Partiality and Adjointness in Modal Logic

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Abstract

Following a proposal of Humberstone, this paper studies a semantics for modal logic based on partial "possibilities" rather than total "worlds." There are a number of reasons, philosophical and mathematical, to find this alternative semantics attractive. Here we focus on the construction of possibility models with a finitary flavor. Our main completeness result shows that for a number of standard modal logics, we can build a canonical possibility model, wherein every logically consistent formula is satisfied, by simply taking each *individual finite formula* (modulo equivalence) to be a possibility, rather than each infinite maximally consistent set of formulas as in the usual canonical world models. Constructing these locally finite canonical models involves solving a problem in general modal logic of independent interest, related to the study of *adjoint* pairs of modal operators: for a given modal logic $\mathbf{L}$, can we find for every formula φ a formula $\mathsf{f}_a^{\mathbf{L}}(\varphi)$ such that for every formula ψ, $\varphi \to \Box_a\psi$ is provable in $\mathbf{L}$ if and only if $\mathsf{f}_a^{\mathbf{L}}(\varphi) \to \psi$ is provable in $\mathbf{L}$? We answer this question for a number of standard modal logics, using model-theoretic arguments with world semantics. This second main result allows us to build for each logic a canonical possibility model out of the lattice of formulas related by provable implication in the logic.

Keywords: possibility semantics, adjointness, completeness, canonical models.

1 Introduction

Humberstone [17] has proposed a semantics for modal logics based on partial "possibilities" rather than total "worlds." One difference between possibility models and world models is that each possibility provides a partial assignment of truth values to atomic sentences, which may leave the truth values of some atomic sentences indeterminate. Unlike standard three-valued semantics, however, Humberstone's semantics still leads to a classical logic because the connectives $\neg$, $\vee$, and $\to$ quantify over *refinements* of the current possibility that resolve its indeterminacies in various ways. Another difference between possibility models and world models, raised toward the end of Humberstone's paper, is that in possibility models a modal operator $\Box$ does not need to quantify over multiple accessible points—a single possibility will do, because a single possibility can leave matters indeterminate in just the way that a set of total worlds can. This idea is especially natural for doxastic and epistemic logic: an agent believes φ at possibility X if and only if φ is true at the single possibility Y that represents *the world as the agent believes it to be* in X.

There are a number of reasons, philosophical and mathematical, to find an alternative semantics based on possibilities attractive. Here we focus on the construction of possibility models with a finitary flavor. Our main completeness result shows that for a number of standard modal logics, we can build a canonical possibility model, wherein every logically consistent formula is satisfied, by simply taking each *individual finite formula* (modulo equivalence) to be a possibility, rather than each infinite maximally consistent set of formulas as in the usual canonical world models. [1] Constructing these locally finite canonical models involves first solving a problem in general modal logic of independent interest, related to the study of *adjoint* pairs [2] of modal operators: for a given modal logic $\mathbf{L}$, can we find for every formula φ a formula $\mathsf{f}_a^{\mathbf{L}}(\varphi)$ such that for every formula ψ, $\varphi \to \Box_a\psi$ is provable in $\mathbf{L}$ if and only if $\mathsf{f}_a^{\mathbf{L}}(\varphi) \to \psi$ is provable in $\mathbf{L}$? We answer this question in §3 for a number of standard modal logics, using model-theoretic arguments with world semantics. [3] This second main result allows us in §4 to build for each logic a canonical possibility model out of the lattice of formulas related by provable implication in the logic.

Given a normal modal logic $\mathbf{L}$, it is a familiar step to consider the lattice $\langle L, \leq\rangle$ where L is the set of equivalence classes of formulas under provable equivalence in $\mathbf{L}$, i.e., $[\varphi] = [\psi]$ iff $\vdash_{\mathbf{L}} \varphi \leftrightarrow \psi$, and $\leq$ is the relation of provable implication in $\mathbf{L}$ lifted to the equivalence classes, i.e., $[\varphi] \leq [\psi]$ iff $\vdash_{\mathbf{L}} \varphi \to \psi$. [4] (Below we will flip and change the relation symbol from '$\leq$' to '$\geqslant$' to match Humberstone.) What we will show is that for a number of modal logics $\mathbf{L}$, we can add to such a lattice functions $f_a\colon L \to L$ such that for all formulas φ and ψ, $[\varphi] \leq [\Box_a\psi]$ iff $f_a([\varphi]) \leq [\psi]$, and that the resulting structure serves as a canonical model for $\mathbf{L}$ according to the functional possibility semantics of §2.

2 Functional Possibility Semantics

We begin with a standard propositional polymodal language. Given a countable set $\mathsf{At} = \{p, q, r, \dots\}$ of atomic sentences and a finite set $I = \{a, b, c, \dots\}$ of

[1] Humberstone [17, p. 326] states a similar result without proof, but see the end of §2 for a problem. For world models, the idea of proving (weak) completeness by constructing models whose points are individual formulas has been carried out in [10,22]. The formulas used there as *worlds* are modal analogues of "state descriptions," characterizing a pointed world model up to n-bisimulation [6, §2.3] for a finite n and a finite set of atomic sentences. By contrast, in our §4, *any* formula (or rather equivalence class thereof) will count as a *possibility*.

[2] If for all formulas φ and ψ, $\vdash_{\mathbf{L}} \varphi \to \Box_1\psi$ iff $\vdash_{\mathbf{L}} \Diamond_2\varphi \to \psi$, then $\Box_1$ and $\Diamond_2$ form an adjoint pair of modal operators (also called a *residuated* pair as in [8, §12.2]). An example is the future box operator G and past diamond operator P of temporal logic. Exploiting such adjointness (or residuation) is the basis of *modal display calculi* (see, e.g., [24]).

[3] After writing a draft of this paper, I learned from Nick Bezhanishvili that Ghilardi [13, Theorem 6.3] proved a similar result for the modal logic $\mathbf{K}$ in an algebraic setting, showing that the finitely generated free algebra of $\mathbf{K}$ is a so-called *tense algebra*, which corresponds to $\mathbf{K}$ having *internal adjointness* as in Definition 3.1 below. Also see [5, Theorem 6.7].

[4] This is the lattice structure of the *Lindenbaum algebra* for $\mathbf{L}$ (see [8]).

modal operator indices, the language $\mathcal{L}$ is defined by

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_a\varphi,$$

where $p \in \mathsf{At}$ and $a \in I$. We define $(\varphi \vee \psi) := \neg(\neg\varphi \wedge \neg\psi)$, $(\varphi \to \psi) := \neg(\varphi \wedge \neg\psi)$, $\Diamond_a\varphi := \neg\Box_a\neg\varphi$, and $\bot := (p \wedge \neg p)$ for some $p \in \mathsf{At}$.

To fix intuitions, it helps to have a specific interpretation of the modal operators in mind. We will adopt a doxastic or epistemic interpretation, according to which $\Box_a$ is the belief or knowledge operator for agent a. This interpretation will also help in thinking about the semantics, but it should be stressed that the approach to follow can be applied to modal logic in general.

By *relational world models*, I mean standard relational structures $\mathfrak{M} = \langle \mathrm{W}, \{\mathrm{R}_a\}_{a\in I}, \mathrm{V}\rangle$ used to interpret $\mathcal{L}$ in the usual way [6]. By *relational possibility models*, I mean Humberstone's [17, §3] models, which we do not have room to review here. Modifying his models, we obtain the following (see [16]).

Definition 2.1 A *functional possibility model* for $\mathcal{L}$ is a tuple $\mathcal{M}$ of the form $\langle W, \geqslant, \{f_a\}_{a\in I}, V\rangle$ where:

1. W is a nonempty set with a distinguished element $\bot_{\mathcal{M}}$;

   Notation: we will use upper-case italic letters for elements of W and upper-case **bold** italic letters for elements of $W - \{\bot_{\mathcal{M}}\}$;

2. $f_a \colon W \to W$;
3. V is a partial function from $\mathsf{At} \times W$ to $\{0,1\}$; [5]
4. $\geqslant$ is a weak partial order on W satisfying the following conditions: [6]
   (a) *persistence* – if $V(p,\boldsymbol{X})\downarrow$ and $\boldsymbol{X'} \geqslant \boldsymbol{X}$, then $V(p,\boldsymbol{X'}) = V(p,\boldsymbol{X})$;
   (b) *refinability* – if $V(p,\boldsymbol{X})\uparrow$, then $\exists \boldsymbol{Y},\boldsymbol{Z} \geqslant \boldsymbol{X}$: $V(p,\boldsymbol{Y}) = 0$, $V(p,\boldsymbol{Z}) = 1$;
   (c) *f-persistence* (*monotonicity*) – if $\boldsymbol{X'} \geqslant \boldsymbol{X}$, then $f_a(\boldsymbol{X'}) \geqslant f_a(\boldsymbol{X})$;
   (d) *f-refinability* – if $\boldsymbol{Y} \geqslant f_a(\boldsymbol{X})$, then $\exists \boldsymbol{X'} \geqslant \boldsymbol{X}$ such that $\forall \boldsymbol{X''} \geqslant \boldsymbol{X'}$: $\boldsymbol{Y}$ and $f_a(\boldsymbol{X''})$ are *compatible*,

   where possibilities $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are compatible iff $\exists \boldsymbol{U}$: $\boldsymbol{U} \geqslant \boldsymbol{Y}$ and $\boldsymbol{U} \geqslant \boldsymbol{Z}$. [7]

These models are defined in the same way as Humberstone's, except where f_a and $\bot_{\mathcal{M}}$ appear. W is the set of *possibilities*, and $\bot_{\mathcal{M}}$ is the totally incoherent "possibility." [8] (Often I will write '$\bot$' instead of '$\bot_{\mathcal{M}}$'.) Unlike worlds, possibilities can be indeterminate in certain respects, so V is a partial function. If $V(p,X)$ is undefined, then possibility X does not determine the truth

[5] As usual, every (total) function is a partial function. To indicate that $V(p,X)$ is defined, I write '$V(p,X)\downarrow$', and to indicate that $V(p,X)$ is undefined, I write '$V(p,X)\uparrow$'.

[6] Another natural condition, though not needed: *unrefinability* – if $Y \geqslant \bot_{\mathcal{M}}$, then $Y = \bot_{\mathcal{M}}$.

[7] So *f-refinability* says: if $\boldsymbol{Y} \geqslant f_a(\boldsymbol{X})$, then $\exists \boldsymbol{X'} \geqslant \boldsymbol{X}\ \forall \boldsymbol{X''} \geqslant \boldsymbol{X'}\ \exists \boldsymbol{Y'} \geqslant \boldsymbol{Y}$: $\boldsymbol{Y'} \geqslant f_a(\boldsymbol{X''})$.

[8] We include $\bot_{\mathcal{M}}$ in order to give semantics for logics that do not extend **KD**. Alternatively, we could drop $\bot_{\mathcal{M}}$ and allow the functions f_a to be partial, so instead of having $f_a(X) = \bot_{\mathcal{M}}$, we would have $f_a(X)\uparrow$. (Then we would modify Definition 2.2 to say that $\mathcal{M},\boldsymbol{X} \Vdash \Box_a\varphi$ iff $f_a(\boldsymbol{X})\uparrow$ or $\mathcal{M}, f_a(\boldsymbol{X}) \Vdash \varphi$.) However, the approach with $\bot_{\mathcal{M}}$ seems to be more convenient.

or falsity of p. For each agent a, the doxastic/epistemic *function* f_a in functional possibility models replaces the doxastic/epistemic accessibility *relation* R_a from relational world models. At any possibility X, $f_a(X)$ represents *the world as agent a believes/knows it to be.* Inspired by Humberstone [17, p. 334], we call $f_a(X)$ agent a's *belief-possibility* at X. As officially stated in Definition 2.2 below, agent a believes/knows φ at X iff φ is true at $f_a(X)$.

All that remains to explain about the models is the *refinement* relation $\geqslant$. Intuitively, $Y \geqslant X$ means that Y is a refinement of X, in the sense that Y makes determinate whatever X makes determinate, and maybe more. (If $Y \geqslant X$ but $X \ngeqslant Y$, then Y is a *proper* refinement of X, written '$Y > X$'.) This explains Humberstone's *persistence* condition, familiar from Kripke semantics for intuitionistic logic [21]. The second condition, *refinability*, says that if a possibility $\boldsymbol{X}$ leaves the truth value of p indeterminate, then some coherent refinement of $\boldsymbol{X}$ decides p negatively and some coherent refinement of $\boldsymbol{X}$ decides p affirmatively. Intuitively, if there is no possible refinement $\boldsymbol{Y}$ of $\boldsymbol{X}$ with $V(p,\boldsymbol{Y}) = 1$ (resp. $V(p,\boldsymbol{Y}) = 0$), then $\boldsymbol{X}$ already determines that p is false (resp. true), so we should already have $V(p,\boldsymbol{X}) = 0$ (resp. $V(p,\boldsymbol{X}) = 1$).

Next are the conditions relating $\geqslant$ to f_a, which simply extend persistence and refinability from atomic to modal facts. First, just as *persistence* ensures that as we go from a possibility $\boldsymbol{X}$ to one of its refinements $\boldsymbol{X'}$, $\boldsymbol{X'}$ determines all of the atomic facts that $\boldsymbol{X}$ did, *f-persistence* ensures that $\boldsymbol{X'}$ determines all of the modal facts that $\boldsymbol{X}$ did, which is just to say that $f_a(\boldsymbol{X'})$ is a refinement of $f_a(\boldsymbol{X})$ for all $a \in I$. Second, just as *refinability* ensures that when $\boldsymbol{X}$ leaves an atomic formula p indeterminate, there are refinements of $\boldsymbol{X}$ that decide p each way, *f-refinability* ensures that when $\boldsymbol{X}$ leaves a modal formula $\Box_a\varphi$ indeterminate, there are refinements of $\boldsymbol{X}$ that decide $\Box_a\varphi$ each way. In fact, just the truth clause for $\neg$ in Definition 2.2 below ensures that if $\mathcal{M},\boldsymbol{X} \nVdash \neg\Box_a\varphi$, then there is a refinement of $\boldsymbol{X}$ that makes $\Box_a\varphi$ true. What *f-refinability* adds is that if $\mathcal{M},\boldsymbol{X} \nVdash \Box_a\varphi$, then there is a refinement of $\boldsymbol{X}$ that makes $\neg\Box_a\varphi$ true. Although it may not be initially obvious that this is the content of *f-refinability*, the proof of Lemma 2.3.2 together with Fig. 1 should make it clear. [9]

We now define truth for formulas of $\mathcal{L}$ in functional possibility models, following Humberstone's clauses for p, $\neg$, and $\wedge$, but changing the clause for $\Box_a$ to use f_a. The idea of using such a function instead of an accessibility relation to give the semantic clause for a modal operator appears in Fine's [9, p. 359] study of relevance logic (also see [18, p. 418], [19, p. 899], and cf. [4]).

Definition 2.2 Given a functional possibility model $\mathcal{M} = \langle W, \geqslant, \{f_a\}_{a\in I}, V\rangle$ with $X \in W$ and $\varphi \in \mathcal{L}$, define $\mathcal{M}, X \Vdash \varphi$ ("φ is true at X in $\mathcal{M}$") as follows:

1. $\mathcal{M}, \bot \Vdash \varphi$ for all φ;
2. $\mathcal{M}, \boldsymbol{X} \Vdash p$ iff $V(p,\boldsymbol{X}) = 1$;

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[9] It is noteworthy that the *f-refinability* assumption is considerably weaker than the functional analogue of Humberstone's [17, 324] relational refinability assumption (**R**), explained at the end of this section. We discuss different strengths of modal refinability in [16].

3. $\mathcal{M}, \boldsymbol{X} \Vdash \neg\varphi$ iff $\forall \boldsymbol{Y} \geqslant \boldsymbol{X}$: $\mathcal{M}, \boldsymbol{Y} \nVdash \varphi$;
4. $\mathcal{M}, \boldsymbol{X} \Vdash (\varphi \wedge \psi)$ iff $\mathcal{M}, \boldsymbol{X} \Vdash \varphi$ and $\mathcal{M}, \boldsymbol{X} \Vdash \psi$;
5. $\mathcal{M}, \boldsymbol{X} \Vdash \Box_a \varphi$ iff $\mathcal{M}, f_a(\boldsymbol{X}) \Vdash \varphi$.

Given $(\varphi \vee \psi) := \neg(\neg\varphi \wedge \neg\psi)$, $(\varphi \to \psi) := \neg(\varphi \wedge \neg\psi)$, and $\Diamond_a \varphi := \neg\Box_a\neg\varphi$, one finds that the truth clauses for $\vee$, $\to$, and $\Diamond_a$ are equivalent to:

1. $\mathcal{M}, \boldsymbol{X} \Vdash (\varphi \vee \psi)$ iff $\forall \boldsymbol{Y} \geqslant \boldsymbol{X}\ \exists \boldsymbol{Z} \geqslant \boldsymbol{Y}$: $\mathcal{M}, \boldsymbol{Z} \Vdash \varphi$ or $\mathcal{M}, \boldsymbol{Z} \Vdash \psi$;
2. $\mathcal{M}, \boldsymbol{X} \Vdash (\varphi \to \psi)$ iff $\forall \boldsymbol{Y} \geqslant \boldsymbol{X}$ with $\mathcal{M}, \boldsymbol{Y} \Vdash \varphi$, $\exists \boldsymbol{Z} \geqslant \boldsymbol{Y}$: $\mathcal{M}, \boldsymbol{Z} \Vdash \psi$;
3. $\mathcal{M}, \boldsymbol{X} \Vdash \Diamond_a \varphi$ iff $\forall \boldsymbol{X'} \geqslant \boldsymbol{X}\ \exists \boldsymbol{Y} \geqslant f_a(\boldsymbol{X'})$: $\mathcal{M}, \boldsymbol{Y} \Vdash \varphi$.

The truth clause for $\vee$ crucially allows a possibility to determine that a disjunction is true without determining which disjunct is true; the clause for $\to$ can be further simplified, as in Lemma 2.3.3 below; and although the clause for $\Diamond_a$ appears unfamiliar, it is quite intuitive—a possibility $\boldsymbol{X}$ determines that φ is compatible with agent a's beliefs iff for any refinement $\boldsymbol{X'}$ of $\boldsymbol{X}$, a's belief-possibility at $\boldsymbol{X'}$ can be refined to a possibility where φ is true.

To get a feel for the semantics, it helps to consider simple models for concrete epistemic examples (see [16]), but we do not have room to do so here. We proceed to general properties of the semantics such as the following from [17].

Lemma 2.3 For any model $\mathcal{M}$, possibilities $\boldsymbol{X}$, $\boldsymbol{Y}$, and formulas φ, ψ:

1. Persistence: if $\mathcal{M}, \boldsymbol{X} \Vdash \varphi$ and $\boldsymbol{Y} \geqslant \boldsymbol{X}$, then $\mathcal{M}, \boldsymbol{Y} \Vdash \varphi$;
2. Refinability: if $\mathcal{M}, \boldsymbol{X} \nVdash \varphi$, then $\exists \boldsymbol{Z} \geqslant \boldsymbol{X}$: $\mathcal{M}, \boldsymbol{Z} \Vdash \neg\varphi$;
3. Implication: $\mathcal{M}, \boldsymbol{X} \Vdash \varphi \to \psi$ iff $\forall \boldsymbol{Z} \geqslant \boldsymbol{X}$: if $\mathcal{M}, \boldsymbol{Z} \Vdash \varphi$, then $\mathcal{M}, \boldsymbol{Z} \Vdash \psi$.

Proof. We treat only the $\Box_a$ case of an inductive proof of part 2 to illustrate *f-refinability* with Fig. 1. If $\mathcal{M}, \boldsymbol{X} \nVdash \Box_a\varphi$, then $\mathcal{M}, f_a(\boldsymbol{X}) \nVdash \varphi$, so by the inductive hypothesis there is some $\boldsymbol{Y} \geqslant f_a(\boldsymbol{X})$ such that $\mathcal{M}, \boldsymbol{Y} \Vdash \neg\varphi$. Now *f-refinability* implies that there is some $\boldsymbol{X'} \geqslant \boldsymbol{X}$ such that for all $\boldsymbol{X''} \geqslant \boldsymbol{X'}$, $\boldsymbol{Y}$ is compatible with $f_a(\boldsymbol{X''})$, which means there is a $\boldsymbol{Y'} \geqslant \boldsymbol{Y}$ with $\boldsymbol{Y'} \geqslant f_a(\boldsymbol{X''})$, which with $\mathcal{M}, \boldsymbol{Y} \Vdash \neg\varphi$ and part 1 implies $\mathcal{M}, f_a(\boldsymbol{X''}) \nVdash \varphi$ and hence $\mathcal{M}, \boldsymbol{X''} \nVdash \Box_a\varphi$. Since this holds for all $\boldsymbol{X''} \geqslant \boldsymbol{X'}$, we have $\mathcal{M}, \boldsymbol{X'} \Vdash \neg\Box_a\varphi$. $\Box$

$$
\begin{tikzcd}
 & & \boldsymbol{Y'} \nVdash \varphi \\
\boldsymbol{X''} \arrow[r, dashed, mapsto] & f_a(\boldsymbol{X''}) \nVdash \varphi \arrow[ur] & \\
\boldsymbol{X'} \Vdash \neg\Box_a\varphi \arrow[u, "\forall"] & & \boldsymbol{Y} \Vdash \neg\varphi \arrow[uu, "\exists"'] \\
\boldsymbol{X} \arrow[r, dashed, mapsto] \arrow[u, "\exists"] & f_a(\boldsymbol{X}) \nVdash \varphi \arrow[ur] &
\end{tikzcd}
$$

Fig. 1. *f-refinability* as used in the proof of Lemma 2.3.2, assuming $\mathcal{M}, \boldsymbol{X} \nVdash \Box_a\varphi$. Solid arrows are for the refinement relation $\geqslant$ and dashed are for the function f_a.

The definition of consequence over possibility models is as for world models.

Definition 2.4 Given a class S of possibility models, $\Sigma \subseteq \mathcal{L}$, and $\varphi \in \mathcal{L}$: $\Sigma \Vdash_{\mathsf{S}} \varphi$ ("φ is a consequence of Σ over S") iff for all $\mathcal{M} \in \mathsf{S}$ and X in $\mathcal{M}$, if $\mathcal{M}, X \Vdash \sigma$ for all $\sigma \in \Sigma$, then $\mathcal{M}, X \Vdash \varphi$; $\Vdash_{\mathsf{S}} \varphi$ ("φ is valid over S") iff $\emptyset \Vdash_{\mathsf{S}} \varphi$; and φ is *satisfiable* in S iff $\nVdash_{\mathsf{S}} \neg\varphi$ (iff there are $\mathcal{M} \in \mathsf{S}$ and $\boldsymbol{X}$ with $\mathcal{M}, \boldsymbol{X} \Vdash \varphi$).

One of Humberstone's insights was that by giving negation the intuitionistic-style clause in Definition 2.2.3, while at the same time defining $\vee$ and $\to$ in terms of $\neg$ and $\wedge$, contrary to intuitionistic logic, we obtain a classical logic from models based on possibilities. Van Benthem [1] also observed that by starting with Kripke models for intuitionistic logic and imposing a *cofinality* condition on the ordering (i.e., if $\forall \boldsymbol{Y} \geqslant \boldsymbol{X}\ \exists \boldsymbol{Z} \geqslant \boldsymbol{Y}$ such that $V(p, \boldsymbol{Z}) = 1$, then $V(p, \boldsymbol{X}) = 1$), one obtains a possibility semantics for (non-modal) classical logic by retaining the intuitionistic semantic clauses for $\neg$, $\to$, and $\wedge$, and defining $(\varphi \vee \psi)$ as $\neg(\neg\varphi \wedge \neg\psi)$. (In fact, van Benthem showed this for first-order logic.) For more on comparisons between classical and intuitionistic logic using possibility models, see [2], [3, Chs. 7-8], [7], [23], and [12, Ch. 8].

Lemma 2.5 If φ is a substitution instance of a classical propositional tautology, then φ is valid over functional possibility models.

Proof. Suppose φ is an instance of a propositional formula δ, where δ contains only the atomic sentences $q_1, \dots, q_n$. Let $\mathcal{L}_{\mathsf{PL}}(q_1, \dots, q_n)$ be the propositional language generated from $q_1, \dots, q_n$. Since φ is an instance of δ, there is some $s\colon \{q_1, \dots, q_n\} \to \mathcal{L}$ such that $\varphi = \hat{s}(\delta)$, where $\hat{s}\colon \mathcal{L}_{\mathsf{PL}}(q_1, \dots, q_n) \to \mathcal{L}$ is the usual extension of s such that $\hat{s}(q_i) = s(q_i)$, $\hat{s}(\neg\alpha) = \neg\hat{s}(\alpha)$, and $\hat{s}((\alpha \wedge \beta)) = (\hat{s}(\alpha) \wedge \hat{s}(\beta))$. Now suppose that φ is not valid, so there is some possibility model $\mathcal{M}$ and $\boldsymbol{X}$ in $\mathcal{M}$ such that $\mathcal{M}, \boldsymbol{X} \nVdash \varphi$, which by Lemma 2.3.2 implies there is a $\boldsymbol{X'} \geqslant \boldsymbol{X}$ such that $\mathcal{M}, \boldsymbol{X'} \Vdash \neg\varphi$. Also by Lemma 2.3.2, for any $\boldsymbol{Y} \in W$ and $\psi \in \mathcal{L}$, we can choose a $\boldsymbol{Y}^{\psi} \geqslant \boldsymbol{Y}$ with $\mathcal{M}, \boldsymbol{Y}^{\psi} \Vdash \psi$ or $\mathcal{M}, \boldsymbol{Y}^{\psi} \Vdash \neg\psi$. Enumerating the formulas of $\mathcal{L}$ as $\psi_1, \psi_2, \dots$, define a sequence $\boldsymbol{X}_0, \boldsymbol{X}_1, \boldsymbol{X}_2, \dots$ such that $\boldsymbol{X}_0 = \boldsymbol{X'}$ and $\boldsymbol{X}_{n+1} = \boldsymbol{X}_n^{\psi_{n+1}}$. Thus, $\boldsymbol{X}_0 \leqslant \boldsymbol{X}_1 \leqslant \boldsymbol{X}_2 \dots$ is a "generic" chain that decides every formula eventually. Define a propositional valuation $v\colon \{q_1, \dots, q_n\} \to \{0, 1\}$ such that $v(q_i) = 1$ if for some $k \in \mathbb{N}$, $\mathcal{M}, \boldsymbol{X}_k \Vdash s(q_i)$, and $v(q_i) = 0$ otherwise. Where $\overline{v}\colon \mathcal{L}_{\mathsf{PL}}(q_1, \dots, q_n) \to \{0, 1\}$ is the usual classical extension of v, one can prove that for all $\alpha \in \mathcal{L}_{\mathsf{PL}}(q_1, \dots, q_n)$,

$$\overline{v}(\alpha) = 1 \text{ iff } \exists k \in \mathbb{N}\colon \mathcal{M}, \boldsymbol{X}_k \Vdash \hat{s}(\alpha), \tag{1}$$

by induction on α. From above, $\mathcal{M}, \boldsymbol{X}_0 \Vdash \neg\hat{s}(\delta)$, i.e., $\mathcal{M}, \boldsymbol{X}_0 \Vdash \hat{s}(\neg\delta)$, and $\neg\delta \in \mathcal{L}_{\mathsf{PL}}(q_1, \dots, q_n)$, so (1) implies $\overline{v}(\neg\delta) = 1$. Thus, δ is not a tautology. $\Box$

Not only is classical propositional logic sound over functional possibility models, but also standard normal modal logics are sound and complete over functional possibility models with constraints on f_a and $\geqslant$ corresponding to the logic's additional axioms. Throughout we adopt the standard nomenclature for normal modal logics, borrowing the names of monomodal logics for their polymodal (fusion) versions. Thus, each $\Box_a$ operator has the same axioms.

The following result raises obvious questions about general correspondence theory for possibility semantics, but we do not have room to discuss them here.

Theorem 2.6 (Soundness and Completeness) For any subset of the axioms $\{\mathrm{D}, \mathrm{T}, 4, \mathrm{B}, 5\}$, [10] the extension of the minimal normal modal logic $\mathbf{K}$ with that set of axioms is sound and strongly complete for the class of functional possibility models satisfying the corresponding constraint for each axiom:

1. D axiom: for all $\boldsymbol{X}$, $f_a(\boldsymbol{X}) \neq \bot$:
2. T axiom: for all $\boldsymbol{X}$, $\boldsymbol{X} \geqslant f_a(\boldsymbol{X})$;
3. 4 axiom: for all $\boldsymbol{X}$, $f_a(f_a(\boldsymbol{X})) \geqslant f_a(\boldsymbol{X})$;
4. B axiom: for all $\boldsymbol{X}, \boldsymbol{Y}$, if $\boldsymbol{Y} \geqslant f_a(\boldsymbol{X})$ then $\exists \boldsymbol{X'} \geqslant \boldsymbol{X}$: $\boldsymbol{X'} \geqslant f_a(\boldsymbol{Y})$;
5. 5 axiom: for all $\boldsymbol{X}, \boldsymbol{Y}$, if $\boldsymbol{Y} \geqslant f_a(\boldsymbol{X})$, then $\exists \boldsymbol{X'} \geqslant \boldsymbol{X}$: $f_a(\boldsymbol{X'}) \geqslant f_a(\boldsymbol{Y})$.

The proof of soundness is straightforward. First, by Lemma 2.5, all tautologies are valid. Second, by Lemma 2.3.3, if φ and $\varphi \to \psi$ are valid, then ψ is valid, so modus ponens is sound; and obviously if φ is valid, then $\Box_a\varphi$ is valid, so the necessitation rule is sound. Next, we check that the K axiom is valid:

> Suppose for reductio that $\mathcal{M}, \boldsymbol{X} \nVdash \Box_a(\varphi \to \psi) \to (\Box_a\varphi \to \Box_a\psi)$, so by Lemma 2.3.3, there is some $\boldsymbol{Y} \geqslant \boldsymbol{X}$ such that $\mathcal{M}, \boldsymbol{Y} \Vdash \Box_a(\varphi \to \psi)$ but $\mathcal{M}, \boldsymbol{Y} \nVdash \Box_a\varphi \to \Box_a\psi$, so by Lemma 2.3.3 again there is some $\boldsymbol{Z} \geqslant \boldsymbol{Y}$ with $\mathcal{M}, \boldsymbol{Z} \Vdash \Box_a\varphi$ but $\mathcal{M}, \boldsymbol{Z} \nVdash \Box_a\psi$, so $\mathcal{M}, f_a(\boldsymbol{Z}) \Vdash \varphi$ but $\mathcal{M}, f_a(\boldsymbol{Z}) \nVdash \psi$. By Lemma 2.3.1, $\mathcal{M}, \boldsymbol{Y} \Vdash \Box_a(\varphi \to \psi)$ and $\boldsymbol{Z} \geqslant \boldsymbol{Y}$ together imply $\mathcal{M}, \boldsymbol{Z} \Vdash \Box_a(\varphi \to \psi)$, so $\mathcal{M}, f_a(\boldsymbol{Z}) \Vdash \varphi \to \psi$. But by Lemma 2.3.3 and the reflexivity of $\geqslant$, we cannot have all of $\mathcal{M}, f_a(\boldsymbol{Z}) \Vdash \varphi \to \psi$, $\mathcal{M}, f_a(\boldsymbol{Z}) \Vdash \varphi$, and $\mathcal{M}, f_a(\boldsymbol{Z}) \nVdash \psi$. Thus, $\mathcal{M}, \boldsymbol{X} \Vdash \Box_a(\varphi \to \psi) \to (\Box_a\varphi \to \Box_a\psi)$.

Using Lemma 2.3.3, it is also easy to check the validity of D, T, 4, B, and 5 over the classes of models with the corresponding constraints.

Completeness can be proved by taking advantage of completeness with respect to relational world models and then showing how to transform any relational world model obeying constraints on R_a corresponding to the axioms into a functional possibility model satisfying the same formulas and obeying constraints of f_a and $\geqslant$ corresponding to the axioms (see [16]). Or completeness can be proved directly with a canonical model construction where the domain is the set of all (equivalence classes of) *sets of formulas* of $\mathcal{L}$ (see [16]).

Here we will prove weak completeness for a selection of the logics covered by Theorem 2.6 using a canonical model construction where the domain is simply the set of all (equivalence classes of) *formulas* of $\mathcal{L}$. In this way, we will prove weak completeness for classes of models obeying the following constraint.

Definition 2.7 A functional possibility model $\mathcal{M}$ is *locally finite* iff for all $\boldsymbol{X} \in W$, the set $\{p \in \mathsf{At} \mid V(p, \boldsymbol{X})\downarrow\}$ is finite.

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[10] As usual, D is $\Box_a\varphi \to \neg\Box_a\neg\varphi$, T is $\Box_a\varphi \to \varphi$, 4 is $\Box_a\varphi \to \Box_a\Box_a\varphi$, B is $\neg\varphi \to \Box_a\neg\Box_a\varphi$ $(\psi \to \Box_a\Diamond_a\psi)$, and 5 is $\neg\Box_a\varphi \to \Box_a\neg\Box_a\varphi$ $(\Diamond_a\psi \to \Box_a\Diamond_a\psi)$.

If At is infinite, then every locally finite model contains infinitely many finite possibilities by *refinability.* Hence the term '*locally* finite', which leads to a distinction. All of the logics considered here have the "finite model property" with respect to possibility semantics (see [16]): any consistent formula is satisfied in a model where W is a finite set. But the elements of such a W are *infinite* possibilities, i.e., each deciding infinitely many atomic sentences. With Definition 2.7, we move from a finite set of infinite possibilities to an infinite set of finite possibilities, a move that has certain philosophical attractions (see [17,16]) and mathematical interest. For the latter, if we wish to build a model in which every consistent formula is satisfied, this inevitably requires an infinite W for the logics with no bound on modal depth. Yet, in a finitary spirit, we may at least aspire to construct such a model to be locally finite, as in §4.

In §3-4, we will work up to Theorem 2.8 below. In [16], we also prove the completeness of **K45** and **KD45** with respect to locally finite models satisfying the appropriate constraints, but space does not permit the proof here.

Theorem 2.8 (Completeness for Locally Finite Models)
Let $\mathbf{L}$ be one of the logics **K**, **KD**, **T**, **K4**, **KD4**, **S4**, or **S5**, and let $\mathsf{S}_{\mathbf{L}}^{LF}$ be the class of *locally finite* functional possibility models satisfying the constraints on f_a and $\geqslant$ corresponding to the axioms of $\mathbf{L}$, as listed in Theorem 2.6. Then $\mathbf{L}$ is weakly complete with respect to $\mathsf{S}_{\mathbf{L}}^{LF}$: for all $\varphi \in \mathcal{L}$, if $\Vdash_{\mathsf{S}_{\mathbf{L}}^{LF}} \varphi$, then $\vdash_{\mathbf{L}} \varphi$.

Humberstone [17, p. 326] also states that one can prove the completeness of some modal logics with respect to classes of his *relational* possibility models, using a canonical model construction in which each possibility is the set of syntactic consequences of a consistent finite set of formulas, but he does not write out a proof. Relational possibility models have relations R_a, instead of functions f_a, so that $\mathcal{M}, \boldsymbol{X} \Vdash \Box_a\varphi$ iff $\mathcal{M}, \boldsymbol{Y} \Vdash \varphi$ for all $\boldsymbol{Y}$ with $\boldsymbol{X} R_a \boldsymbol{Y}$. Here it is relevant to consider Humberstone's [17, p. 324-5] refinability condition (**R**). According to (**R**), if $\boldsymbol{X} R_a \boldsymbol{Y}$, then $\exists \boldsymbol{X'} \geqslant \boldsymbol{X}\ \forall \boldsymbol{X''} \geqslant \boldsymbol{X'}$: $\boldsymbol{X''} R_a \boldsymbol{Y}$. This is very strong. Given $\boldsymbol{X''} R_a \boldsymbol{Y}$, it must be that for every formula φ that is not true at $\boldsymbol{Y}$, $\Box_a\varphi$ is not true at $\boldsymbol{X''}$. Then since this holds for all $\boldsymbol{X''} \geqslant \boldsymbol{X'}$, $\neg\Box_a\varphi$ must be true at $\boldsymbol{X'}$. Thus, if $\boldsymbol{Y}$ makes only finitely many atomic sentences p true, then $\boldsymbol{X'}$ must make infinitely many formulas $\neg\Box_a p$ true. But then $\boldsymbol{X'}$ cannot be the set of consequences of a consistent finite set of formulas, because no consistent finite set entails infinitely many formulas of the form $\neg\Box_a p$.

3 Internal Adjointness

Our goal is to construct a canonical model for $\mathbf{L}$ in which each possibility is the equivalence class of a single formula φ, such that $\mathcal{M}, [\varphi] \Vdash \psi$ iff $\vdash_{\mathbf{L}} \varphi \to \psi$. To do so, we need to define functions f_a such that $\mathcal{M}, [\varphi] \Vdash \Box_a\psi$ iff $\mathcal{M}, f_a([\varphi]) \Vdash \psi$, which means we need functions f_a such that $\vdash_{\mathbf{L}} \varphi \to \Box_a\psi$ iff $\vdash_{\mathbf{L}} \mathsf{f}_a(\varphi) \to \psi$. Intuitively, for a finite "possibility" φ, we want a finite "belief-possibility" $\mathsf{f}_a(\varphi)$ such that whatever is *believed* according to φ is *true* according to $\mathsf{f}_a(\varphi)$. It is an independently natural question whether such functions f_a exist for $\mathbf{L}$.

Definition 3.1 A modal logic $\mathbf{L}$ has *internal adjointness* iff for all $\varphi \in \mathcal{L}$ and

$a \in I$, there is a $\mathsf{f}_a^{\mathbf{L}}(\varphi) \in \mathcal{L}$ such that for all $\psi \in \mathcal{L}$:

$$\vdash_{\mathbf{L}} \varphi \to \Box_a \psi \text{ iff } \vdash_{\mathbf{L}} \mathsf{f}_a^{\mathbf{L}}(\varphi) \to \psi.$$

Not every modal logic has internal adjointness. For example:

Proposition 3.2 **K5**, **K45**, **KD5**, and **KD45** lack internal adjointness.

Proof. Let $\mathbf{L}$ be any of the logics listed. Suppose there is a formula $\mathsf{f}_a^{\mathbf{L}}(\top)$ such that for all formulas ψ, $\vdash_{\mathbf{L}} \top \to \Box_a \psi$ iff $\vdash_{\mathbf{L}} \mathsf{f}_a^{\mathbf{L}}(\top) \to \psi$. Then since $\nvdash_{\mathbf{L}} \top \to \Box_a \bot$, we have $\nvdash_{\mathbf{L}} \mathsf{f}_a^{\mathbf{L}}(\top) \to \bot$, which we will show below to imply $\nvdash_{\mathbf{L}} \mathsf{f}_a^{\mathbf{L}}(\top) \to (\Box_a p \to p)$ for an atomic p that does not occur in $\mathsf{f}_a^{\mathbf{L}}(\top)$. But $\vdash_{\mathbf{L}} \Box_a(\Box_a p \to p)$ and hence $\vdash_{\mathbf{L}} \top \to \Box_a(\Box_a p \to p)$, so taking $\psi := \Box_a p \to p$ refutes the supposition. Thus, $\mathbf{L}$ lacks internal adjointness.

Given $\nvdash_{\mathbf{L}} \mathsf{f}_a^{\mathbf{L}}(\top) \to \bot$, it follows by the completeness of $\mathbf{L}$ with respect to the class $\mathsf{C}_{\mathbf{L}}$ of Euclidean/transitive/serial relational world models that there is a world model $\mathfrak{M} \in \mathsf{C}_{\mathbf{L}}$ such that $\mathfrak{M}, x \vDash \mathsf{f}_a^{\mathbf{L}}(\top)$. Define a new model $\mathfrak{M}'$ to be like $\mathfrak{M}$ except that (a) there is a new world x' that can "see" via each R_b for $b \in I$ all and only the worlds that x can see via R_b and (b) p is true everywhere in $\mathfrak{M}'$ except at x', which otherwise agrees with x on atomic sentences. Since p does not occur in $\mathsf{f}_a^{\mathbf{L}}(\top)$, and $\mathfrak{M}, x$ and $\mathfrak{M}', x'$ are bisimilar with respect to the language without p, from $\mathfrak{M}, x \vDash \mathsf{f}_a^{\mathbf{L}}(\top)$ we have $\mathfrak{M}', x' \vDash \mathsf{f}_a^{\mathbf{L}}(\top)$, and by construction we have $\mathfrak{M}', x' \vDash \Box_a p \wedge \neg p$. Also by construction, $\mathfrak{M}' \in \mathsf{C}_{\mathbf{L}}$ (for if $\mathrm{R}_b^{\mathfrak{M}}$ is Euclidean/transitive/serial, then so is $\mathrm{R}_b^{\mathfrak{M}'}$), so by the soundness of $\mathbf{L}$ with respect to $\mathsf{C}_{\mathbf{L}}$, we have $\nvdash_{\mathbf{L}} \mathsf{f}_a^{\mathbf{L}}(\top) \to (\Box_a p \to p)$, as claimed above. □

The problem is that with the logics in Proposition 3.2, the shift-reflexivity axiom $\Box_a(\Box_a p \to p)$ is derivable for all p, but a consistent formula entails $\Box_a p \to p$ only if it contains p, and no formula contains infinitely many p. If we wish to overcome Proposition 3.2, we must extend our language and logics. [11]

Yet Theorem 3.9 will show that for a number of logics $\mathbf{L}$, we already have the ability to find an appropriate $\mathsf{f}_a^{\mathbf{L}}(\varphi)$ in our original language $\mathcal{L}$. [12] In order to define $\mathsf{f}_a^{\mathbf{L}}(\varphi)$, we first need the following standard definition and result.

[11] For example, consider an expanded language that includes all the formulas of $\mathcal{L}$ plus a new formula $\mathbf{loop}_a$ for each $a \in I$, such that in relational world models, $\mathfrak{M}, w \vDash \mathbf{loop}_a$ iff $w\mathrm{R}_a w$ (as in [11, §3]), and in functional possibility models, $\mathcal{M}, \boldsymbol{X} \Vdash \mathbf{loop}_a$ iff $\boldsymbol{X} \geqslant f_a(\boldsymbol{X})$. Intuitively, $\mathbf{loop}_a$ says that agent a's beliefs are compatible with the facts. The key axiom schema for $\mathbf{loop}_a$ is $\mathbf{loop}_a \to (\Box_a \varphi \to \varphi)$, and the shift-reflexivity axiom $\Box_a(\Box_a \varphi \to \varphi)$ can be captured by $\Box_a \mathbf{loop}_a$, which says that the agent believes that her beliefs are compatible with the facts. As shown in [16], with logics **K45loop** and **KD45loop**, the problem of Proposition 3.2 does not arise. Moreover, a detour through these logics for the expanded language allows one to prove the completeness with respect to locally finite models of **K45** and **KD45** for $\mathcal{L}$, despite Proposition 3.2 [16]. Another approach to overcoming Proposition 3.2, which has greater generality but less doxastic/epistemic motivation than the approach with $\mathbf{loop}_a$, is to add a backward-looking operator $\overleftarrow{\Diamond}_a$ to our language with the truth clause: $\mathcal{M}, \boldsymbol{X} \Vdash \overleftarrow{\Diamond}_a \varphi$ iff for some $\boldsymbol{Y} \in W$, $\boldsymbol{X} \geqslant f_a(\boldsymbol{Y})$ and $\mathcal{M}, \boldsymbol{Y} \Vdash \varphi$. In world models, the clause is: $\mathfrak{M}, w \vDash \overleftarrow{\Diamond}_a \varphi$ iff for some $v \in \mathrm{W}$, $v\mathrm{R}_a w$ and $\mathfrak{M}, v \vDash \varphi$. So $\overleftarrow{\Diamond}_a$ is the existential modality for the converse relation. Then it is easy to see that $\varphi \to \Box_a \psi$ is valid iff $\overleftarrow{\Diamond}_a \varphi \to \psi$ is.

[12] Compare our definition of *internal adjointness* to that of *indigenous inverses* in [20, §6.2].

Definition 3.3 A $\varphi \in \mathcal{L}$ is in *modal disjunctive normal form* (DNF) iff it a disjunction of conjunctions, each conjunct of which is either (a) a propositional formula α (whose form will not matter here), (b) of the form $\Box_a \beta$ for some $a \in I$ and β in DNF, or (c) of the form $\Diamond_a \gamma$ for some $a \in I$ and γ in DNF.

Lemma 3.4 For any normal modal logic $\mathbf{L}$ and $\varphi \in \mathcal{L}$, there is a $\varphi' \in \mathcal{L}$ in DNF such that $\vdash_{\mathbf{L}} \varphi \leftrightarrow \varphi'$.

Another useful definition and result that will help us prove Theorem 3.9 for logics with the T axiom involves the idea of a *T-unpacked* formula from [15].

Definition 3.5 If $\varphi \in \mathcal{L}$ is in DNF, then a disjunct δ_φ of φ is T-unpacked iff for all $a \in I$ and formulas β, if $\Box_a \beta$ is a conjunct of δ_φ, then there is a disjunct δ_β of β such that every conjunct of δ_β is a conjunct of δ_φ.

The formula φ itself is T-unpacked iff every disjunct of φ is T-unpacked.

For example, one can check that $\varphi := \Box_a p \vee \big(p \wedge \Box_a(q \vee \Box_b r) \wedge \Diamond_a s\big)$ is not T-unpacked. By contrast, the following formula, which is equivalent to φ in the logic $\mathbf{T}$, is T-unpacked, as highlighted by the boldface type:

$$\begin{aligned}\varphi^* := {} & \big(\Box_a p \wedge \boldsymbol{p}\big) \vee \big(p \wedge \Box_a(q \vee \Box_b r) \wedge \boldsymbol{q} \wedge \Diamond_a s\big) \vee \\ & \big(p \wedge \Box_a(q \vee \Box_b r) \wedge \Box_b \boldsymbol{r} \wedge \boldsymbol{r} \wedge \Diamond_a s\big).\end{aligned}$$

This kind of transformation between φ and φ^* can be carried out in general.

Lemma 3.6 For every extension $\mathbf{L}$ of $\mathbf{K}$ containing the T axiom and $\varphi \in \mathcal{L}$, there is a T-unpacked DNF $\varphi^* \in \mathcal{L}$ such that $\vdash_{\mathbf{L}} \varphi \leftrightarrow \varphi^*$.

Proof. Transform φ into DNF and then apply to each disjunct the following provable equivalences in $\mathbf{L}$:

$$\begin{aligned}\big(\psi \wedge \cdots \wedge \Box_a(\bigvee_{\delta \in \Delta} \delta) \wedge \cdots \wedge \chi\big) & \Leftrightarrow \big(\psi \wedge \cdots \wedge \Box_a(\bigvee_{\delta \in \Delta} \delta) \wedge (\bigvee_{\delta \in \Delta} \delta) \wedge \cdots \wedge \chi\big) \\ & \Leftrightarrow \bigvee_{\delta' \in \Delta} \big(\psi \wedge \cdots \wedge \Box_a(\bigvee_{\delta \in \Delta} \delta) \wedge \delta' \wedge \cdots \wedge \chi\big),\end{aligned}$$

where the first step uses the T axiom and the second uses propositional logic. Repeated transformations of this kind produce a T-unpacked DNF formula. □

Henceforth, for every logic $\mathbf{L}$ and $\varphi \in \mathcal{L}$, we fix an $\mathbf{L}$-equivalent $NF_{\mathbf{L}}(\varphi)$ in DNF which is T-unpacked if $\mathbf{L}$ contains the T axiom and each disjunct of which is $\mathbf{L}$-consistent if φ is, since we may always drop inconsistent disjuncts.

We can now define the belief-possibility $\mathsf{f}_a^{\mathbf{L}}(\varphi)$ of agent a according to φ and logic $\mathbf{L}$. Since the definition of $\mathsf{f}_a^{\mathbf{L}}(\varphi)$ depends on the specific logic $\mathbf{L}$, for the sake of space I will restrict attention to the standard doxastic and epistemic logics not excluded by Proposition 3.2: $\mathbf{K}$, $\mathbf{KD}$, $\mathbf{T}$, $\mathbf{K4}$, $\mathbf{KD4}$, $\mathbf{S4}$, and $\mathbf{S5}$ (and any other extension of $\mathbf{KB}$). For each $\mathbf{L}$ that does not extend $\mathbf{KB}$, our $\mathsf{f}_a^{\mathbf{L}}$ is a *non-connectival operation* on formulas in the terminology of [19, p. 49]

Definition 3.7 Consider an $\mathbf{L}$-consistent formula $NF_{\mathbf{L}}(\varphi) := \delta_1 \vee \cdots \vee \delta_n$. For $a \in I$ and $\mathbf{L} \in \{\mathbf{K}, \mathbf{KD}, \mathbf{T}\}$, define

$$\mathsf{f}_a^{\mathbf{L}}(\delta_i) := \bigwedge\{\beta \mid \Box_a \beta \text{ a conjunct of } \delta_i\}.$$

For $\mathbf{L} \in \{\mathbf{K4}, \mathbf{KD4}, \mathbf{S4}\}$, define

$$\mathsf{f}^{\mathbf{L}}_a(\delta_i) := \bigwedge\{\beta, \Box_a\beta \mid \Box_a\beta \text{ a conjunct of } \delta_i\}.$$

For all of the above $\mathbf{L}$,[13] define

$$\mathsf{f}^{\mathbf{L}}_a(\delta_1 \vee \cdots \vee \delta_n) := \mathsf{f}^{\mathbf{L}}_a(\delta_1) \vee \cdots \vee \mathsf{f}^{\mathbf{L}}_a(\delta_n).$$

For any $\mathbf{L}$-consistent formula φ not in normal form, let $\mathsf{f}^{\mathbf{L}}_a(\varphi) = \mathsf{f}^{\mathbf{L}}_a(NF_{\mathbf{L}}(\varphi))$.[14]

For any $\mathbf{L}$-inconsistent formula φ, let $\mathsf{f}^{\mathbf{L}}_a(\varphi) = \bot$.

Finally, for any extension of **KB** and any φ, simply let $\mathsf{f}^{\mathbf{L}}_a(\varphi) = \Diamond_a\varphi$.

To see the need for the assumption in Definition 3.7 that $NF_{\mathbf{L}}(\varphi)$ is T-unpacked if $\mathbf{L}$ contains the T axiom, suppose $\mathbf{L}$ is $\mathbf{T}$ and δ is $\neg q \wedge \Box_a(\Box_a p \vee \Box_a q)$, which is not T-unpacked. Then $\mathsf{f}^{\mathbf{L}}_a(\delta)$ would be $\Box_a p \vee \Box_a q$, and we would have $\vdash_{\mathbf{T}} \delta \to \Box_a p$ but $\nvdash_{\mathbf{T}} \mathsf{f}^{\mathbf{L}}_a(\delta) \to p$, contrary to our desired Theorem 3.9. If we T-unpack $\neg q \wedge \Box_a(\Box_a p \vee \Box_a q)$, we first obtain

$$\big(\neg q \wedge \Box_a(\Box_a p \vee \Box_a q) \wedge \Box_a p \wedge p\big) \vee \big(\neg q \wedge \Box_a(\Box_a p \vee \Box_a q) \wedge \Box_a q \wedge q\big),$$

the right disjunct of which is inconsistent, so we drop it to obtain the T-unpacked $\delta' := \neg q \wedge \Box_a(\Box_a p \vee \Box_a q) \wedge \Box_a p \wedge p$. Now $\mathsf{f}^{\mathbf{L}}_a(\delta')$ is $(\Box_a p \vee \Box_a q) \wedge p$, so we have $\vdash_{\mathbf{T}} \delta' \to \Box_a p$ and $\vdash_{\mathbf{T}} \mathsf{f}^{\mathbf{L}}_a(\delta') \to p$, as desired.

Next, note that each "possibility" φ determines that a believes $\mathsf{f}^{\mathbf{L}}_a(\varphi)$.

Lemma 3.8 For every $\mathbf{L}$ in Definition 3.7, $\varphi \in \mathcal{L}$, and $a \in I$: $\vdash_{\mathbf{L}} \varphi \to \Box_a\mathsf{f}^{\mathbf{L}}_a(\varphi)$.

Proof. For extensions of **KB**, $\varphi \to \Box_a\mathsf{f}^{\mathbf{L}}_a(\varphi)$ is $\varphi \to \Box_a\Diamond_a\varphi$, which is the B axiom. For the other logics, it suffices to show $\vdash_{\mathbf{L}} \varphi \to \Box_a\mathsf{f}^{\mathbf{L}}_a(\varphi)$ where φ is $NF_{\mathbf{L}}(\psi)$ for some ψ. So φ is of the form $\delta_1 \vee \cdots \vee \delta_n$. For each disjunct δ_i of φ,

$$\vdash_{\mathbf{L}} \delta_i \to \bigwedge_{\psi \text{ a conjunct of } \mathsf{f}^{\mathbf{L}}_a(\delta_i)} \Box_a\psi,$$

which for any normal modal logic implies

$$\vdash_{\mathbf{L}} \delta_i \to \Box_a \bigwedge_{\psi \text{ a conjunct of } \mathsf{f}^{\mathbf{L}}_a(\delta_i)} \psi, \text{ i.e., } \vdash_{\mathbf{L}} \delta_i \to \Box_a\mathsf{f}^{\mathbf{L}}_a(\delta_i),$$

which for any normal modal logic implies

$$\vdash_{\mathbf{L}} \delta_i \to \Box_a(\mathsf{f}^{\mathbf{L}}_a(\delta_1) \vee \cdots \vee \mathsf{f}^{\mathbf{L}}_a(\delta_n)), \text{ i.e., } \vdash_{\mathbf{L}} \delta_i \to \Box_a\mathsf{f}^{\mathbf{L}}_a(\varphi).$$

Since the above holds for all disjuncts δ_i of φ, we have $\vdash_{\mathbf{L}} \varphi \to \Box_a\mathsf{f}^{\mathbf{L}}_a(\varphi)$. $\Box$

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[13] For the logics **K45loop** and **KD45loop** mentioned in footnote 11, we would define

$$\mathsf{f}^{\mathbf{L}}_a(\delta_i) := \mathbf{loop}_a \wedge \bigwedge\{\beta, \Box_a\beta \mid \Box_a\beta \text{ a conjunct of } \delta_i\} \cup \{\Diamond_a\gamma \mid \Diamond_a\gamma \text{ a conjunct of } \delta_i\}.$$

[14] Note that by Lemma 3.8 and Theorem 3.9, if $\vdash_{\mathbf{L}} \varphi \leftrightarrow \psi$, then $\vdash_{\mathbf{L}} \mathsf{f}^{\mathbf{L}}_a(\varphi) \leftrightarrow \mathsf{f}^{\mathbf{L}}_a(\psi)$.

We are now ready to prove our first main result, Theorem 3.9, which shows that our selected logics have internal adjointness. The proof involves the gluing together of relational world models in the style of the completeness proofs in [15]. These constructions are interesting, as are the ways that the definition of $\mathsf{f}_a^{\mathbf{L}}(\varphi)$ is used, but for the sake of space we give the proof in the Appendix.

Theorem 3.9 (Internal Adjointness) For any $\mathbf{L}$ among $\mathbf{K}$, $\mathbf{KD}$, $\mathbf{T}$, $\mathbf{K4}$, $\mathbf{KD4}$, $\mathbf{S4}$, and $\mathbf{S5}$ (or any other extension of $\mathbf{KB}$), $\varphi, \psi \in \mathcal{L}$, and $a \in I$:

$$\vdash_{\mathbf{L}} \varphi \to \Box_a \psi \text{ iff } \vdash_{\mathbf{L}} \mathsf{f}_a^{\mathbf{L}}(\varphi) \to \psi.$$

Note that the right to left direction is straightforward: if $\vdash_{\mathbf{L}} \mathsf{f}_a^{\mathbf{L}}(\varphi) \to \psi$, then $\vdash_{\mathbf{L}} \Box_a \mathsf{f}_a^{\mathbf{L}}(\varphi) \to \Box_a \psi$ since $\mathbf{L}$ is normal, so $\vdash_{\mathbf{L}} \varphi \to \Box_a \psi$ by Lemma 3.8.

The following Lemma will be used to prove Lemma 4.4 in §4.

Lemma 3.10 Let $\mathbf{L}$ be one of the logics in Definition 3.7 and $\varphi, \psi \in \mathcal{L}$.

1. If $\mathbf{L}$ contains the D axiom and φ is $\mathbf{L}$-consistent, then $\mathsf{f}_a^{\mathbf{L}}(\varphi)$ is $\mathbf{L}$-consistent;
2. If $\mathbf{L}$ contains the T axiom, then $\vdash_{\mathbf{L}} \varphi \to \mathsf{f}_a^{\mathbf{L}}(\varphi)$;
3. If $\mathbf{L}$ contains the 4 axiom, then $\vdash_{\mathbf{L}} \mathsf{f}_a^{\mathbf{L}}(\varphi) \to \Box_a \mathsf{f}_a^{\mathbf{L}}(\varphi)$, which by Theorem 3.9 is equivalent to $\vdash_{\mathbf{L}} \mathsf{f}_a^{\mathbf{L}}(\mathsf{f}_a^{\mathbf{L}}(\varphi)) \to \mathsf{f}_a^{\mathbf{L}}(\varphi)$;
4. If $\mathbf{L}$ contains the B axiom, φ and ψ are $\mathbf{L}$-consistent, and $\vdash_{\mathbf{L}} \varphi \to \mathsf{f}_a^{\mathbf{L}}(\psi)$, then $\psi \wedge \mathsf{f}_a^{\mathbf{L}}(\varphi)$ is $\mathbf{L}$-consistent;
5. If $\mathbf{L}$ contains the 5 axiom, then $\vdash_{\mathbf{L}} \Diamond_a \varphi \to \Box_a \mathsf{f}_a^{\mathbf{L}}(\varphi)$, which by Theorem 3.9 is equivalent to $\vdash_{\mathbf{L}} \mathsf{f}_a^{\mathbf{L}}(\Diamond_a \varphi) \to \mathsf{f}_a^{\mathbf{L}}(\varphi)$.

Proof. For part 1, for any normal modal logic $\mathbf{L}$, if $\vdash_{\mathbf{L}} \mathsf{f}_a^{\mathbf{L}}(\varphi) \to \bot$, then $\vdash_{\mathbf{L}} \Box_a \mathsf{f}_a^{\mathbf{L}}(\varphi) \to \Box_a \bot$, which with Lemma 3.8 implies $\vdash_{\mathbf{L}} \varphi \to \Box_a \bot$, which for $\mathbf{L}$ with the D axiom implies $\vdash_{\mathbf{L}} \varphi \to \bot$. For part 2, given $\vdash_{\mathbf{L}} \varphi \to \Box_a \mathsf{f}_a^{\mathbf{L}}(\varphi)$ by Lemma 3.8, it follows for any $\mathbf{L}$ with the T axiom that $\vdash_{\mathbf{L}} \varphi \to \mathsf{f}_a^{\mathbf{L}}(\varphi)$.

For part 3, if $\mathbf{L}$ is $\mathbf{S5}$, then the claim is immediate given $\mathsf{f}_a^{\mathbf{L}}(\varphi) = \Diamond_a \varphi$ from Definition 3.7. Let us consider the other logics in Definition 3.7 with the 4 axiom. We can assume without loss of generality that φ is a formula in DNF of the form $\delta_1 \vee \cdots \vee \delta_n$, and $\mathsf{f}_a^{\mathbf{L}}(\varphi) = \mathsf{f}_a^{\mathbf{L}}(\delta_1) \vee \cdots \vee \mathsf{f}_a^{\mathbf{L}}(\delta_n)$ by Definition 3.7. Observe that for each of the $\mathbf{L}$ in Definition 3.7 with the 4 axiom and each δ_i,

$$\vdash_{\mathbf{L}} \mathsf{f}_a^{\mathbf{L}}(\delta_i) \to \bigwedge_{\psi \text{ a conjunct of } \mathsf{f}_a^{\mathbf{L}}(\delta_i)} \Box_a \psi.$$

Now the proof that $\vdash_{\mathbf{L}} \mathsf{f}_a^{\mathbf{L}}(\varphi) \to \Box_a \mathsf{f}_a^{\mathbf{L}}(\varphi)$ follows the pattern for Lemma 3.8.

Given Definition 3.7, part 4 is equivalent to the claim that for any $\mathbf{L}$-consistent φ and ψ, if $\vdash_{\mathbf{L}} \varphi \to \Diamond_a \psi$, then $\psi \wedge \Diamond_a \varphi$ is $\mathbf{L}$-consistent. If $\psi \wedge \Diamond_a \varphi$ is $\mathbf{L}$-inconsistent, then $\vdash_{\mathbf{L}} \Diamond_a \varphi \to \neg \psi$, which implies $\vdash_{\mathbf{L}} \Box_a \Diamond_a \varphi \to \Box_a \neg \psi$ for a normal $\mathbf{L}$. Then since $\mathbf{L}$ has the B axiom, $\vdash_{\mathbf{L}} \varphi \to \Box_a \Diamond_a \varphi$, so we have $\vdash_{\mathbf{L}} \varphi \to \Box_a \neg \psi$, which with $\vdash_{\mathbf{L}} \varphi \to \Diamond_a \psi$ contradicts the $\mathbf{L}$-consistency of φ.

For part 5, by Lemma 3.8, $\vdash_{\mathbf{L}} \varphi \to \Box_a \mathsf{f}_a^{\mathbf{L}}(\varphi)$, which for a normal $\mathbf{L}$ implies $\vdash_{\mathbf{L}} \Diamond_a \varphi \to \Diamond_a \Box_a \mathsf{f}_a^{\mathbf{L}}(\varphi)$, so $\vdash_{\mathbf{L}} \Diamond_a \varphi \to \Box_a \mathsf{f}_a^{\mathbf{L}}(\varphi)$ for $\mathbf{L}$ with the 5 axiom. $\Box$

4 Canonical Models of Finite Possibilities

We can now construct locally finite canonical possibility models for the logics $\mathbf{L}$ in Theorem 2.8. For $\varphi \in \mathcal{L}$, let $[\varphi]_{\mathbf{L}} = \{\psi \in \mathcal{L} \mid \vdash_{\mathbf{L}} \varphi \leftrightarrow \psi\}$. Fix an enumeration $\varphi_1, \varphi_2, \ldots$ of the formulas of $\mathcal{L}$, and for every $\varphi \in \mathcal{L}$, let $\varphi_{\mathbf{L}}$ be the member of $[\varphi]_{\mathbf{L}}$ that occurs first in the enumeration. We do this so that our possibilities can simply be formulas, rather than their equivalence classes.[15] This simplifies the presentation, but nothing important turns on it.

Definition 4.1 For each logic $\mathbf{L}$ in Theorem 2.8, define the canonical functional finite-possibility model $\mathbb{M}^{\mathbf{L}} = \langle W^{\mathbf{L}}, \geqslant^{\mathbf{L}}, \{f_a^{\mathbf{L}}\}_{a \in I}, V^{\mathbf{L}} \rangle$ as follows:

1. $W^{\mathbf{L}} = \{\sigma_{\mathbf{L}} \mid \sigma \in \mathcal{L}\}$; $\bot_{\mathbb{M}^{\mathbf{L}}} = \bot_{\mathbf{L}}$;
2. $\sigma' \geqslant^{\mathbf{L}} \sigma$ iff $\vdash_{\mathbf{L}} \sigma' \to \sigma$;
3. $f_a^{\mathbf{L}}(\sigma) = \mathsf{f}_a^{\mathbf{L}}(\sigma)_{\mathbf{L}}$;
4. $V^{\mathbf{L}}(p, \sigma) = 1$ iff $\vdash_{\mathbf{L}} \sigma \to p$; $V^{\mathbf{L}}(p, \sigma) = 0$ iff $\vdash_{\mathbf{L}} \sigma \to \neg p$.

Following our earlier convention, we will use boldface letters for the consistent formulas in $W^{\mathbf{L}} - \{\bot_{\mathbf{L}}\}$. In some of the text in the rest of this section, to reduce clutter we will leave the sub/superscript for $\mathbf{L}$ implicit.

Our first job is to check that $\mathbb{M}^{\mathbf{L}}$ is indeed a functional possibility model.

Lemma 4.2 (Canonical Model is a Model) For each logic $\mathbf{L}$ in Theorem 2.8, $\mathbb{M}^{\mathbf{L}}$ is a functional possibility model according to Definition 2.1, and $\mathbb{M}^{\mathbf{L}}$ is locally finite according to Definition 2.7.

Proof. The conditions of *persistence*, *refinability*, and f*-persistence* are all easy to check for $\mathbb{M}^{\mathbf{L}}$. It is also clear that for any $\boldsymbol{\sigma} \in W$, $\{p \in \mathsf{At} \mid \vdash \boldsymbol{\sigma} \to \pm p\}$ is finite, so $\mathbb{M}^{\mathbf{L}}$ is *locally finite*. Let us verify that f*-refinability* holds:

> For all consistent $\boldsymbol{\sigma}, \boldsymbol{\gamma} \in W$, if $\boldsymbol{\gamma} \geqslant f_a(\boldsymbol{\sigma})$, then there is a $\boldsymbol{\sigma'} \geqslant \boldsymbol{\sigma}$ such that for all $\boldsymbol{\sigma''} \geqslant \boldsymbol{\sigma'}$ there is a $\boldsymbol{\gamma'} \geqslant \boldsymbol{\gamma}$ such that $\boldsymbol{\gamma'} \geqslant f_a(\boldsymbol{\sigma''})$.

Given $\boldsymbol{\gamma} \geqslant f_a(\boldsymbol{\sigma})$, we have $\vdash \boldsymbol{\gamma} \to \mathsf{f}_a(\boldsymbol{\sigma})$. Now since $\boldsymbol{\gamma}$ is consistent, $\nvdash \boldsymbol{\gamma} \to \neg\boldsymbol{\gamma}$, which with $\vdash \boldsymbol{\gamma} \to \mathsf{f}_a(\boldsymbol{\sigma})$ implies $\nvdash \mathsf{f}_a(\boldsymbol{\sigma}) \to \neg\boldsymbol{\gamma}$, which with Theorem 3.9 implies $\nvdash \boldsymbol{\sigma} \to \Box_a\neg\boldsymbol{\gamma}$. Thus, $\boldsymbol{\sigma'} = \boldsymbol{\sigma} \wedge \Diamond_a\boldsymbol{\gamma}$ is consistent. Now for any consistent $\boldsymbol{\sigma''} \geqslant \boldsymbol{\sigma'}$, i.e., $\vdash \boldsymbol{\sigma''} \to \boldsymbol{\sigma'}$, we claim that $\boldsymbol{\gamma'} = \boldsymbol{\gamma} \wedge \mathsf{f}_a(\boldsymbol{\sigma''})$ is consistent. If not, then $\vdash \mathsf{f}_a(\boldsymbol{\sigma''}) \to \neg\boldsymbol{\gamma}$, which for any normal modal logic implies $\vdash \Box_a\mathsf{f}_a(\boldsymbol{\sigma''}) \to \Box_a\neg\boldsymbol{\gamma}$, which with Lemma 3.8 implies $\vdash \boldsymbol{\sigma''} \to \Box_a\neg\boldsymbol{\gamma}$. But given $\boldsymbol{\sigma'} = \boldsymbol{\sigma} \wedge \Diamond_a\boldsymbol{\gamma}$ and $\vdash \boldsymbol{\sigma''} \to \boldsymbol{\sigma'}$, we have $\vdash \boldsymbol{\sigma''} \to \Diamond_a\boldsymbol{\gamma}$, which with $\vdash \boldsymbol{\sigma''} \to \Box_a\neg\boldsymbol{\gamma}$ contradicts the consistency of $\boldsymbol{\sigma''}$. Thus, $\boldsymbol{\gamma'}$ is consistent. Then since $\vdash \boldsymbol{\gamma'} \to \mathsf{f}_a(\boldsymbol{\sigma''})$, we have $\boldsymbol{\gamma'} \geqslant f_a(\boldsymbol{\sigma''})$. Hence we have shown that there is a $\boldsymbol{\sigma'} \geqslant \boldsymbol{\sigma}$ such that for all $\boldsymbol{\sigma''} \geqslant \boldsymbol{\sigma}$ there is a $\boldsymbol{\gamma'} \geqslant \boldsymbol{\gamma}$ such that $\boldsymbol{\gamma'} \geqslant f_a(\boldsymbol{\sigma''})$. $\Box$

Our next job is to show that for any formulas φ and σ, φ being *true* at the possibility σ in $\mathbb{M}^{\mathbf{L}}$ is equivalent to $\sigma \to \varphi$ being *derivable* in $\mathbf{L}$.

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[15] The reason for dealing with equivalence classes and representatives at all is so that the relation $\geqslant$ in the canonical model will be antisymmetric, as Humberstone [17, p. 318] requires. If we had instead allowed $\geqslant$ to be a preorder—which would not have changed any of our results—then we could take our domain to be the set of all consistent formulas plus $\bot$.

Lemma 4.3 (Truth) For any logic $\mathbf{L}$ in Theorem 2.8, $\sigma \in W^{\mathbf{L}}$, and $\varphi \in \mathcal{L}$: $\mathbb{M}^{\mathbf{L}}, \sigma \Vdash \varphi$ iff $\vdash_{\mathbf{L}} \sigma \to \varphi$.

Proof. The claim is immediate for $\sigma = \bot$, given Definition 2.2.1. For $\sigma \neq \bot$, we prove the claim by induction on φ. The atomic case is by definition of $V^{\mathbf{L}}$, and the $\wedge$ case is routine. For the $\neg$ case, if $\nvdash \boldsymbol{\sigma} \to \neg\varphi$, then $(\boldsymbol{\sigma} \wedge \varphi)$ is consistent. Then since $\vdash (\boldsymbol{\sigma} \wedge \varphi) \to \boldsymbol{\sigma}$, i.e., $(\boldsymbol{\sigma} \wedge \varphi) \geqslant \boldsymbol{\sigma}$, we have a $\boldsymbol{\sigma'} \geqslant \boldsymbol{\sigma}$ such that $\vdash \boldsymbol{\sigma'} \to \varphi$, which by the inductive hypothesis implies that $\mathbb{M}, \boldsymbol{\sigma'} \Vdash \varphi$, which implies $\mathbb{M}, \boldsymbol{\sigma} \nVdash \neg\varphi$. In the other direction, if $\vdash \boldsymbol{\sigma} \to \neg\varphi$, then for all $\boldsymbol{\sigma'} \geqslant \boldsymbol{\sigma}$, i.e., $\vdash \boldsymbol{\sigma'} \to \boldsymbol{\sigma}$, we have $\vdash \boldsymbol{\sigma'} \to \neg\varphi$, so $\nvdash \boldsymbol{\sigma'} \to \varphi$ by the consistency of $\boldsymbol{\sigma'}$, so $\mathbb{M}, \boldsymbol{\sigma'} \nVdash \varphi$ by the inductive hypothesis. Thus, $\mathbb{M}, \boldsymbol{\sigma} \Vdash \neg\varphi$.

For the $\Box_a$ case, given $f_a(\boldsymbol{\sigma}) = \mathsf{f}_a(\boldsymbol{\sigma})$, we have the following equivalences: $\vdash \boldsymbol{\sigma} \to \Box_a\varphi$ iff $\vdash \mathsf{f}_a(\boldsymbol{\sigma}) \to \varphi$ (by Theorem 3.9) iff $\mathbb{M}, f_a(\boldsymbol{\sigma}) \Vdash \varphi$ (by the inductive hypothesis) iff $\mathbb{M}, \boldsymbol{\sigma} \Vdash \Box_a\varphi$ (by the truth definition). □

If we only wished to prove the case of Theorem 2.8 for $\mathbf{K}$, then with Lemmas 4.2 and 4.3 we would be done. However, to prove Theorem 2.8 for the various extensions of $\mathbf{K}$, we need to make sure that $\mathbb{M}^{\mathbf{L}}$ satisfies the conditions on f_a and $\geqslant$ corresponding to the extra axioms of $\mathbf{L}$, given in Theorem 2.6.

Lemma 4.4 (Canonicity) The model $\mathbb{M}^{\mathbf{L}}$ is such that:

1. If $\mathbf{L}$ contains the D axiom, then for all $\boldsymbol{\sigma} \in W^{\mathbf{L}}$, $f_a(\boldsymbol{\sigma}) \neq \bot$;
2. If $\mathbf{L}$ contains the T axiom, then for all $\boldsymbol{\sigma} \in W^{\mathbf{L}}$, $\boldsymbol{\sigma} \geqslant f_a(\boldsymbol{\sigma})$;
3. If $\mathbf{L}$ contains the 4 axiom, then for all $\boldsymbol{\sigma} \in W^{\mathbf{L}}$, $f_a(f_a(\boldsymbol{\sigma})) \geqslant f_a(\boldsymbol{\sigma})$;
4. If $\mathbf{L}$ contains the B axiom, then for all $\boldsymbol{\sigma}, \boldsymbol{\gamma} \in W^{\mathbf{L}}$, if $\boldsymbol{\gamma} \geqslant f_a(\boldsymbol{\sigma})$, then $\exists \boldsymbol{\sigma'} \geqslant \boldsymbol{\sigma} \colon \boldsymbol{\sigma'} \geqslant f_a(\boldsymbol{\gamma})$.
5. If $\mathbf{L}$ contains the 5 axiom, then for all $\boldsymbol{\sigma}, \boldsymbol{\gamma} \in W^{\mathbf{L}}$, if $\boldsymbol{\gamma} \geqslant f_a(\boldsymbol{\sigma})$, then $\exists \boldsymbol{\sigma'} \geqslant \boldsymbol{\sigma} \colon f_a(\boldsymbol{\sigma'}) \geqslant f_a(\boldsymbol{\gamma})$.

Proof. Each part follows from the corresponding part of Lemma 3.10. For part 1, we need that if $\boldsymbol{\sigma} \in W^{\mathbf{L}}$ is $\mathbf{L}$-consistent, then so is $\mathsf{f}_a^{\mathbf{L}}(\boldsymbol{\sigma})$, which is given by Lemma 3.10.1. For part 2, we need that for all $\boldsymbol{\sigma} \in W^{\mathbf{L}}$, $\vdash_{\mathbf{L}} \boldsymbol{\sigma} \to \mathsf{f}_a^{\mathbf{L}}(\boldsymbol{\sigma})$, which is given by Lemma 3.10.2. For part 3, we need that for all $\boldsymbol{\sigma} \in W^{\mathbf{L}}$, $\vdash_{\mathbf{L}} \mathsf{f}_a^{\mathbf{L}}(\mathsf{f}_a^{\mathbf{L}}(\boldsymbol{\sigma})) \to \mathsf{f}_a^{\mathbf{L}}(\boldsymbol{\sigma})$, which is given by Lemma 3.10.3.

For part 4, we need that for all $\mathbf{L}$-consistent $\boldsymbol{\sigma}, \boldsymbol{\gamma} \in W^{\mathbf{L}}$, if $\vdash_{\mathbf{L}} \boldsymbol{\gamma} \to \mathsf{f}_a^{\mathbf{L}}(\boldsymbol{\sigma})$, then there is some $\mathbf{L}$-consistent $\boldsymbol{\sigma'}$ with (i) $\vdash_{\mathbf{L}} \boldsymbol{\sigma'} \to \boldsymbol{\sigma}$ and (ii) $\vdash_{\mathbf{L}} \boldsymbol{\sigma'} \to \mathsf{f}_a^{\mathbf{L}}(\boldsymbol{\gamma})$. Setting $\boldsymbol{\sigma'} := \boldsymbol{\sigma} \wedge \mathsf{f}_a^{\mathbf{L}}(\boldsymbol{\gamma}) = \boldsymbol{\sigma} \wedge \Diamond_a\boldsymbol{\gamma}$, then (i) and (ii) are immediate, and the $\mathbf{L}$-consistency of $\boldsymbol{\sigma'}$ is given by Lemma 3.10.4.

For part 5, we need that for all $\mathbf{L}$-consistent $\boldsymbol{\sigma}, \boldsymbol{\gamma} \in W^{\mathbf{L}}$, if $\vdash_{\mathbf{L}} \boldsymbol{\gamma} \to \mathsf{f}_a^{\mathbf{L}}(\boldsymbol{\sigma})$, then there is some $\mathbf{L}$-consistent $\boldsymbol{\sigma'}$ such that (iii) $\vdash_{\mathbf{L}} \boldsymbol{\sigma'} \to \boldsymbol{\sigma}$ and (iv) $\vdash_{\mathbf{L}} \mathsf{f}_a^{\mathbf{L}}(\boldsymbol{\sigma'}) \to \mathsf{f}_a^{\mathbf{L}}(\boldsymbol{\gamma})$. Setting $\boldsymbol{\sigma'} := \boldsymbol{\sigma} \wedge \Diamond_a\boldsymbol{\gamma}$, then (iii) is immediate. By Lemma 3.10.5, we have $\vdash_{\mathbf{L}} \Diamond_a\boldsymbol{\gamma} \to \Box_a\mathsf{f}_a^{\mathbf{L}}(\boldsymbol{\gamma})$ and hence $\vdash_{\mathbf{L}} \boldsymbol{\sigma'} \to \Box_a\mathsf{f}_a^{\mathbf{L}}(\boldsymbol{\gamma})$, which by Theorem 3.9 implies (iv). Finally, suppose for reductio that $\boldsymbol{\sigma'}$ is $\mathbf{L}$-inconsistent, so $\vdash_{\mathbf{L}} \boldsymbol{\sigma} \to \neg\Diamond_a\boldsymbol{\gamma}$. Then $\vdash_{\mathbf{L}} \boldsymbol{\sigma} \to \Box_a\neg\boldsymbol{\gamma}$, which by Theorem 3.9 implies $\vdash_{\mathbf{L}} \mathsf{f}_a^{\mathbf{L}}(\boldsymbol{\sigma}) \to \neg\boldsymbol{\gamma}$, which with $\vdash_{\mathbf{L}} \boldsymbol{\gamma} \to \mathsf{f}_a^{\mathbf{L}}(\boldsymbol{\sigma})$ implies that $\boldsymbol{\gamma}$ is $\mathbf{L}$-inconsistent, contradicting our initial assumption. Thus, $\boldsymbol{\sigma'}$ is $\mathbf{L}$-consistent. □

We have now shown that lattices $\langle L, \leq \rangle$ as in §1, equipped with functions f_a exhibiting $\mathbf{L}$'s internal adjointness, can be viewed as canonical possibility models. This illustrates the closeness of possibility *semantics* to modal *syntax*.

Finally, we put all of the pieces together for our culminating result. [16]

Theorem 2.8 (Completeness for Locally Finite Models)
Let $\mathbf{L}$ be one of the logics $\mathbf{K}$, $\mathbf{KD}$, $\mathbf{T}$, $\mathbf{K4}$, $\mathbf{KD4}$, $\mathbf{S4}$, or $\mathbf{S5}$, and let $\mathsf{S}^{LF}_{\mathbf{L}}$ be the class of *locally finite* functional possibility models satisfying the constraints on f_a and $\geqslant$ corresponding to the axioms of $\mathbf{L}$, as listed in Theorem 2.6. Then $\mathbf{L}$ is weakly complete with respect to $\mathsf{S}^{LF}_{\mathbf{L}}$: for all $\varphi \in \mathcal{L}$, if $\Vdash_{\mathsf{S}^{LF}_{\mathbf{L}}} \varphi$, then $\vdash_{\mathbf{L}} \varphi$.

Proof. By Lemmas 4.2 and 4.4, $\mathbb{M}^{\mathbf{L}} \in \mathsf{S}^{LF}_{\mathbf{L}}$, and by the definition of $\mathbb{M}^{\mathbf{L}}$, $\neg\varphi_{\mathbf{L}} \in W^{\mathbf{L}}$. Assuming $\nvdash_{\mathbf{L}} \varphi$, we have $\neg\varphi_{\mathbf{L}} \neq \bot_{\mathbf{L}}$. Then by Lemma 4.3, $\mathbb{M}^{\mathbf{L}}, \neg\varphi_{\mathbf{L}} \nVdash \varphi$, which with $\mathbb{M}^{\mathbf{L}} \in \mathsf{S}^{LF}_{\mathbf{L}}$ implies $\nVdash_{\mathsf{S}^{LF}_{\mathbf{L}}} \varphi$ by Definition 2.4. $\Box$

5 Conclusion

A Humberstonian model theory for modal logic, based on partial possibilities instead of total worlds, involves not only a different intuitive picture of modal models, but also a different mathematical approach to their construction. The infinitary staples of completeness proofs for world semantics—maximally consistent sets, Lindenbaum's Lemma—are not needed for possibility semantics. This may be considered an advantage, [17] but the purpose of this paper was not to advocate for possibilities *over* worlds. Nor was it to advocate for functions *over* relations. Modal reasoning with relational world models is natural and powerful, as our own Appendix shows. The purpose of this paper was instead to suggest how modal reasoning with functional possibility models is also natural and powerful, and how this reasoning leads to the independently interesting issue of *internal adjointness* for modal logics. There are many other interesting issues around the corner, such as the study of transformations between possibility models and world models (see [16,14]). Hopefully, however, we have already seen enough to motivate further study of possibilities for modal logic.

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[16] Of course, we do not have *strong* completeness over locally finite models. For an infinite set $\Sigma \subseteq \mathsf{At}$ of atomic sentences, there is no locally finite model containing a possibility that makes all of Σ true, so $\Sigma \Vdash_{\mathsf{S}^{LF}_{\mathbf{L}}} \bot$; yet for every finite subset $\Sigma_0 \subseteq \Sigma$, we have $\Sigma_0 \nVdash_{\mathsf{S}^{LF}_{\mathbf{L}}} \bot$ (so the consequence relation $\Vdash_{\mathsf{S}^{LF}_{\mathbf{L}}}$ is not compact) and hence $\Sigma_0 \nvdash_{\mathbf{L}} \bot$ by soundness, so $\Sigma \nvdash_{\mathbf{L}} \bot$.

[17] Van Benthem [3, p. 78] remarks: "There is something inelegant to an ordinary Henkin argument. One has a consistent set of sentences S, perhaps quite small, that one would like to see satisfied semantically. Now, some arbitrary *maximal* extension S^+ of S is to be taken to obtain a model (for S^+, and hence for S)—but the added part $S^+ - S$ plays no role subsequently. We started out with something partial, but the method forces us to be total." This "problem of the 'irrelevant extension' " [1, p. 1] is solved by possibility semantics.

Appendix

In this appendix, we prove Theorem 3.9 from §3. In the proof, which uses standard relational semantics, we invoke the completeness of the logics listed with respect to their corresponding classes of relational world models. Let $\mathsf{C}_{\mathbf{L}}$ be the class of relational world models determined by logic $\mathbf{L}$, so, e.g., $\mathsf{C}_{\mathbf{K}}$ is the class of all relational world models, $\mathsf{C}_{\mathbf{T}}$ is the class of reflexive relational world models, $\mathsf{C}_{\mathbf{K4}}$ is the class of transitive relational world models, etc.

Theorem 3.9 (Internal Adjointness) For any $\mathbf{L}$ among $\mathbf{K}$, $\mathbf{KD}$, $\mathbf{T}$, $\mathbf{K4}$, $\mathbf{KD4}$, $\mathbf{S4}$, and $\mathbf{S5}$ (or any other extension of $\mathbf{KB}$), $\varphi, \psi \in \mathcal{L}$, and $a \in I$:

$$\vdash_{\mathbf{L}} \varphi \to \Box_a \psi \text{ iff } \vdash_{\mathbf{L}} \mathsf{f}^{\mathbf{L}}_a(\varphi) \to \psi.$$

Proof. From right to left, if $\vdash \mathsf{f}^{\mathbf{L}}_a(\varphi) \to \psi$, then $\vdash \Box_a \mathsf{f}^{\mathbf{L}}_a(\varphi) \to \Box_a \psi$ since $\mathbf{L}$ is normal, so $\vdash \varphi \to \Box_a \psi$ by Lemma 3.8.

From left to right, the proof for logics that contain the B axiom, such as $\mathbf{S5}$, is simple: if $\nvdash_{\mathbf{L}} \mathsf{f}^{\mathbf{L}}_a(\varphi) \to \psi$, then by the completeness of $\mathbf{L}$ with respect to $\mathsf{C}_{\mathbf{L}}$, there is a relational world model $\mathfrak{M} \in \mathsf{C}_{\mathbf{L}}$ with world w such that $\mathfrak{M}, w \vDash \mathsf{f}^{\mathbf{L}}_a(\varphi) \wedge \neg\psi$, which by the definition of $\mathsf{f}^{\mathbf{L}}_a(\varphi)$ for logics containing B means $\mathfrak{M}, w \vDash \Diamond_a \varphi \wedge \neg\psi$. Hence there is some v with $w\mathrm{R}_a v$ such that $\mathfrak{M}, v \vDash \varphi$. By the symmetry of R_a, we also have $v\mathrm{R}_a w$, so $\mathfrak{M}, v \vDash \Diamond_a \neg\psi$. Finally, by the soundness of $\mathbf{L}$ with respect $\mathsf{C}_{\mathbf{L}}$, $\mathfrak{M}, v \vDash \varphi \wedge \Diamond_a \neg\psi$ implies $\nvdash_{\mathbf{L}} \varphi \to \Box_a \psi$.

From left to right for logics without the B axiom, if φ is $\mathbf{L}$-inconsistent, then $\vdash_{\mathbf{L}} \varphi \to \Box_a \psi$ for all ψ, and $\mathsf{f}^{\mathbf{L}}_a(\varphi) = \bot$ by Definition 3.7, so $\vdash_{\mathbf{L}} \mathsf{f}^{\mathbf{L}}_a(\varphi) \to \psi$ for all ψ. So suppose that φ is $\mathbf{L}$-consistent. Given our definition of $\mathsf{f}^{\mathbf{L}}_a$ in terms of $NF_{\mathbf{L}}(\varphi)$, we can assume φ is in DNF and each of its disjuncts is $\mathbf{L}$-consistent; moreover, we can assume that φ is T-unpacked if $\mathbf{L}$ contains the T axiom.

Now if $\nvdash_{\mathbf{L}} \mathsf{f}^{\mathbf{L}}_a(\varphi) \to \psi$, then there is a $\mathbf{L}$-consistent disjunct δ of φ such that $\nvdash_{\mathbf{L}} \mathsf{f}^{\mathbf{L}}_a(\delta) \to \psi$. Since δ is $\mathbf{L}$-consistent, by the completeness of $\mathbf{L}$ with respect to $\mathsf{C}_{\mathbf{L}}$ there is a relational world model $\mathfrak{A} = \langle \mathrm{W}^{\mathfrak{A}}, \{\mathrm{R}^{\mathfrak{A}}_a\}_{a \in I}, \mathrm{V}^{\mathfrak{A}} \rangle \in \mathsf{C}_{\mathbf{L}}$ with $x \in \mathrm{W}^{\mathfrak{A}}$ such that $\mathfrak{A}, x \vDash \delta$. Now define a model $\mathfrak{A}' = \langle \mathrm{W}^{\mathfrak{A}'}, \{\mathrm{R}^{\mathfrak{A}'}_a\}_{a \in I}, \mathrm{V}^{\mathfrak{A}'} \rangle$, shown in Fig. 2 below, that is just like $\mathfrak{A}$ except with one new world x' that can "see" all and only the worlds that x can see (so x' cannot see itself):

- $\mathrm{W}^{\mathfrak{A}'} = \mathrm{W}^{\mathfrak{A}} \cup \{x'\}$ for $x' \notin \mathrm{W}^{\mathfrak{A}}$;
- for all $b \in I$, $w\mathrm{R}^{\mathfrak{A}'}_b v$ iff either $w\mathrm{R}^{\mathfrak{A}}_b v$ or $[w = x'$ and $x\mathrm{R}^{\mathfrak{A}}_b v]$;
- $\mathrm{V}^{\mathfrak{A}'}(p, w) = 1$ iff either $\mathrm{V}^{\mathfrak{A}}(p, w) = 1$ or $[w = x'$ and $\mathrm{V}^{\mathfrak{A}}(p, x) = 1]$.

Define $E \subseteq \mathrm{W}^{\mathfrak{A}'} \times \mathrm{W}^{\mathfrak{A}}$ such that wEv iff $[w = x'$ and $v = x]$ or $[w \neq x'$ and $w = v]$. Then E is a *bisimulation* relating $\mathfrak{A}', x'$ and $\mathfrak{A}, x$, so by the invariance of modal truth under bisimulation [6, §2.2], $\mathfrak{A}, x \vDash \delta$ implies $\mathfrak{A}', x' \vDash \delta$.

Since $\nvdash_{\mathbf{L}} \mathsf{f}^{\mathbf{L}}_a(\delta) \to \psi$, by the completeness of $\mathbf{L}$ with respect to $\mathsf{C}_{\mathbf{L}}$ there is a relational world model $\mathfrak{B} = \langle \mathrm{W}^{\mathfrak{B}}, \{\mathrm{R}^{\mathfrak{B}}_a\}_{a \in I}, \mathrm{V}^{\mathfrak{B}} \rangle \in \mathsf{C}_{\mathbf{L}}$ with $y \in \mathrm{W}^{\mathfrak{B}}$ such that $\mathfrak{B}, y \vDash \mathsf{f}^{\mathbf{L}}_a(\delta) \wedge \neg\psi$. Without loss of generality, we can assume that the domains of $\mathfrak{A}'$ and $\mathfrak{B}$ are disjoint. Define a new model $\mathfrak{C} = \langle \mathrm{W}^{\mathfrak{C}}, \{\mathrm{R}^{\mathfrak{C}}_a\}_{a \in I}, \mathrm{V}^{\mathfrak{C}} \rangle$, shown in Fig. 2 below, by first taking the disjoint union of $\mathfrak{A}'$ and $\mathfrak{B}$ and then connecting x' from $\mathfrak{A}'$ to y from $\mathfrak{B}$ by an accessibility arrow for agent a:

- $W^{\mathfrak{C}} = W^{\mathfrak{A}'} \cup W^{\mathfrak{B}}$; $V^{\mathfrak{C}}(p,w) = 1$ iff $V^{\mathfrak{A}'}(p,w) = 1$ or $V^{\mathfrak{B}}(p,w) = 1$.
- for a in the lemma, $w\mathrm{R}^{\mathfrak{C}}_a v$ iff either $w\mathrm{R}^{\mathfrak{A}'}_a v$, $w\mathrm{R}^{\mathfrak{B}}_a v$, or $[w = x'$ and $v = y]$;
- for $b \neq a$, $w\mathrm{R}^{\mathfrak{C}}_b v$ iff either $w\mathrm{R}^{\mathfrak{A}'}_b v$ or $w\mathrm{R}^{\mathfrak{B}}_b v$.

The identity relation on $W^{\mathfrak{A}'} \setminus \{x'\}$ is a bisimulation between $\mathfrak{A}'$ and $\mathfrak{C}$, so

$$\forall w \in W^{\mathfrak{A}'} \setminus \{x'\}\ \forall \chi \in \mathcal{L}\colon \mathfrak{A}', w \vDash \chi \text{ iff } \mathfrak{C}, w \vDash \chi. \tag{2}$$

Similarly, the identity relation on $W^{\mathfrak{B}}$ is a bisimulation between $\mathfrak{B}$ and $\mathfrak{C}$, so $\forall w \in W^{\mathfrak{B}}\ \forall \chi \in \mathcal{L}\colon \mathfrak{B}, w \vDash \chi$ iff $\mathfrak{C}, w \vDash \chi$. Given $\mathfrak{B}, y \vDash \mathsf{f}^{\mathbf{L}}_a(\delta) \wedge \neg\psi$, it follows that $\mathfrak{C}, y \vDash \mathsf{f}^{\mathbf{L}}_a(\delta) \wedge \neg\psi$. Then since $x'\mathrm{R}^{\mathfrak{C}}_a y$, we have $\mathfrak{C}, x' \vDash \Diamond_a\neg\psi$.

Now we claim that given $\mathfrak{A}', x' \vDash \delta$, also $\mathfrak{C}, x' \vDash \delta$. Recall that δ is a conjunction that has as conjuncts zero or more propositional formulas α, formulas of the form $\Box_b\beta$, and formulas of the form $\Diamond_b\gamma$ for various $b \in I$, including a. The propositional part of δ is still true at x' in $\mathfrak{C}$, since the valuation on x' has not changed from $\mathfrak{A}'$ to $\mathfrak{C}$. For the modal parts, we use the following facts:[18]

$$\mathrm{R}^{\mathfrak{C}}_b[x'] = \mathrm{R}^{\mathfrak{A}'}_b[x'] \text{ for all } b \in I \setminus \{a\}; \tag{3}$$

$$\mathrm{R}^{\mathfrak{C}}_a[x'] = \mathrm{R}^{\mathfrak{A}'}_a[x'] \cup \{y\}. \tag{4}$$

For any $j \in I$ and conjunct of δ of the form $\Diamond_j\gamma$, given $\mathfrak{A}', x' \vDash \Diamond_j\gamma$, there is a $v \in \mathrm{R}^{\mathfrak{A}'}_j[x']$ such that $\mathfrak{A}', v \vDash \gamma$, which implies $\mathfrak{C}, v \vDash \gamma$ by (2), given $x' \notin \mathrm{R}^{\mathfrak{A}'}_j[x']$. Then since $\mathrm{R}^{\mathfrak{A}'}_j[x'] \subseteq \mathrm{R}^{\mathfrak{C}}_j[x']$ by (3) and (4), $\mathfrak{C}, x' \vDash \Diamond_j\gamma$.

For any $b \in I \setminus \{a\}$ and conjunct of δ of the form $\Box_b\beta$, given $\mathfrak{A}', x' \vDash \Box_b\beta$, we have that for all $v \in \mathrm{R}^{\mathfrak{A}'}_b[x']$, $\mathfrak{A}', v \vDash \beta$, which implies $\mathfrak{C}, v \vDash \beta$ by (2), given $x' \notin \mathrm{R}^{\mathfrak{A}'}_b[x']$. Then since $\mathrm{R}^{\mathfrak{C}}_b[x'] \subseteq \mathrm{R}^{\mathfrak{A}'}_b[x']$ by (3), $\mathfrak{C}, x' \vDash \Box_b\beta$.

Finally, for any conjunct of δ of the form $\Box_a\beta$, given $\mathfrak{A}', x' \vDash \Box_a\beta$, we have that for all $v \in \mathrm{R}^{\mathfrak{A}'}_a[x']$, $\mathfrak{A}', v \vDash \beta$, which implies $\mathfrak{C}, v \vDash \beta$ by (2), given $x' \notin \mathrm{R}^{\mathfrak{A}'}_a[x']$. Now since $\Box_a\beta$ is a conjunct of δ, β is a conjunct of $\mathsf{f}^{\mathbf{L}}_a(\delta)$, so given $\mathfrak{C}, y \vDash \mathsf{f}^{\mathbf{L}}_a(\delta)$, we have $\mathfrak{C}, y \vDash \beta$. Combining this with the fact that for all $v \in \mathrm{R}^{\mathfrak{A}'}_a[x']$, $\mathfrak{C}, v \vDash \beta$, it follows by (4) that $\mathfrak{C}, x' \vDash \Box_a\beta$. Thus, $\mathfrak{C}, x' \vDash \delta$.

Putting together the previous arguments, we have shown $\mathfrak{C}, x' \vDash \delta \wedge \Diamond_a\neg\psi$ and hence $\mathfrak{C}, x' \vDash \varphi \wedge \Diamond_a\neg\psi$. Now if $\mathbf{L}$ is $\mathbf{K}$ (resp. $\mathbf{KD}$), then given that $\mathfrak{C} \in \mathsf{C}_{\mathbf{K}}$ (resp. $\mathfrak{C} \in \mathsf{C}_{\mathbf{KD}}$) by construction, it follows by soundness that $\nvdash_{\mathbf{L}} \varphi \to \Box_a\psi$. This completes the proof of the theorem for $\mathbf{K}$ and $\mathbf{KD}$.

Now for $\mathbf{L} \in \{\mathbf{T}, \mathbf{K4}, \mathbf{KD4}, \mathbf{S4}\}$, define $\mathfrak{C}_{\mathbf{L}}$ to be exactly like $\mathfrak{C}$ except that for every $b \in I$, $\mathrm{R}^{\mathfrak{C}_{\mathbf{L}}}_b$ is the *reflexive* and/or *transitive closure* of $\mathrm{R}^{\mathfrak{C}}_b$, depending on whether T and/or 4 are axioms of $\mathbf{L}$.[19] Thus, $\mathfrak{C}_{\mathbf{L}} \in \mathsf{C}_{\mathbf{L}}$. For example, see $\mathfrak{C}_{\mathbf{K4}}$ at the bottom of Fig. 2. Now we must check that we still have $\mathfrak{C}_{\mathbf{L}}, x' \vDash \delta$ and $\mathfrak{C}_{\mathbf{L}}, y \vDash \mathsf{f}^{\mathbf{L}}_a(\delta) \wedge \neg\psi$. Since $W^{\mathfrak{C}} = W^{\mathfrak{C}_{\mathbf{L}}}$ and $\forall z \in W^{\mathfrak{C}} \setminus \{x'\}\colon x' \notin \mathrm{R}^{\mathfrak{C}_{\mathbf{L}}}_b[z]$, the identity relation on $W^{\mathfrak{C}} \setminus \{x'\}$ is a bisimulation between $\mathfrak{C}$ and $\mathfrak{C}_{\mathbf{L}}$, so

$$\forall w \in W^{\mathfrak{C}} \setminus \{x'\}\ \forall \chi \in \mathcal{L}\colon \mathfrak{C}, w \vDash \chi \text{ iff } \mathfrak{C}_{\mathbf{L}}, w \vDash \chi. \tag{5}$$

[18] For a world model $\mathfrak{M}$, $w \in W^{\mathfrak{M}}$, and $i \in I$, let $\mathrm{R}^{\mathfrak{M}}_i[w] = \{v \in W^{\mathfrak{M}} \mid w\mathrm{R}_i v\}$.

[19] Note that for $b \neq a$, the transitive closure of $\mathrm{R}^{\mathfrak{C}}_b$ is just $\mathrm{R}^{\mathfrak{C}}_b$ itself. However, the reflexive closure of $\mathrm{R}^{\mathfrak{C}}_b$ is not $\mathrm{R}^{\mathfrak{C}}_b$ itself, because we do not have $x'\mathrm{R}^{\mathfrak{C}}_b x'$.

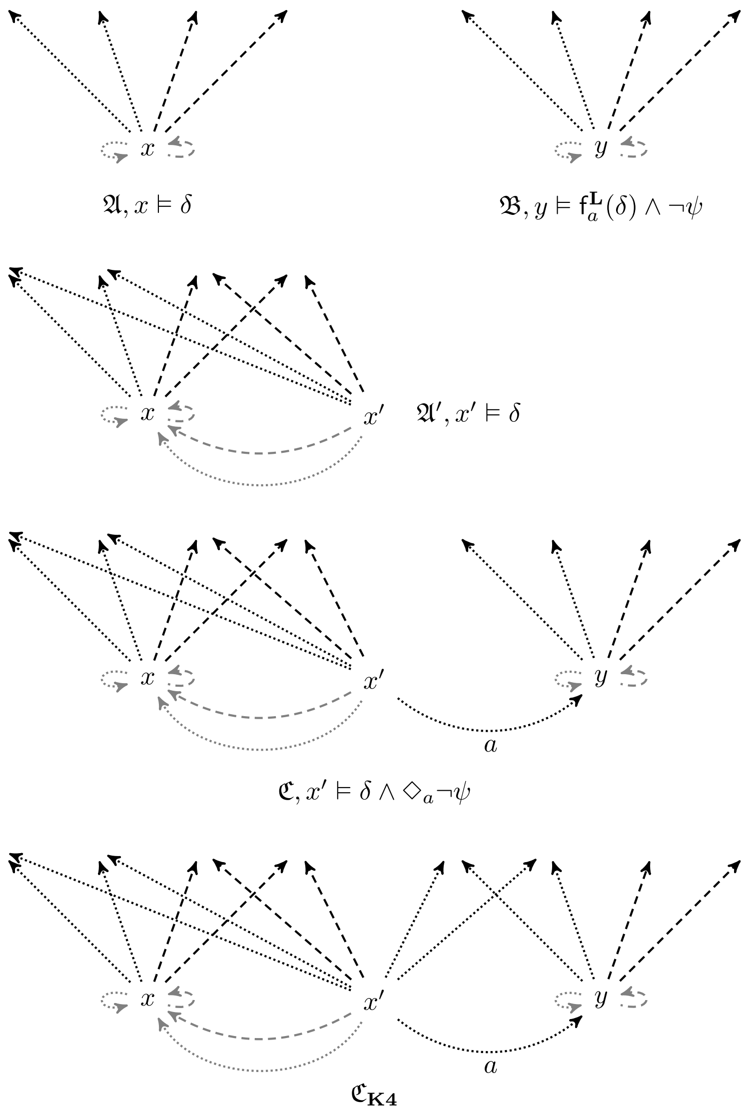

Fig. 2. models $\mathfrak{A}$ (upper left), $\mathfrak{B}$ (upper right), $\mathfrak{A}'$ (below $\mathfrak{A}$), $\mathfrak{C}$ (below $\mathfrak{A}'$), and $\mathfrak{C}_{\mathbf{K4}}$ (below $\mathfrak{C}$). Gray arrows might be included, depending on the initial models $\mathfrak{A}$ and $\mathfrak{B}$.

Thus, $\mathfrak{C}, y \vDash \mathsf{f}_a^{\mathbf{L}}(\delta) \wedge \neg\psi$ implies $\mathfrak{C}_{\mathbf{L}}, y \vDash \mathsf{f}_a^{\mathbf{L}}(\delta) \wedge \neg\psi$. It remains to show $\mathfrak{C}_{\mathbf{L}}, x' \vDash \delta$.

The propositional part of δ is still true at x' in $\mathfrak{C}_{\mathbf{L}}$, since the valuation on x' has not changed from $\mathfrak{C}$ to $\mathfrak{C}_{\mathbf{L}}$. From (5) and the fact that for every $j \in I$, $x' \notin \mathrm{R}_j^{\mathfrak{C}}[x']$ and $\mathrm{R}_j^{\mathfrak{C}}[x'] \subseteq \mathrm{R}_j^{\mathfrak{C}_{\mathbf{L}}}[x']$, it follows that the conjuncts of δ of the form $\Diamond_j \gamma$ are still true at x'. We need only check that every conjunct of δ of the form $\Box_j \beta$ is still true at x'. The argument depends on the choice of $\mathbf{L}$.

Let us begin with $\mathbf{T}$, so we can assume by Lemma 3.6 that φ is T-unpacked. Since for all $j \in I$, $\mathrm{R}_j^{\mathfrak{C}_\mathbf{T}}$ is the reflexive closure of $\mathrm{R}_j^{\mathfrak{C}}$, it follows that

$$\mathrm{R}_j^{\mathfrak{C}_\mathbf{T}}[x'] = \mathrm{R}_j^{\mathfrak{C}}[x'] \cup \{x'\}. \tag{6}$$

For any $j \in I$ and conjunct of δ of the form $\Box_j\beta$, given $\mathfrak{C}, x' \vDash \Box_j\beta$, we have that for all $v \in \mathrm{R}_j^{\mathfrak{C}}[x']$, $\mathfrak{C}, v \vDash \beta$. It follows given (5) and $x' \notin \mathrm{R}_j^{\mathfrak{C}}[x']$ that

$$\text{for any } j \in I, \text{ conjunct of } \delta \text{ of the form } \Box_j\beta, \text{ and } v \in \mathrm{R}_j^{\mathfrak{C}}[x']\colon \mathfrak{C}_\mathbf{T}, v \vDash \beta. \tag{7}$$

Thus, by (6), to show $\mathfrak{C}_\mathbf{T}, x' \vDash \Box_j\beta$ it only remains to show that $\mathfrak{C}_\mathbf{T}, x' \vDash \beta$. Since φ is T-unpacked, for each $\Box_j\beta$ conjunct of δ, there is some disjunct δ_β of β such that every conjunct of δ_β is a conjunct of δ. Given this fact, we can prove by induction on the modal depth $d(\beta)$ of β that $\mathfrak{C}_\mathbf{T}, x' \vDash \beta$.

If $d(\beta) = 0$, so β is propositional, then δ_β is a propositional conjunct of δ, so $\mathfrak{C}, x' \vDash \delta$ implies $\mathfrak{C}, x' \vDash \delta_\beta$, which implies $\mathfrak{C}_\mathbf{T}, x' \vDash \delta_\beta$, since δ_β is propositional, which implies $\mathfrak{C}_\mathbf{T}, x' \vDash \beta$, since δ_β is a disjunct of β.

If $d(\beta) = n + 1$, then by the inductive hypothesis, for every $\Box_j\chi$ conjunct of δ with $d(\chi) \leq n$, $\mathfrak{C}_\mathbf{T}, x' \vDash \chi$. Since φ is T-unpacked, there is a disjunct δ_β of β such that every conjunct of δ_β is a conjunct of δ. As shown above, every propositional conjunct of δ is true at $\mathfrak{C}_\mathbf{T}, x'$, and every conjunct of δ of the form $\Diamond_j\gamma$ is true at $\mathfrak{C}_\mathbf{T}, x'$, so every propositional conjunct of δ_β and every conjunct of δ_β of the form $\Diamond_j\gamma$ is true at $\mathfrak{C}_\mathbf{T}, x'$. Thus, to establish $\mathfrak{C}_\mathbf{T}, x' \vDash \delta_\beta$, it only remains to show that every conjunct of δ_β of the form $\Box_j\chi$ is true at $\mathfrak{C}_\mathbf{T}, x'$. Since $d(\beta) = n + 1$ and $\Box_j\chi$ is a conjunct of δ_β, $d(\chi) \leq n$, so by the inductive hypothesis, $\mathfrak{C}_\mathbf{T}, x' \vDash \chi$; and since $\Box_j\chi$ is a conjunct of δ, we have from (7) that for all $v \in \mathrm{R}_j^{\mathfrak{C}}[x'] = \mathrm{R}_j^{\mathfrak{C}_\mathbf{T}}[x'] \setminus \{x'\}$, $\mathfrak{C}_\mathbf{T}, v \vDash \chi$. Putting these two facts together, it follows from (6) that $\mathfrak{C}_\mathbf{T}, x' \vDash \Box_j\chi$. This completes the proof of $\mathfrak{C}_\mathbf{T}, x' \vDash \delta_\beta$ and hence $\mathfrak{C}_\mathbf{T}, x' \vDash \beta$, which is all that was left to show $\mathfrak{C}_\mathbf{T}, x' \vDash \Box_j\beta$.

Let us now show for **K4**/**KD4** that every conjunct of δ of the form $\Box_j\beta$ is true at x'. Since for all $j \in I$, $\mathrm{R}_j^{\mathfrak{C}_\mathbf{K4}}$ is the transitive closure of $\mathrm{R}_j^{\mathfrak{C}}$, we have:

$$\mathrm{R}_b^{\mathfrak{C}_\mathbf{K4}}[x'] = \mathrm{R}_b^{\mathfrak{C}}[x'] \text{ for } b \in I \setminus \{a\}; \tag{8}$$

$$\mathrm{R}_a^{\mathfrak{C}_\mathbf{K4}}[x'] = \mathrm{R}_a^{\mathfrak{C}}[x'] \cup \mathrm{R}_a^{\mathfrak{C}}[y]. \tag{9}$$

For any $b \in I \setminus \{a\}$ and conjunct of δ of the form $\Box_b\beta$, given $\mathfrak{C}, x' \vDash \Box_b\beta$, we have that for all $v \in \mathrm{R}_b^{\mathfrak{C}}[x']$, $\mathfrak{C}, v \vDash \beta$, which implies $\mathfrak{C}_\mathbf{K4}, v \vDash \beta$ by (5), given $x' \notin \mathrm{R}_b^{\mathfrak{C}}[x']$. Then by (8), $\mathfrak{C}_\mathbf{K4}, x' \vDash \Box_b\beta$.

For any conjunct of δ of the form $\Box_a\beta$, given $\mathfrak{C}, x' \vDash \Box_a\beta$, we have that for all $v \in \mathrm{R}_a^{\mathfrak{C}}[x']$, $\mathfrak{C}, v \vDash \beta$, which implies $\mathfrak{C}_\mathbf{K4}, v \vDash \beta$ by (5), given $x' \notin \mathrm{R}_a^{\mathfrak{C}}[x']$. Now since $\Box_a\beta$ is a conjunct of δ, $\Box_a\beta$ is also a conjunct of $\mathsf{f}_a^{\mathbf{K4}}(\delta)$, so given $\mathfrak{C}, y \vDash \mathsf{f}_a^{\mathbf{K4}}(\delta)$, we have $\mathfrak{C}, y \vDash \Box_a\beta$. Thus, for all $u \in \mathrm{R}_a^{\mathfrak{C}}[y]$, $\mathfrak{C}, u \vDash \beta$, which implies $\mathfrak{C}_\mathbf{K4}, u \vDash \beta$ by (5), given $x' \notin \mathrm{R}_a^{\mathfrak{C}}[y]$. Combining this with the fact that for all $v \in \mathrm{R}_a^{\mathfrak{C}}[x']$, $\mathfrak{C}_\mathbf{K4}, v \vDash \beta$, it follows by (9) that $\mathfrak{C}_\mathbf{K4}, x' \vDash \Box_a\beta$.

This completes the proof for **K4**, and the same applies to **KD4**.

For **S4**, a combination of the arguments above for **KT** and **K4** works.

We have now shown that for all $\mathbf{L} \in \{$**K**, **KD**, **T**, **K4**, **KD4**, **S4**$\}$, there is a model $\mathfrak{M} \in \mathsf{C}_\mathbf{L}$ (i.e., $\mathfrak{C}$ for **K**/**KD** or $\mathfrak{C}_\mathbf{L}$ for the others) such that $\mathfrak{M}, x' \vDash \delta$,

$\mathfrak{M}, y \vDash \neg\psi$, and $x' \mathrm{R}_a^{\mathfrak{M}} y$, which implies $\mathfrak{M}, x' \vDash \varphi \wedge \Diamond_a \neg\psi$. Then since $\mathbf{L}$ is sound with respect to $\mathsf{C}_\mathbf{L}$ and $\mathfrak{M} \in \mathsf{C}_\mathbf{L}$, it follows that $\nvdash_\mathbf{L} \varphi \to \Box_a \psi$. $\Box$

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