r, R_1^{-1}, \dots, R_r^{-1})$. The well-known logic **S5** is determined by *clusters*, i.e., frames of the form $(W, W \times W)$.

Definition 2.1 A *p-morphism* from a frame $F = (W, R_1, \dots, R_r)$ onto $F' = (W', R_1', \dots, R_r')$ is a surjective map $f : W \longrightarrow W'$ satisfying the conditions

- $xR_iy \Longrightarrow f(x)R_i'f(y)$ (monotonicity),
- $f(x)R_i'z \Longrightarrow \exists y\ (xR_iy\ \&\ f(y) = z)$ (the lift property).

A p-morphsim of a Kripke model $M = (F, \theta)$ onto $M' = (F', \theta')$ should also satisfy the condition

$$\theta(p) = f^{-1}(\theta'(p))$$

for any $i \le r$, $p \in PL$ (or $PL\lceil k$ if the models are k-weak).

$f : F \twoheadrightarrow F'$ denotes that f is a p-morphism from F onto F'; the same notation is used for Kripke frames.

Lemma 2.2 *Let F, F' be r-modal Kripke frames, M, M' Kripke models over them, A an r-modal formula.*

(i) If $f : F \twoheadrightarrow F'$, then $\mathbf{L}(F) \subseteq \mathbf{L}(F')$.

(ii) For any x in F, $M, x \vDash A$ iff $M', f(x) \vDash A$.

(iii) If A is closed, then $F \vDash A$ iff $F' \vDash A$.

Definition 2.3 Let $F = (W, R_1, \ldots, R_r)$ be a frame, $u, v \in W$, $m \geq 1$. A *path of length m* from u to v is a sequence $(u_0, j_0, u_1, \ldots, j_{m-1}, u_m)$ such that $u = u_0$, $v = u_m$ and for all $i < m$, $u_i R_{j_i} u_{i+1}$. A singleton sequence (u) is the *path of length 0* (from u to u).

An r-temporal frame $(W, R_1, \ldots, R_r, R_1^{-1}, \ldots, R_r^{-1})$ will be denoted by $(W, R_1, \ldots, R_r, R_{-1}, \ldots, R_{-r})$ (where $R_{-j} := R_j^{-1}$). Then paths are sequences $(u_0, j_0, u_1, \ldots, j_{m-1}, u_m)$, in which $j_0, \ldots, j_m$ are integers.

Definition 2.4 A path $(u_0, j_0, u_1, \ldots, j_{m-1}, u_m)$ in an r-temporal frame $(W, R_1, \ldots, R_r, R_{-1}, \ldots, R_{-r})$ is called *reduced* if it does not contain adjacent opposite arrows; speaking precisely, if there is no j such that $u_{j-1} = u_{j+1}$ and $i_j = -i_{j+1}$.

Definition 2.5 The *depth* of a point x in a frame F (denoted by $d(x)$) is the maximum of lengths of paths in F beginning from x (if this maximum exists), or ∞ otherwise.

The depth of x w.r.t. to the relation R_i (denoted by $d_i(x)$) is the depth of x in the frame (W, R_i).

Similarly in a temporal frame we define the *reduced depth* of x (denoted by $rd(x)$) as the maximum of lengths of reduced paths beginning from x.

The depth of a frame F (denoted by $d(F)$) is the maximal depth of its points (if it exists) and ∞ otherwise; similarly for the reduced depth in a temporal frame.

Definition 2.6 A *cone in a frame F with root u* (notation: $F{\uparrow}u$) is the restriction of F to the set of all points, to which there exists a path from u; similarly a *cone in a Kripke model $M{\uparrow}u$* is defined.

Lemma 2.7 *(Generation Lemma)*

(i) $\mathbf{L}(F) = \bigcap\limits_{u \in F} \mathbf{L}(F{\uparrow}u)$.

(ii) $M{\uparrow}u$ is a reliable submodel of M.

Definition 2.8 A *tree* with root u is a frame F such that $F = F{\uparrow}u$ and for every $v \in F$ there exists a unique path from u to v. The length of this path is called the *height* of v and denoted by $h(v)$. The height of F ($h(F)$) is the maximal $h(v)$ (if it exists), or ∞ otherwise.

Definition 2.9 For a $2r$-modal tree $G = (W, S_1, \ldots, S_{2r})$, the frame $F = (W, R_1, \ldots, R_r, R_1^{-1}, \ldots, R_r^{-1})$, where $R_i = S_i \cup S_{r+i}^{-1}$, is called the *r-temporal tree (with the pattern G)*. The height function in F is then defined as the height function in G.

Speaking informally, a temporal tree is a modal tree, in which some of the arrows are inverted.

There is an equivalent definition: an n-temporal tree with root r is an n-temporal frame, in which for every point x there exists a unique reduced path from r to x.

Recall the standard unravelling construction (cf. [6]).

Definition 2.10 Let $F = (W, R_1, \ldots, R_r)$ be a cone with root u. The *unravelling* of F is the frame $F^\sharp = (W^\sharp, R_1^\sharp, \ldots, R_r^\sharp)$, in which $W^\sharp$ is the set of all paths from u to points in F, and $\alpha R_i^\sharp \beta$ iff $\beta = (\alpha, i, x)$ for some x.

Lemma 2.11 *$F^\sharp$ is a tree. The map π sending every path to its endpoint is a p-morphism $F^\sharp \twoheadrightarrow F$.*

A similar construction exists in the temporal case [14]:

Definition 2.12 Let $F = (W, R_1, \ldots, R_r, R_1^{-1}, \ldots, R_r^{-1})$ be a cone with root u. The *temporal unravelling* of F is the frame

$$F^{t\sharp} = (W^\sharp, R_1^{t\sharp}, \ldots, R_r^{t\sharp}, R_{-1}^{t\sharp}, \ldots, R_{-r}^{t\sharp}),$$

in which $W^{t\sharp}$ is the set of all reduced paths from u to points in F, and $\alpha R_i^{t\sharp} \beta$ iff ($\beta = (\alpha, i, x)$ or $\alpha = (\beta, -i, x)$) for some x.

Lemma 2.13 *$F^{t\sharp}$ is a temporal tree. The map π sending every path to its endpoint is a p-morphism $F^{t\sharp} \twoheadrightarrow F$.*

Definition 2.14 The *canonical frame* for an r-modal logic (maybe weak) L is $F_L = (W_L,\ R_{1,L}, \ldots, R_{r,L})$, where W_L is the set of all maximal L-consistent sets of formulas in the language of L; $xR_{i,L}y$ iff for any A, $\Box_i A \in x$ implies $A \in y$.

The *canonical model* for L is $M_L = (F_L, \theta_L)$, where $\theta_L(p_i) = \{x \mid p_i \in x\}$.

Theorem 2.15 *(Canonical model theorem) For any formula A in the language of L,*

(1) $M_L, x \models A$ *iff* $A \in x$;

(2) $M_L \models A$ *iff* $A \in L$.

Lemma 2.16 *(Rigidity lemma) In a canonical model, if there is an isomorphism of two cones $M_L{\uparrow}x$, $M_L{\uparrow}y$ sending x to y, then $x = y$.*

Proof. Since an isomorphism preserves the truth values of formulas, the same formulas (in the language of L) are true in x and y. Hence $x = y$ by 2.15(1). ■

Definition 2.17 A modal logic L is called *canonical* if $F_L \models L$ (or equivalently, $L = \mathbf{L}(F_L)$) and *weakly canonical* if $F_{L\lceil k} \models L$ for any finite k.

Definition 2.18 An r-modal logic L is called *locally tabular* if for any finite k there exist finitely many r-modal k-formulas up to equivalence in L.

The local tabularity of L is obviously equivalent to the local finiteness of the variety of L-algebras (which means finiteness of all finitely generated L-algebras, cf. [10], Ch.6, Sec. 14).

Definition 2.19 An r-modal logic L is called *tabular* if $L = \mathbf{L}(F)$ for some finite r-modal frame F. L has the *finite model property (fmp)* if it is an intersection of tabular logics.

The following simple facts are well-known:

Lemma 2.20 *(1) A modal logic L is locally tabular iff every weak canonical model $M_{L\lceil k}$ is finite.*

(2) Every extension of a locally tabular modal logic in the same language is locally tabular.

(3) Every tabular logic is locally tabular.

(4) Every locally tabular logic has the fmp.

Definition 2.21 The *product* of Kripke frames $F = (W, R_1, \dots, R_r)$, $G = (V, S_1, \dots, S_m)$ is the frame

$$F \times G = (W \times V, R_{11}, \dots, R_{r1}, S_{12}, \dots, S_{m2})$$

such that

$$(x, y)R_{i1}(x', y') \Leftrightarrow xR_ix' \ \&\ y = y';$$

$$(x, y)S_{j2}(x', y') \Leftrightarrow x = x' \ \&\ yS_jy'.$$

Definition 2.22 The *product* of an r-modal logic L_1 and an m-modal logic L_2 is the $(r+m)$-modal logic

$$L_1 \times L_2 := \mathbf{L}(\{F_1 \times F_2 \mid F_1 \vDash L_1,\ F_2 \vDash L_2\}).$$

Definition 2.23 The *commutative join* of an r-modal logic L_1 and an m-modal logic L_2 is obtained from their fusion $L_1 * L_2$ by adding the axioms

$$\Diamond_i \Box_{r+j} p \to \Box_{r+j} \Diamond_i p,\ \Box_i \Box_{r+j} p \leftrightarrow \Box_{r+j} \Box_i p$$

for $1 \le i \le r$, $1 \le j \le m$.

Recall that the corresponding frame conditions are:

$$R_i^{-1} \circ R_{r+j} \subseteq R_{r+j} \circ R_i^{-1},\ R_{r+j} \circ R_i = R_i \circ R_{r+j}.$$

Definition 2.24 Logics $L_1,\ L_2$ are called *product-matching* if $L_1 \times L_2 = [L_1, L_2]$.

Recall a sufficient condition for the product matching property.

Definition 2.25 A modal formula is called *Horn* if the class of its frames is first-order definable by a universal Horn sentence.

A modal logic is *Horn axiomatizable* if it is axiomatized by by adding closed or Horn modal formulas.

Theorem 2.26 *Every two complete Horn axiomatizable modal logics are product-matching.*

For the proof cf. Theorem 7.12 from [6] (a slightly weaker claim) or Theorem 5.9 from [5] (for 1-modal logics).

Definition 2.27 Let $M = (W, R_1, \dots, R_r, \theta)$ be an n-modal Kripke model, Ψ a set of r-modal formulas closed under subformulas. For $x \in W$ let $\Psi_x := \{A \in \Psi \mid M, x \vDash A\}$. Two worlds $x, y \in W$ are called Ψ-*equivalent in* M (notation: $(M, x) \equiv_\Psi (M, y)$, or just $x \equiv_\Psi y$) if $\Psi_x = \Psi_y$. The map $h : x \mapsto x/\equiv_\Psi$ sending every world to its Ψ-equivalence class is called the *filtration map* (through Ψ).

Definition 2.28 (cf. [6]) Under the assumptions of Definition 2.27, a Kripke model $M' = (W', R'_1, ..., R'_r, \theta')$ is called a *filtration of* M *through* Ψ if for any $x, y \in W$, for any formula A, $1 \leq i \leq r$:

(f1) $W' = W/\equiv_\Psi$;

(f2) $xR_iy \implies h(x)R'_ih(y)$;

(f3) $h(x)R'_ih(y)$ & $M, x \vDash \Box_i A$ & $\Box_i A \in \Psi \implies M, y \vDash A$;

(f4) if $q \in \Psi \cap PL$, then $M, x \vDash q \iff M', h(x) \vDash q$.

The *greatest filtration* of M through Ψ is defined by the conditions (f1), (f4), and

(f3^+) $h(x)R'_ih(y)$ iff for any A,
$M, x \vDash \Box_i A$ & $\Box_i A \in \Psi \implies M, y \vDash A$.

Lemma 2.29 *(Filtration Lemma). Let M' be a filtration of M through Ψ. Then for any $x \in W$, for any $A \in \Psi$*

$$M, x \vDash A \textit{ iff } M', h(x) \vDash A.$$

Definition 2.30 Let M be an n-modal Kripke model, Ψ the set of all n-modal k-formulas. The greatest filtration of M through Ψ is called *canonical.*

For the canonical filtration we can obviously identify $h(x)$ with Ψ_x, i.e., the set of all k-formulas true at M, x. For any modal logic L true in M, the set Ψ_x is maximal L-consistent, i.e., $\Psi_x \in W_L$ (cf. Definition 2.14). So M' is isomorphic to a reliable submodel of the canonical model $M_{L\lceil k}$.

In fact every p-morphism onto a reliable submodel of a canonical model is a canonical filtration:

Lemma 2.31 *Suppose $h : M \twoheadrightarrow M'$ for a reliable submodel M' of a weak canonical model $M_{L\lceil k}$ for some modal logic L. Then M' is a canonical filtration of M through $\Psi = \mathcal{L}_n\lceil k$ and h is the filtration map.*

Proof. For any k-formula A

$$M, x \vDash A \text{ iff } M', h(x) \vDash A \text{ iff } M, h(x) \vDash A \text{ iff } A \in h(x)$$

by Lemma 2.2, the reliability of M', and Theorem 2.15. Thus $h(x) = \Psi_x$. ■

Proposition 2.32 *If the canonical filtration M' is finite, then the filtration map $h : M \longrightarrow M'$ is a p-morphism.*

Proof. By definition and the Filtration Lemma, $h(x) = h(y)$ iff for any $A \in \Psi$, $M', h(x) \vDash A \Leftrightarrow M', h(y) \vDash A$. So every two different points in M' are distinguished by a formula from Ψ. Since M' is finite, for any $u \in M'$ there exists $A_u \in \Psi$ which is true exactly at u.

Now suppose $h(x)R'_i h(y)$. Then $h(x) \vDash \Diamond_i A_{h(y)}$, and so $x \vDash \Diamond_i A_{h(y)}$. Thus there exists $z \in R_i(x)$ such that $z \vDash A_{h(y)}$. Hence $h(z) \vDash A_{h(y)}$ implying that $h(z) = h(y)$. ■

3 Modal logics of finite depth

In the r-modal language we introduce the total box and diamond as abbreviations:

$$\Box A := \Box_1 A \wedge \ldots \wedge \Box_r A, \ \Diamond A := \Diamond_1 A \vee \ldots \vee \Diamond_r A.$$

Lemma 3.1 *Let $F = (W, R_1, \ldots, R_r)$ be a frame, $k \geq 1$. Then (in any model over F), for any $x \in W$*

$$x \vDash \Box^{k+1}\bot \text{ iff } d(x) \leq k.$$

The proof is by induction, cf. [6], Lemma 9.2.

Theorem 3.2 *Every logic $\mathbf{K}_r + \Box^k\bot$ is locally tabular.*

This theorem was proved in [6] by examining weak canonical models: a simple inductive argument shows that for any finite d there are finitely many points of depth d in every weak canonical model $M_{L\lceil m}$. Since $\Box^k\bot$ holds in M, all points are of depth less than k; therefore M is finite.

However, let us sketch another proof in the style of the present paper. Consider a cone M' in $M_{L\lceil m}$ and its unravelling M (which is a model over an r-modal tree of depth $(k-1)$ $(W, R_1, \ldots, R_r)$). The canonical map $h : M \longrightarrow M'$ is a p-morphism.

We define the equivalence relations $\equiv_n$ on W by induction:

- $x \equiv_0 y$ iff $x \vDash q \Leftrightarrow y \vDash q$ for any $q \in PL\lceil m$,
- $x \equiv_{n+1} y$ iff $x \equiv_0 y$ & $\forall i\ (R_i(x)/\equiv_n) = (R_i(y)/\equiv_n)$.

Also put

$$x \sim_n y := (x \vDash A \Leftrightarrow y \vDash A \text{ for any } m\text{-formula } A \text{ of depth } \leq n).$$

Lemma 3.3 *If $x \equiv_n y$, then $x \sim_n y$.*

Proof. The proof is straightforward by induction. ■

Lemma 3.4 *If $d(x), d(y) \leq n$ and $x \equiv_n y$, then $x \equiv_{n+1} y$.*

Proof. By induction on n.

The case $n = 0$ is trivial, since $R_i(x) = R_i(y) = \varnothing$.

Suppose the claim holds for $n > 0$, and consider points x, y of depth $\leq n+1$ such that $x \equiv_{n+1} y$. Then for any $z \in R_i(x)$ there exists $z' \in R_i(y)$ such that $z \equiv_n z'$. Then $d(z), d(z') \leq n$, so by IH $z \equiv_{n+1} z'$. It follows that $(R_i(x)/\equiv_{n+1}) \subseteq (R_i(y)/\equiv_{n+1})$. The converse follows by symmetry. ■

Thus by Lemma 3.4 $x \equiv_k y$ implies $x \equiv_n y$ for any $n \geq k$; hence $h(x) = h(y)$ whenever $x \equiv_k y$. But the number of $\equiv_k$-classes is finite; this follows easily by induction.

So every cone in the weak canonical model is finite of limited size. Since this model is distinguishable, it is rigid in the following sense: every two points with isomorphic cones and the same truth values of the proposition letters $p_1, \ldots, p_m$ must coincide. Therefore the whole model is finite.

Let us use a similar method to prove a stronger result.

Consider the logics $\mathbf{S5} \times (\mathbf{K}_r + \Box^s\bot)$. First note that $\mathbf{S5} \times (\mathbf{K}_r + \Box^s\bot) = [\mathbf{S5}, \mathbf{K}_r + \Box^s\bot]$ by Theorem 2.26. The axioms of this logic are Sahlqvist formulas, so it is canonical.

Proposition 3.5 *Every cone validating* $\mathbf{K}_n \times \mathbf{K}_m = [\mathbf{K}_n, \mathbf{K}_m]$ *is a p-morphic image of a product of an n-modal tree and an m-modal tree.*

Proof. The proof is by applying a transfinite version of the "rectification game". Such a game for the countable case is constructed in the proof Lemma 5.2 from [5]. For a transfinite game just add the requirement that the network at the limit stage is the union of all earlier networks. ■

Lemma 3.6 *Every cone validating* $\mathbf{S5} \times (\mathbf{K}_r + \Box^s\bot)$ *is a p-morphic image of a product of a cluster and an r-modal tree of depth $\leq s - 1$.*

Proof. The argument is the same as in the proof of theorem 7.2 from [6]. Let $F = (W, R_0, R_1, \ldots, R_r)$ be a given cone; then $F \vDash [\mathbf{K}, \mathbf{K}_r]$. So by Proposition 3.5, there is a p-morphism $f : F_1 \times F_2 \twoheadrightarrow F$, where F_1, F_2 are trees (respectively, 1-modal and r-modal). Let C be the cluster with the set of worlds of F_1. Since R_0 is an equivalence, it follows that $f : C \times F_2 \twoheadrightarrow F$. By lemma 2.2, the validity of the closed formula $\Box^s\bot$ is preserved in $F_1 \times F_2$. Hence by Lemma 3.1, F_2 is of depth $\leq s - 1$. ■

Theorem 3.7 *Every logic* $\mathbf{S5} \times (\mathbf{K}_r + \Box^s\bot)$ *is locally tabular.*

Proof. Let $L = \mathbf{S5} \times (\mathbf{K}_r + \Box^s\bot)$, and again let us show that all the cones in $M_{L\lceil m}$ are finite.

Consider a cone $M_1 = M_{L\lceil m}{\uparrow}u$; let $M_1 = (F_1, \theta_1)$. By Lemma 3.6, F_1 is a p-morphic image of a product $C \times F$ of a cluster C and an r-modal tree F of depth $\leq s - 1$. So M_1 is a p-morphic image of a model M over $C \times F$.

Let R_0, $R_1, \ldots, R_r$ be the relations in M (so R_0 is an equivalence).

We define the equivalence relations $\equiv_n$ on M by induction:

- $x \equiv_0 y$ iff $x \vDash q \Leftrightarrow y \vDash q$ for any $q \in PL\lceil m$,
- $x \equiv_{2n+1} y$ iff $x \equiv_{2n} y$ & $(R_0(x)/\equiv_{2n}) = (R_0(y)/\equiv_{2n})$.
- $x \equiv_{2n+2} y$ iff $x \equiv_{2n+1} y$ & $\forall i > 0$ $(R_i(x)/\equiv_{2n+1}) = (R_i(y)/\equiv_{2n+1})$.

Lemma 3.8 *The number of $\equiv_n$-classes in M is finite.*

Proof. By induction we show that the set $W_n := W/\equiv_n$ is finite (where W is the set of worlds in M).

Obviously W_0 is finite, of cardinality at most 2^m.

Suppose W_{2n} is finite. Note that every class $x/\equiv_{2n+1}$ is fully determined by the pair $(x/\equiv_{2n}, R_0(x)/\equiv_{2n})$. Thus

$$|W_{2n+1}| \leq |W_{2n}| \cdot 2^{|W_{2n}|}.$$

(where $|\dots|$ denotes the cardinality).

Similarly, $(x/\equiv_{2n+2})$ is fully determined by the tuple $(x/\equiv_{2n+1}, R_1(x)/\equiv_{2n+1}, \dots, R_r(x)/\equiv_{2n+1})$. Hence

$$|W_{2n+2}| \leq |W_{2n+1}| \cdot 2^{r|W_{2n+1}|}.$$

■

Lemma 3.9 *If $x \equiv_{2n} y$, then $x \sim_n y$.*

Proof. By induction. The base is trivial. For the step, suppose $x \equiv_{2n+2} y$.

If $x \vDash \Diamond_i A,\ i > 0,\ md(\Diamond_i A) \leq n+1$, then $z \vDash A$ for some $z \in R_i(x)$, and $d(A) \leq n$. Since $x \equiv_{2n+2} y$, there is $z' \in R_i(y)$ such that $z' \equiv_{2n+1} z$, and so $z' \equiv_{2n} z$. By the IH, $z' \vDash A$. It follows that $y \vDash \Diamond_i A$.

If $x \vDash \Diamond_0 A,\ md(\Diamond_0 A) \leq n+1$, then $z \vDash A$ for some $z \in R_0(x)$, and $d(A) \leq n$. $x \equiv_{2n+2} y$ implies $x \equiv_{2n+1} y$, so there is $z' \in R_0(y)$ such that $z' \equiv_{2n} z$. By the IH, $z' \vDash A$. It follows that $y \vDash \Diamond_0 A$. ■

By $d(x)$ we denote the depth of a point of a point $x \in M$ over the second coordinate. More precisely, if $x = (a, b)$, then $d(x)$ is $d(b)$ (in the tree F).

Lemma 3.10 *If $d(x), d(y) \leq n$ and $x \equiv_{2n+1} y$, then $x \equiv_k y$ for any $k > 2n+1$.*

Proof. By induction on n.

(a) Consider the case $n = 0$. Suppose $d(x) = d(y) = 0$, $x \equiv_1 y$ and show that $x \equiv_k y$ for any $k > 1$ by induction on k.

$x \equiv_{2j+1} y$ clearly implies $x \equiv_{2j+2} y$, since $R_i(x) = R_i(y) = \varnothing$ for $i > 0$.

On the other hand, if $x \equiv_{2j+2} y$, then $x \equiv_{2j+1} y$, so $R_0(x)/\equiv_{2j} = R_0(y)/\equiv_{2j}$. Let us show that $R_0(x)/\equiv_{2j+2} = R_0(y)/\equiv_{2j+2}$. In fact, since $x \equiv_{2j+1} y$, for any $z \in R_0(x)$ there is $z' \in R_0(y)$ such that $z \equiv_{2j} z'$. Since $R_0(z) = R_0(x)$ and $R_0(z') = R_0(y)$, we also have $z \equiv_{2j+1} z'$. But $d(z) = d(z') = 0$, so as we have noticed above, $z \equiv_{2j+2} z'$. It follows that $R_0(x)/\equiv_{2j+2} \subseteq R_0(y)/\equiv_{2j+2}$, and we obtain the converse by symmetry. Therefore $x \equiv_{2j+2} y$ implies $x \equiv_{2j+3} y$.

(b) Now consider the induction step for the main induction on n. Suppose $d(x), d(y) \leq n$, $x \equiv_{2n+1} y$ and show that $x \equiv_k y$ for any $k > 2n+1$ by induction on k.

Suppose $k = 2j+1 > 2n+1$ and the claim is proved for less k. $x \equiv_{2n+1} y$ implies $R_0(x)/\equiv_{2n} = R_0(y)/\equiv_{2n}$, so for any $z \in R_0(x)$ there is $z' \in R_0(y)$ such that $z \equiv_{2n} z'$. Then $z \equiv_{2n+1} z'$ (since $R_0(z) = R_0(x),\ R_0(z') = R_0(y)$). Since $d(z), d(z') \leq n$, by the IH (applied to z, z') it follows that $z \equiv_{2j} z'$. Thus we have proved $R_0(x)/\equiv_{2j} \subseteq R_0(y)/\equiv_{2j}$, and the converse follows by symmetry. So we obtain $x \equiv_k y$.

Suppose $k = 2j+2 > 2n+1$ and the claim is proved for less k. $x \equiv_{2n+1} y$ implies $x \equiv_{2n} y$ and thus $R_i(x)/\equiv_{2n-1} = R_i(y)/\equiv_{2n-1}$ for any $i > 0$. Now for any $z \in R_i(x)$ there is $z' \in R_i(y)$ such that $z \equiv_{2n-1} z'$. But $d(z), d(z') \leq n-1$, so by the IH of the main induction, $z \equiv_{2n-1} z'$ implies $z \equiv_{2j+1} z'$. So we obtain $R_i(x)/\equiv_{2j+1} \subseteq R_i(y)/\equiv_{2j+1}$, and the converse follows by symmetry. Thus $x \equiv_k y$. ■

The argument in the proof of the theorem is now as in the case of $\mathbf{K}_r + \Box^s\bot$. Since $d(x), d(y) \leq s-1$, by Lemma 3.10 $x \equiv_{2s} y$ implies $x \equiv_k y$ for any $k \geq 2s$; hence by Lemma 3.9, $h(x) = h(y)$ whenever $x \equiv_{2s} y$. Now Lemma 3.8 implies that M_1 is finite of limited size. Therefore up to isomorphism, there are finitely many cones in the weak canonical model, so it is finite by rigidity. ■

Theorem 3.11 *Every logic* $\mathbf{S5} \times \mathbf{K}^m$ *has the fmp.*

Proof. First note that

$$\mathbf{S5} \times \mathbf{K}^m = \mathbf{L}(\{C \times F_1 \times \ldots \times F_m \mid C \text{ is a cluster}, F_1, \ldots, F_m \text{ are trees}\}.$$

This is proved similarly to proposition 4.10 from [6]. In fact, every cone in a product $G_0 \times G_1 \times \ldots G_m$ has the form $C \times H_1 \times \ldots \times H_m$, where C is a cluster, H_i are cones, so it is a p-morphic image of $C \times F_1 \times \ldots \times F_m$, where $F_i = H_i^\sharp$.

Every formula refutable in $C \times F_1 \times \ldots \times F_m$ is also refutable in a product $C \times F_1^- \times \ldots \times F_m^-$, where F_i^- is a tree of finite depth obtained by truncation of F_i; this is similar to Lemma 9.11 from [6]. The product $F_1^- \times \ldots \times F_m^-$ is also of finite depth: if $d(F_i^-) < s$, then $d(F_1^- \times \ldots \times F_m^-) < ms$. Therefore

$$\mathbf{S5} \times \mathbf{K}^m = \bigcap_s (\mathbf{S5} \times (\mathbf{K}^m + \Box^s\bot)).$$

Note that the logic $\mathbf{S5} \times (\mathbf{K}^m + \Box^s\bot)$ contains $\mathbf{S5} \times (\mathbf{K}_m + \Box^s\bot)$, so it is locally tabular by Theorem 3.2 and Lemma 2.20. Then it has the fmp (Lemma 2.20) and eventually $\mathbf{S5} \times \mathbf{K}^m$ has the fmp as an intersection of logics with the fmp. ■

The fmp for the logic $\mathbf{S5} \times \mathbf{K}$ was proved in [6] by another method giving a better upper bound for the size of countermodels. The above theorem for $m > 1$ seems new; however, all these logics are undecidable (this follows from a general result by R. Hirsch, I. Hodkinson, and A. Kurucz, cf. Theorem 8.28 from [5]).

Now consider the difference logic $\mathbf{DL}$. Recall that

$$\mathbf{DL} = \mathbf{K} + \Diamond\Box p \to p + p \wedge \Box \to \Box\Box p.$$

$\mathbf{DL}$-cones are of the form (W, R), where R contains the inequality relation $\neq_W := \{(x, y) \in W^2 \mid x \neq y\}$ (cf. [3]).

Lemma 3.12 *Every* $\mathbf{DL}$*-cone* (W, R) *is a p-morphic image of some inequality frame* $(V, \neq_V)$.

Proof. By a well-known construction: to obtain V duplicate the reflexive points of W and make them irreflexive. ■

Theorem 3.13 *Every logic* $\mathbf{DL} \times (\mathbf{K}_r + \Box^s\bot)$ *has the fmp.*

Proof. Almost the same as in Theorem 3.7, but with another starting point. Suppose a formula A in letters $p_1, \ldots, p_m$ is not in $L = \mathbf{DL} \times (\mathbf{K}_r + \Box^s\bot)$. Then it is refuted in a weak model M_0 over a product of cones $G \times F$, where $G \vDash \mathbf{DL}$, F is of depth $< s$.

By Lemma 3.12 G is a p-morphic image of an inequality frame C; so A is refuted in $C \times F$, and thus in $C \times F^\sharp$; $F^\sharp$ is a tree of depth $< s$.

Next, we define the relations $\equiv_n$ in the corresponding model M exactly as in the proof of 3.7 and repeat the further proof (with a slight change in the proof of 3.10: at the induction step for $k = 2j+1$ instead of $R_0(z) = R_0(x)$, $R_0(z') = R_0(y)$ we have $R_0(z) \cup \{z\} = R_0(x) \cup \{x\}$, $R_0(z') \cup \{z\} = R_0(y) \cup \{y\}$). Therefore the canonical filtration M' of M is finite, and we can apply Proposition 2.32. Thus A is refuted in a finite L-frame. ■

Note that now we cannot claim the local tabularity, because we obtain a p-morphism onto some submodel of $M_{L\lceil m}$, but not onto an arbitrary cone.

Theorem 3.14 *Every logic* $\mathbf{DL} \times \mathbf{K}^m$ *has the fmp.*

Proof. Similar to 3.11. By truncation, refutability of a formula in a product $C \times F_1 \times \ldots \times F_m$, where C is an inequality frame, F_i are trees, is reduced to refutability in $C \times F_1^- \times \ldots \times F_m^-$, where F_i^- are trees of finite depth. Hence

$$\mathbf{DL} \times \mathbf{K}^m = \bigcap_s (\mathbf{DL} \times (\mathbf{K}^m + \Box^s\bot)),$$

and we can apply Theorem 3.13. ■

4 Temporal logics of finite depth

Now let us modify some results of the previous section for temporal logics.

Definition 4.1 Consider the following r-temporal formulas

$$Rd_{i_1\ldots i_n} := \neg(P_0 \wedge \Diamond_{i_1}(P_1 \wedge \Diamond_{i_2}(P_2 \wedge \ldots \wedge \Diamond_{i_n} P_n)\ldots))),$$

where $i_j \in \{\pm 1, \ldots, \pm r\}$,

$$P_0 := p_0,$$

$$P_{j+1} := \begin{cases} p_{j+1} \wedge \neg p_{j-1} & \text{if } i_{j+1} = -i_j, \\ p_{j+1} & \text{otherwise,} \end{cases}$$

$$Rd_n := \bigwedge \{Rd_{i_1\ldots i_n} \mid i_1, \ldots, i_n \in \{\pm 1, \ldots, \pm r\}\}.$$

Proposition 4.2 *For an* r*-temporal frame* F

$$F \vDash Rd_n \textit{ iff } rd(F) < n.$$

Proof. (If.) Suppose in a model over F we have $u_0 \vDash \neg Rd_{i_1\ldots i_n}$. Then there are $u_1, \ldots, u_n$ such that for any $j \geq 0$, $u_j \vDash P_j$ and $u_j R_{i_j} u_{j+1}$. So $u_0, \ldots, u_n$ is a path in F. Note that $i_{j+1} = -i_j$ implies $u_{j+1} \neq u_{j-1}$, since in this case $P_{j+1} = p_{j+1} \wedge \neg p_{j-1}$ and P_{j-1} implies p_{j-1}. Thus $(u_0, i_1 \ldots, i_n, u_n)$ is a reduced path of length n.

(Only if.) Suppose there is a reduced path $(u_0, i_1 \ldots, i_n, u_n)$. Consider a valuation θ in F such that $\theta(p_j) = \{u_j\}$. Then we obtain a Kripke model, in which $u_j \vDash P_j$ and $u_0 \vDash \neg Rd_{i_1\ldots i_n}$. Thus $F \nvDash Rd_n$. ■

Now consider the logics $\mathbf{K.t}_r + Rd_n$.

Proposition 4.3 *$\mathbf{K.t}_r + Rd_n$ is weakly canonical.*

Proof. Let $L = \mathbf{K.t}_r + Rd_n$. Consider a weak canonical frame $F_{L\lceil k} = (W, R_1, \ldots, R_r, R_{-1}, \ldots, R_{-r})$ and suppose it has a reduced path $(u_0, i_1 \ldots, i_n, u_n)$. Let A_j be a formula true at u_j and false at all the u_m differing from u_j. Then in $M_{L\lceil k}$ for any j $u_j \vDash A_j$, and $u_{j+1} \vDash A_{j+1} \wedge \neg A_{j-1}$. It follows that $u_0 \vDash \neg Rd_{i_1\ldots i_n}(A_0, \ldots, A_n)$. At the same time $L\lceil k \vdash Rd_{i_1\ldots i_n}(A_0, \ldots, A_n)$ contradicting the Canonical model theorem 2.15. Therefore $d(F_{L\lceil k}) < n$, and thus L is weakly canonical by Proposition 4.2. ■

Theorem 4.4 *Every logic $\mathbf{K.t}_r + Rd_n$ is locally tabular.*

Proof. As we have just proved, the weak canonical frames are of reduced depth $< n$. So by Lemma 2.13, every cone M' in a weak canonical model can be unravelled into a model M over an r-temporal tree of height $< n$.

It remains to show that the canonical filtration of M (which coincides with M' by Lemma 2.31) is finite. We do this again by an appropriate stratification.

Let R_i, $i = \pm 1, \ldots \pm r$, be the accessibility relations in M. For a point $x \in M$ (which is not a root) let x^- be its predecessor in the tree, $R_i^\bullet(x) := R_i(x) - \{x^-\}$. For $x, y \in M$ we put

$$x \approx y \text{ iff } \forall i \in \{\pm 1, \ldots, \pm r\}(x^- R_i x \Leftrightarrow y^- R_i y).$$

For $x, y \in M$ we define $x \equiv_n y$ by induction: $x \equiv_0 y$ is the same as above; $x \equiv_{n+1} y$ iff

$$x \equiv_n y \;\&\; x \approx y \;\&\; x^- \equiv_n y^- \;\&\; \forall i\ (R_i^\bullet(x)/\equiv_n) = (R_i^\bullet(y)/\equiv_n).$$

Lemma 4.5 *$x \equiv_n y$ implies $x \sim_n y$*

Proof. By induction. For the induction step: suppose the claim holds for n and $x \equiv_{n+1} y$; consider a formula $\Diamond_i A$ of depth $(n+1)$. If $x \vDash \Diamond_i A$, then $z \vDash A$ for some $z \in R_i(x)$. Note that then there is $z' \in R_i(y)$ such that $z \equiv_n z'$. In fact, if $xR_i^\bullet z$, this follows from $(R_i^\bullet(x)/\equiv_n) = (R_i^\bullet(y)/\equiv_n)$. If $xR_i z = x^-$, then $x \approx y$ implies $yR_i y^-$; since $x^- \equiv_n y^-$, we can take $z' = y^-$.

Thus we have $z' \vDash A$, $y \vDash \Diamond_i A$.

The proof for $\Diamond_{-i} A$ is similar. ■

$d(x)$ denotes the depth and $h(x)$ the height of a point x in the tree M (more precisely, in its pattern).

Lemma 4.6 *If $h(x) = h(y)$, $\max(d(x), d(y)) \leq n$, $x \equiv_n y$, $x \approx y$ and $x^- \equiv_n y^-$, then $x \equiv_{n+1} y$.*

Proof. By induction on $k := \max(d(x), d(y))$. The case $k = 0$ is obvious.

Suppose the claim holds for k, and $\max(d(x), d(y)) = k + 1$. To show that $x \equiv_{n+1} y$, we have to check $(R_i^\bullet(x)/\equiv_n) = (R_i^\bullet(y)/\equiv_n)$.

First suppose $n > 0$. If $xR_i^\bullet z$, then $d(z) < d(x)$ and there exists $z' \in R_i^\bullet(y)$ such that $z \equiv_{n-1} z'$ (since $x \equiv_n y$). Then $\max(d(z), d(z')) \leq k$, so by IH, $z \equiv_n z'$ (note that $z \approx z'$, since $z^- = x$, $z'^- = y$, xR_iz, yR_iz'). Thus $(R_i^\bullet(x)/\equiv_n) \subseteq (R_i^\bullet(y)/\equiv_n)$; and the converse holds by symmetry.

Now suppose $n = 0$, i.e., $d(x) = d(y) = 0$. Then $R_i^\bullet(x) = R_i^\bullet(y) = \varnothing$, and we readily obtain $x \equiv_1 y$. ■

Next we define x^{-k} as the $k - th$ predecessor of x (and $x^{-0} = x$). For x, y of the same height h put

$$x \approx^+ y \text{ iff } \forall m < h(x^{-m} \approx y^{-m} \;\&\; x^{-m} \equiv_r y^{-m})$$

Lemma 4.7 *If $x \approx^+ y$, then $x \equiv_\Psi y$.*

Proof. By induction on h. If $h = 0$, then we have the root $x = y$.

Suppose the claim holds for $h - 1$. By induction we prove that $x \equiv_n y$ for $n \geq r$. The base is given. Supposing $x \equiv_n y$ let us check $x \equiv_{n+1} y$.

In fact, since $x \approx^+ y$, we have $x \approx y$ and $x^- \equiv_r y^-$. Hence $x^- \equiv_n y^-$ by the IH of the main induction (on h). Therefore $x \equiv_{n+1} y$ by Lemma 4.6. ■

Since the height of M is finite, Lemma 4.7 implies the finiteness of M'. ■

5 Conclusion

The method developed in this paper can probably be modified for different kinds of logics: intuitionistic, intuitionistic modal and maybe others. There are more applications within modal logic as well; for example, theorems 3.7, 3.13 can be extended to the temporal case. We hope to publish further results in the sequel.

The study of locally tabular logics can be made within a general context of locally finite varieties of algebras. Cf. [1], where in particular, an algebraic proof of Segerberg's theorem is proposed. It is likely that the results of the present paper can also be proved using the technique from [1].

The paper [2] proves other interesting results on local tabularity using an algebraic technique; in particular, it shows that every proper extension of $\mathbf{S5}^2$ is locally tabular. Our model-theoretic method is probably applicable to this case as well, but this not so obvious.

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