On dynamic topological and metric logics

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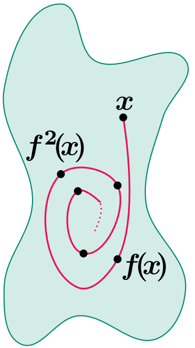
joint work with

Boris Konev, Frank Wolter and Michael Zakharyaschev

Dynamic systems

'space' + f



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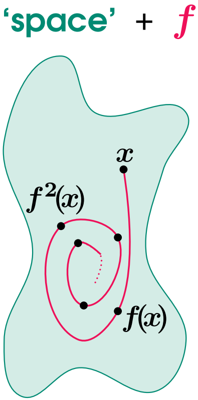

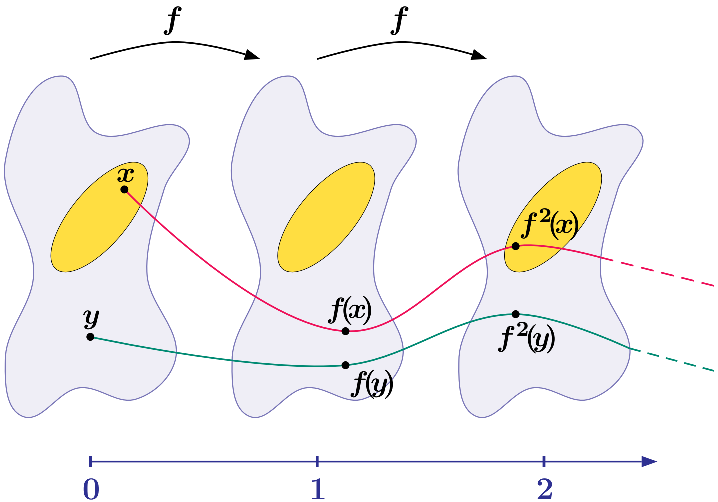

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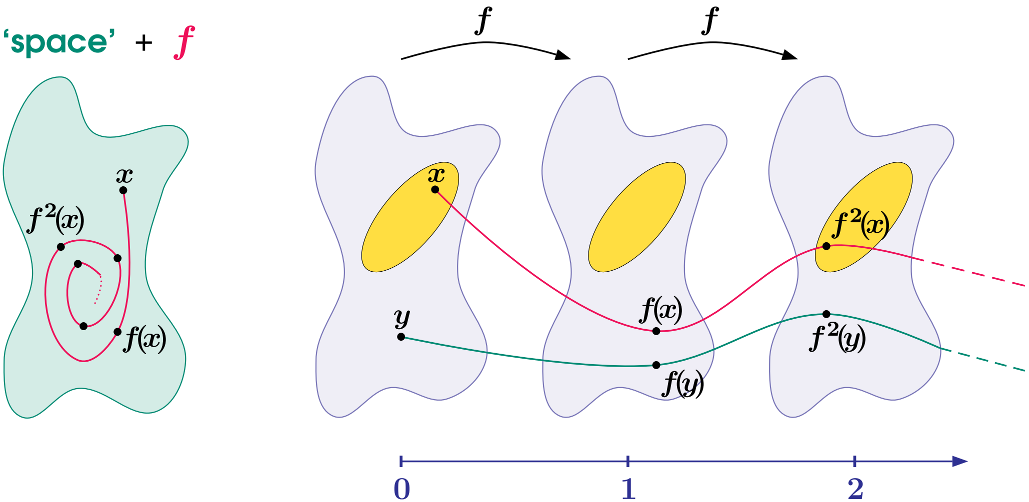

Temporal logic to describe and reason about behaviour of dynamic systems:

- variables p are interpreted by sets of points, i.e., point x is in p: $x \in p$
- x always stays in p: $x \in \Box_F p$
- x occurs in p infinitely often: $x \in \Box_F \Diamond_F p$
- ...

Dynamic topological logic

Dynamic topological structure $\mathfrak{F} = \langle \mathfrak{T}, f \rangle$

$\mathfrak{T} = \langle T, \mathbb{I} \rangle$ a *topological space*

T is the universe of $\mathfrak{T}$
$\mathbb{I}$ is the interior operator on $\mathfrak{T}$
$\mathbb{C}$ is the closure operator on $\mathfrak{T}$
$(\mathbb{C}X = -\mathbb{I} - X)$

$f : T \to T$ a *total continuous function* (X open $\Rightarrow f^{-1}(X)$ open)

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$$\mathfrak{V}(\Box_F\varphi) = \bigcap_{n=1}^{\infty} f^{-n}(\mathfrak{V}(\varphi)) \quad \text{and} \quad \mathfrak{V}(\Diamond_F\varphi) = \bigcup_{n=1}^{\infty} f^{-n}(\mathfrak{V}(\varphi))$$

Classes of dynamic topological structures

Topological spaces $\mathfrak{T} = \langle T, \mathbb{I} \rangle$

- arbitrary topologies
- Aleksandrov: **arbitrary** (not only finite) intersections of open sets are open

  — every Kripke frame $\mathfrak{G} = \langle U, R \rangle$, where R is a **quasi-order**,
  induces the Aleksandrov topological space $\langle U, \mathbb{I}_{\mathfrak{G}} \rangle$:

  $$\mathbb{I}_{\mathfrak{G}} X = \{x \in U \mid \forall y \, (xRy \to y \in X)\}$$

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Functions $f : T \to T$

- continuous
- homeomorphisms: continuous bijections with continuous inverses
- ...

Known results

$\mathcal{DTL}_{\bigcirc}$ — subset of $\mathcal{DTL}$ containing **no 'infinite'** operators ($\Box_F$ and $\Diamond_F$)

Artemov, Davoren & Nerode (1997): The two dynamic topo-logics

$$\mathrm{Log}_{\bigcirc}\{\langle \mathfrak{F}, f\rangle\} \qquad \text{and} \qquad \mathrm{Log}_{\bigcirc}\{\langle \mathfrak{F}, f\rangle \mid \mathfrak{F} \text{ an Aleksandrov space}\}$$

coincide, have the **fmp**, are finitely **axiomatisable**, and so decidable.

NB. $\mathrm{Log}_{\bigcirc}\{\langle \mathfrak{F}, f\rangle\} \subsetneq \mathrm{Log}_{\bigcirc}\{\langle \mathbb{R}, f\rangle\}$ (Slavnov 2003, Kremer & Mints 2003)

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Open problem: axiomatisations and algorithmic properties of the **full $\mathcal{DTL}$?**

Homeomorphisms: bad news

Theorem 1. **No logic** from the list below is **recursively enumerable**:

- Log $\{\langle \mathfrak{F}, f \rangle \mid f$ a homeomorphism$\}$,
- Log $\{\langle \mathfrak{F}, f \rangle \mid \mathfrak{F}$ an Aleksandrov space, f a homeomorphism$\}$,
- Log $\{\langle \mathbb{R}^n, f \rangle \mid f$ a homeomorphism, $n \geq 1\}$.

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NB. All these logics are **different**.

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Finite iterations:

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Theorem 2. The two topo-logics

$$\mathrm{Log}^* \{\langle \mathfrak{F}, f\rangle\} \quad \text{and} \quad \mathrm{Log}^* \{\langle \mathfrak{F}, f\rangle \mid \mathfrak{F} \text{ an Aleksandrov space}\}$$

coincide and are **decidable**, but **not in primitive recursive** time.

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<u>However:</u>

Theorem 3. The two topo-logics

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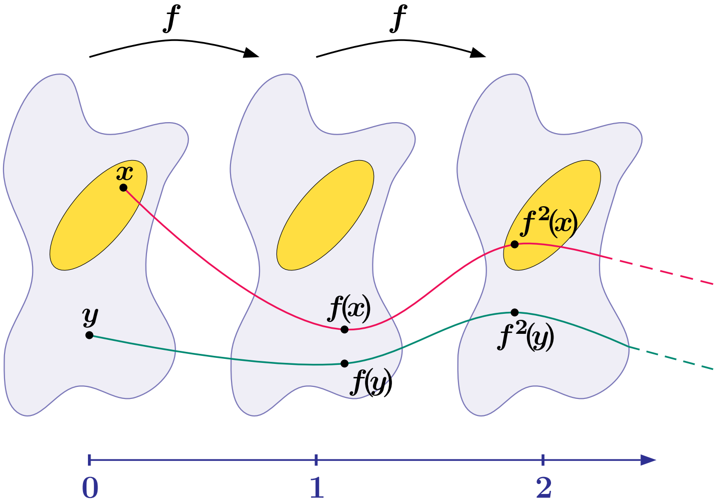
coincide but are **not recursively enumerable**.

Dynamics in metric spaces



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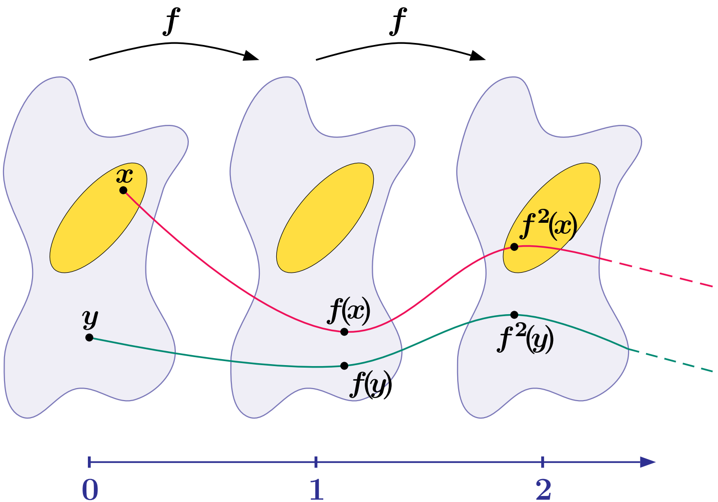

A **metric space** $\mathfrak{D} = \langle W, d \rangle$, where $d \colon W \times W \to \mathbb{R}^+$ is a metric, induces the **topological space** $\mathfrak{T}_d = \langle W, \mathbb{I}_d \rangle$:

$$\mathbb{I}_d X = \{x \in W \mid \exists \delta > 0 \, \forall y \in W \; (d(x, y) < \delta \;\; \to \;\; y \in X)\}$$

Logics of metric spaces

$\mathfrak{D} = \langle W, d \rangle$ a *metric space*

- propositional variables $p, q, \ldots$
- the Booleans $\neg$, $\wedge$ and $\vee$
- interior **I** and closure **C** operators
- universal $\forall$ and existential $\exists$ modalities
- metric operators $\exists^{\leq a}$ and $\forall^{\leq a}$, for $a \in \mathbb{Q}^+$

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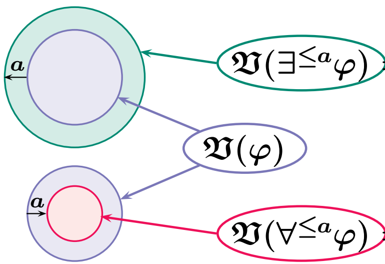

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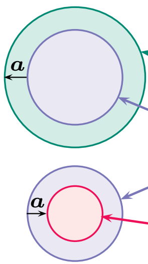

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Wolter & Zakharyaschev (2004):

The set of valid $\mathcal{MT}$-formulas is **axiomatisable**.

The satisfiability of $\mathcal{MT}$-formulas is **EXPTIME-complete**.

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Theorem 4. The set of valid $\mathcal{DML}$-formulas is **decidable**.

However, the decision problem is **not elementary**.

Proof. Quasimodels, reduction to monadic second-order logic and yardsticks.

Open problems and future research

The field still remains a big research challenge...

- Axiomatisation of decidable logics/fragments
- Various topological and metric spaces: Euclidean, compact, etc.
- functions: Lipschitz continuous, contracting maps, etc.
- Model checking
- ...