Topological logics with connectedness predicates

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joint work with

Ian Pratt-Hartmann, Frank Wolter and **Michael Zakharyaschev**

Topological logics

terms:

$$\tau \quad ::= \quad r_i \quad | \quad \mathbf{0} \quad | \quad \overline{\tau} \quad | \quad \tau_1 \cap \tau_2 \quad | \quad \tau_1 \cup \tau_2 \quad | \quad \tau^{\circ} \quad | \quad \tau^{-} \quad | \quad \dots$$

formulas:

$$\varphi \quad ::= \quad \tau_1 = \tau_2 \quad | \quad \tau_1 \subseteq \tau_2 \quad | \quad c(\tau) \quad | \quad \neg\varphi \quad | \quad \varphi_1 \wedge \varphi_2 \quad | \quad \dots$$

Topological logics

topological model $\mathfrak{M} = (\boldsymbol{T}, \cdot^{\mathfrak{M}})$
$\boldsymbol{T}$ a topological space
$\cdot^{\mathfrak{M}}$ a valuation

terms: **subsets of T**

$$\boldsymbol{\tau} \quad ::= \quad \boldsymbol{r_i} \quad | \quad \underset{\text{empty set}}{\mathbf{0}} \quad | \quad \underset{\text{complement}}{\overline{\boldsymbol{\tau}}} \quad | \quad \boldsymbol{\tau_1} \cap \boldsymbol{\tau_2} \quad | \quad \boldsymbol{\tau_1} \cup \boldsymbol{\tau_2} \quad | \quad \underset{\text{interior}}{\boldsymbol{\tau}^{\circ}} \quad | \quad \underset{\text{closure}}{\boldsymbol{\tau}^{-}} \quad | \quad \dots$$

formulas: **true** or **false**

$$\varphi \quad ::= \quad \boldsymbol{\tau_1} = \boldsymbol{\tau_2} \quad | \quad \boldsymbol{\tau_1} \subseteq \boldsymbol{\tau_2} \quad | \quad c(\boldsymbol{\tau}) \quad | \quad \neg\varphi \quad | \quad \varphi_1 \land \varphi_2 \quad | \quad \dots$$

e.g., $\mathfrak{M} \models \boldsymbol{\tau_1} = \boldsymbol{\tau_2}$ iff $\boldsymbol{\tau}_1^{\mathfrak{M}} = \boldsymbol{\tau}_2^{\mathfrak{M}}$

$\mathfrak{M} \models c(\boldsymbol{\tau})$ iff $\boldsymbol{\tau}^{\mathfrak{M}}$ is connected

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empty set complement interior closure

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Examples:

$$c(r_1) \ \wedge \ c(r_2) \ \wedge (r_1 \cap r_2 \neq \mathbf{0}) \ \rightarrow \ c(r_1 \cup r_2)$$

'the union of two intersecting connected sets r_1 and r_2 is connected'

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'if r_1 is a connected set, and r_2 is sandwiched between r_1 and its closure,
then r_2 is also connected'

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Let $\mathcal{L}$ a language with functions F and predicates P and $\mathcal{K}$ be a class of models

Sat$(\mathcal{L}, \mathcal{K})$ is the set of $\mathcal{L}$-formulas satisfiable in models over $\mathcal{K}$

$\mathcal{S}4_u$ as a topological logic

$\mathcal{S}4_u$-terms: $\quad \tau \quad ::= \quad r_i \mid \overline{\tau} \mid \tau_1 \cap \tau_2 \mid \tau_1 \cup \tau_2 \mid \tau^{\circ} \mid \tau^{-}$

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NB. This definition (although it does not allows nested universal modalities) is as expressive as the ‘standard’ one

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(Shehtman 99, Areces *et. al* 00): **Sat**$(\mathcal{S}4_u$, ALL$)$ $=$ **Sat**$(\mathcal{S}4_u$, ALEK$)$,
and this set is **PSPACE**-complete

(Aleksandrov spaces = quasi-ordered Kripke frames)

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NB. **Sat**$(\mathcal{S}4_u, \text{ALL}) \neq$ **Sat**$(\mathcal{S}4_u, \mathbb{R}^n)$ (in contrast with $\mathcal{S}4$)

Example:

$$(r_1 \neq \mathbf{0}) \;\wedge\; (r_2 \neq \mathbf{0}) \;\wedge\; (r_1 \cup r_2 = \mathbf{1}) \;\wedge\; (r_1^- \cap r_2 = \mathbf{0}) \;\wedge\; (r_1 \cap r_2^- = \mathbf{0})$$

is satisfiable in a topological space T iff T is not connected

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is satisfiable in a topological space T iff T is not connected

but **Sat**$(\mathcal{S}4_u, \mathbb{R}^n) =$ **Sat**$(\mathcal{S}4_u, \textsc{Con}) =$ **Sat**$(\mathcal{S}4_u, \textsc{Con} \cap \textsc{Alek})$ and this set is **PSPACE**-complete

$\mathcal{S}4_u$ over connected topological spaces

Aleksandrov spaces = quasi-ordered Kripke frames

connectedness = connectedness in the **undirected** graph
(induced by the quasi-order)

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Example: generating all numbers from $\mathbf{0}$ to $\mathbf{2}^n - \mathbf{1}$:

- $\mathbf{0}$ and $\mathbf{2}^n - \mathbf{1}$ are non-empty:

$$\overline{v_n} \cap \dots \cap \overline{v_1} \neq \mathbf{0} \qquad\qquad v_n \cap \dots \cap v_1 \neq \mathbf{0}$$

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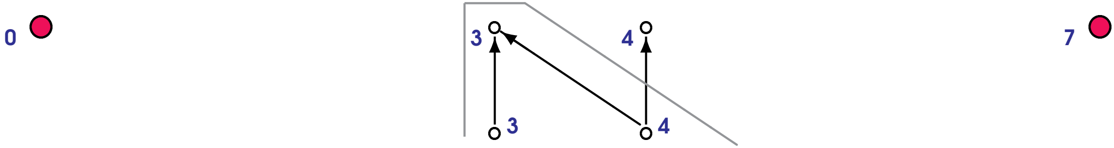

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- the closure of $\boldsymbol{m}$ can share points only with $\boldsymbol{m} + \mathbf{1}$, for $\mathbf{0} \leq m < \mathbf{2}^n - \mathbf{1}$:

$$(v_j \cap \overline{v_k})^- \subseteq v_j, \qquad (\overline{v_j} \cap \overline{v_k})^- \subseteq \overline{v_j}, \qquad \text{for } n \geq j > k \geq 1$$

$$(\overline{v_k} \cap v_{k-1} \cap \cdots \cap v_1)^- \subseteq (v_k \cap \overline{v_i}) \cup (\overline{v_k} \cap v_i), \qquad \text{for } n \geq \quad k > i \geq 1$$

$S4_u$ over connected topological spaces

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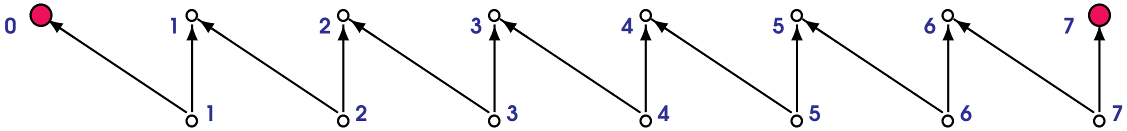

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- $\mathbf{2}^n - \mathbf{1}$ is a closed set (and thus its closure shares no points with **0**):

$$(v_n \cap \dots \cap v_1)^- \subseteq v_n \cap \dots \cap v_1$$

$\mathcal{S}4_u c$ = $\mathcal{S}4_u$ + connectedness predicate (1)

$\mathcal{S}4_u c$-terms: $\tau \quad ::= \quad \mathcal{S}4_u$-terms

$\mathcal{S}4_u c$-formulas: $\varphi \quad ::= \quad \tau_1 = \tau_2 \quad | \quad c(\tau) \quad | \quad \neg\varphi \quad | \quad \varphi_1 \wedge \varphi_2 \quad | \quad \varphi_1 \wedge \varphi_2$

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$\downarrow$ one occurrence of c

Theorem. **Sat**$(\mathcal{S}4_u c^1, \textsc{All})$ is **PSPACE**-complete

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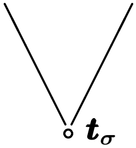
Theorem. **Sat**$(\mathcal{S}4_u c^1, \textsc{All})$ is **PSpace**-complete

Proof. Let $\psi = (\tau_0 = \mathbf{0}) \wedge \bigwedge_{i=1}^{m} (\tau_i \neq \mathbf{0}) \wedge \big(c(\sigma) \wedge (\sigma \neq \mathbf{0})\big)$ (conjunct of a full DNF)

1. guess a type (Hintikka set) t_σ containing σ and $\overline{\tau_0}^{\circ}$
   and expand the tableau branch by branch (all points with σ are to be connected to t_σ)



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Theorem. **Sat**$(\mathcal{S}4_u c^1$, ALL) is PSPACE-complete

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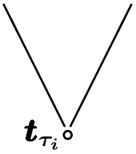

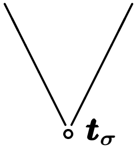

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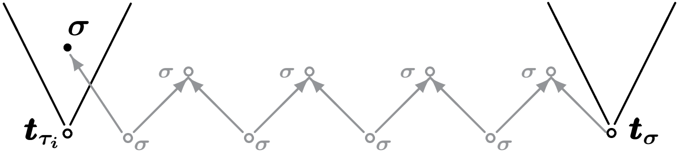
– if σ appears in the tableau
then we construct a path to t_σ
(by "divide and conquer")



path of length $\mathbf{2}^{|\psi|}$

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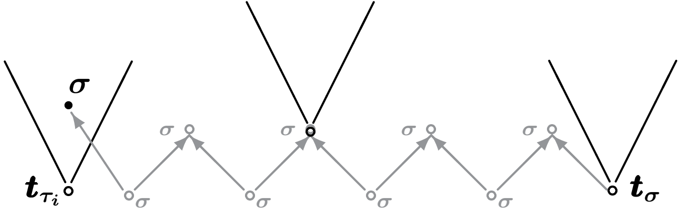

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Theorem. **Sat**$(\mathcal{S}4_u c^1,$ ALL$)$ is **PSPACE**-complete

Proof. Let $\psi = (\tau_0 = \mathbf{0}) \wedge \bigwedge_{i=1}^{m} (\tau_i \neq \mathbf{0}) \wedge \big(c(\sigma) \wedge (\sigma \neq \mathbf{0})\big)$ (conjunct of a full DNF)

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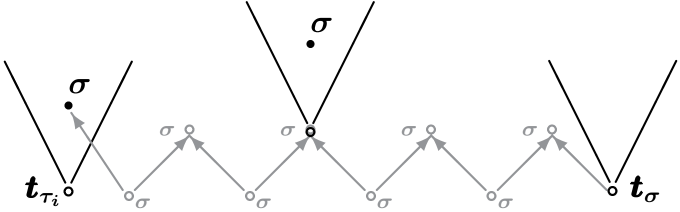

path of length $2^{|\psi|}$

$\mathcal{S}4_u c$ = $\mathcal{S}4_u$ + connectedness predicate (2)

Theorem. **Sat**$(\mathcal{S}4_u c$, ALL) is EXPTIME-complete

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Theorem. **Sat**$(\mathcal{S}4_u c$, ALL) is **EXPTIME**-complete

Proof. (upper bound)

Let $\psi = (\tau_0 = \mathbf{0}) \wedge \bigwedge_{i=1}^{m} (\tau_i \neq \mathbf{0}) \wedge \bigwedge_{i=1}^{k} \big(c(\sigma_i) \wedge (\sigma_i \neq \mathbf{0})\big)$ (conjunct of a full DNF)

The proof is by reduction to $\mathcal{PDL}$ with converse and nominals **(De Giacomo 95)**

Let α and β be atomic programs and ℓ_i a nominal, for each σ_i

- the $\mathcal{S}4$-box is simulated by $[\alpha^*]$:

  $\tau^\dagger$ is the result of replacing in τ each sub-term ϑ° with $[\alpha^*]\vartheta$

- the universal box is simulated by $[\gamma]$, where $\gamma = (\beta \cup \beta^- \cup \alpha \cup \alpha^-)^*$

$\mathcal{S}4_u c$ = $\mathcal{S}4_u$ + connectedness predicate (2)

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  $\tau^\dagger$ is the result of replacing in τ each sub-term ϑ° with $[\alpha^*]\vartheta$

- the universal box is simulated by $[\gamma]$, where $\gamma = (\beta \cup \beta^- \cup \alpha \cup \alpha^-)^*$

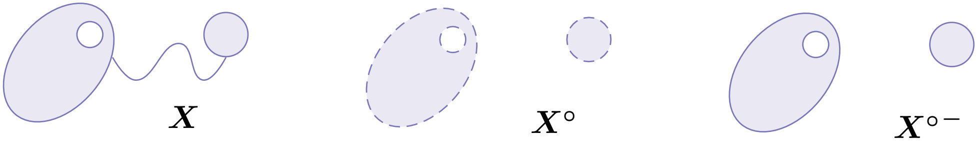
$$\psi' = [\gamma]\neg\tau_0^\dagger \ \wedge \ \bigwedge_{i=1}^{m} \langle\gamma\rangle\tau_i^\dagger \ \wedge \ \bigwedge_{i=1}^{k} \Big(\langle\gamma\rangle(\ell_i \wedge \sigma_i^\dagger) \ \wedge \ [\gamma](\sigma_i^\dagger \to \langle(\alpha \cup \alpha^-; \sigma_i^\dagger?)^*\rangle \ell_i) \Big)$$

ψ' is satisfiable iff ψ is satisfiable

Regular closed sets and $\mathcal{B}$

$X \subseteq T$ is **regular closed** if $X = X^{\circ -}$

RC(T) = sets of the form $X^{\circ -}$, for $X \subseteq T$



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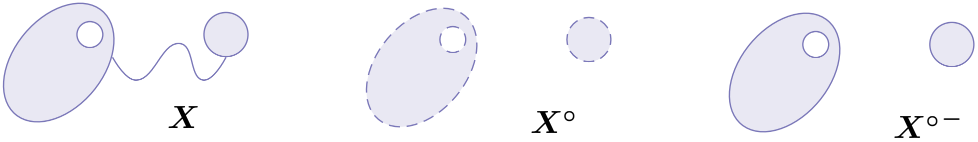

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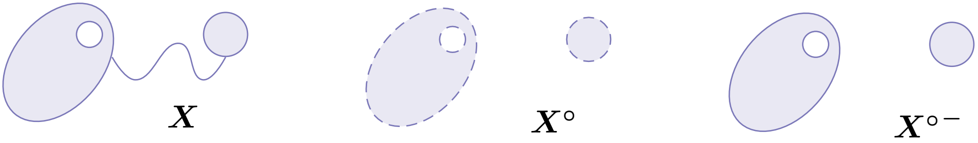

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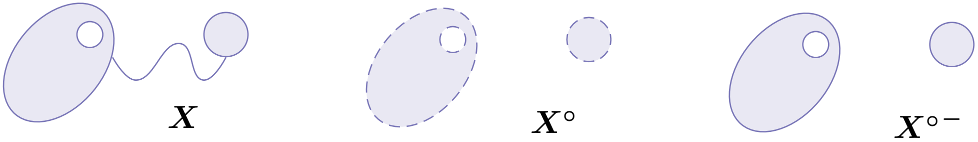

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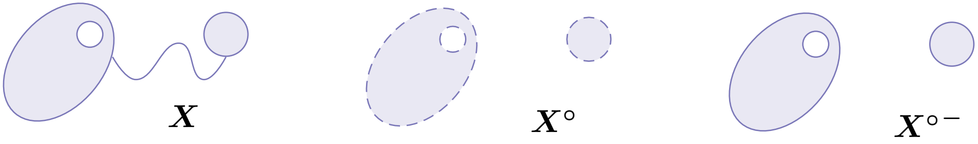

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no topology!

Regular closed sets and RCC-8

(Egenhofer & Franzosa, 91) and (Randell, Rui & Cohn, 92):

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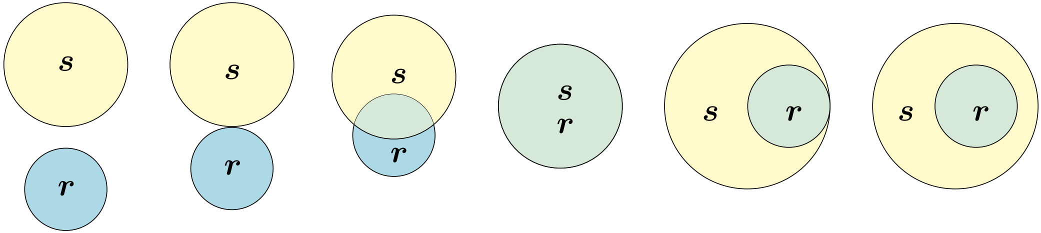

DC(r, s) **EC**(r, s) **PO**(r, s) **EQ**(r, s) **TPP**(r, s) **NTPP**(r, s)

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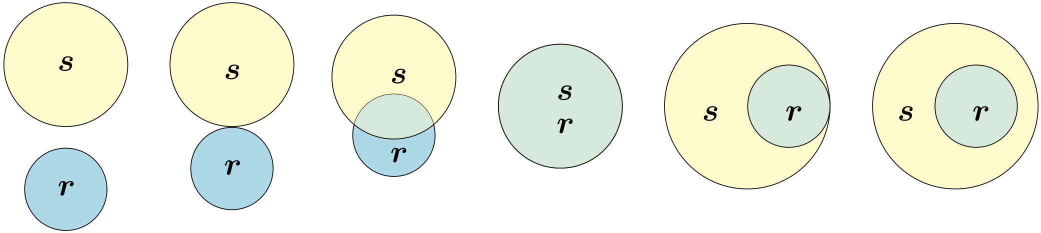

(Bennett 94): $\mathcal{RCC}$-8 is a **fragment** of $\mathcal{S}4_u$:

$r \cap s = \mathbf{0}$	$r \cdot s = \mathbf{0}$ $r \cap s \neq \mathbf{0}$	$\neg(r \subseteq s)$ $\neg(s \subseteq r)$ $r \cdot s \neq \mathbf{0}$	$r = s$	$r \subseteq s$ $r \cap (-s) \neq \mathbf{0}$ $\neg(s \subseteq r)$	$r \cap (-s) = \mathbf{0}$ $\neg(s \subseteq r)$

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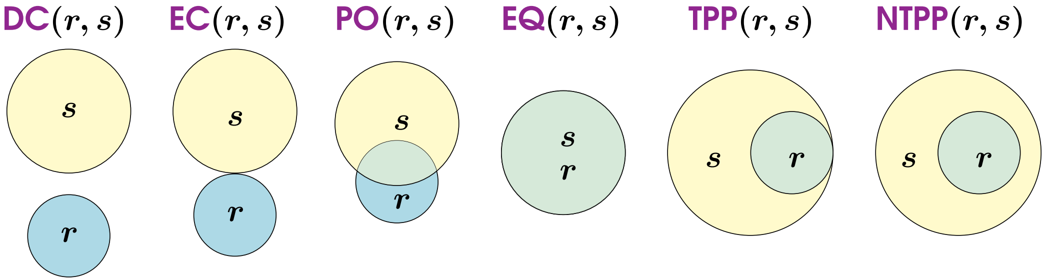

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Contact predicate

$\mathcal{C}$-terms: $\quad \tau \quad ::= \quad \mathcal{B}$-terms

$\downarrow$Whitehead's `connection' relation

$\mathcal{C}$-formulas: $\quad \varphi \quad ::= \quad \tau_1 = \tau_2 \quad | \quad C(\tau_1, \tau_2) \quad | \quad \neg\varphi \quad | \quad \varphi_1 \wedge \varphi_2$

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Theorem. **Sat**$(\mathcal{C}c, \textsc{Reg})$ is **ExpTime**-complete

Sat$(\mathcal{C}c,$ **RC**$(\mathbb{R}^n))$, $n \geq 2$, is **ExpTime**-hard

Proof. Hardness by reduction of the global consequence relation for the modal logic **K**

Reduction from $\mathcal{C}c$ to $\mathcal{B}c$

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$\mathcal{S}4_u c$ in Euclidean spaces

- satisfiable in $\mathbb{R}^2$ but not in $\mathbb{R}$:

$$\bigwedge_{1 \leq i \leq 3} c(r_i) \quad \wedge \bigwedge_{1 \leq i < j \leq 3} (r_i \cap r_j \neq \mathbf{0}) \quad \wedge \quad (r_1 \cap r_2 \cap r_3 = \mathbf{0})$$

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$$\bigwedge_{i\in\{j,k\}} (v_i \subseteq e^{\circ}_{j,k}) \ \wedge \bigwedge_{1\leq i\leq 5} (v_i \neq \mathbf{0}) \ \wedge \bigwedge_{\{i,j\}\cap\{k,l\}=\emptyset} (e_{i,j} \cap e_{k,l} = \mathbf{0}) \ \wedge \bigwedge_{1\leq i<j\leq 5} c(e^{\circ}_{i,j})$$

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Theorem. **Sat**$(\mathcal{S}4_u c, \mathbb{R})$ is **PSPACE**-complete

Proof. Embedding into temporal logic with $\mathcal{S}$ and $\mathcal{U}$ over $(\mathbb{R}, <)$, which is PSpace-complete **(Reynolds, 99)**

Summary of the results

<table>
<tr><th>language</th><th>Reg</th><th>ConReg</th><th>RC$(\mathbb{R}^n)$
$n > 2$</th><th>RC$(\mathbb{R}^2)$</th><th>RC$(\mathbb{R})$</th></tr>
<tr><td>$\mathcal{RCC}$-8</td><td colspan="5">NP</td></tr>
<tr><td>$\mathcal{RCC}$-8c</td><td colspan="3">NP</td><td>NP</td><td>≤PSpace,≥NP</td></tr>
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</table>

- Upper bounds for satisfiability over $\mathbb{R}^n$, $n > 1$, are not known (even decidability)
- Component counting predicates $c^{\leq k}(\tau)$: NExpTime instead of ExpTime
- k-contact relations $C^k(\tau_1, \dots, \tau_k)$ do not increase complexity

Infinite vs. finite number of components

$\mathbb{R}^1$: $\mathcal{RCC}$-$8c$-formula satisfiable over **RC**$(\mathbb{R})$ but not over **RCP**$(\mathbb{R})$

(**RCP**$(\mathbb{R}^n)$ = regular closed, semi-linear subsets of $\mathbb{R}^n$)

r_1 is connected and

any two of r_1, r_2, r_3, r_4 touch at their boundaries without overlapping:

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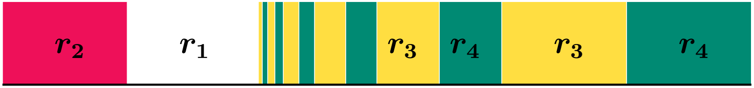

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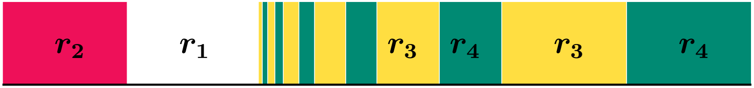

$\mathbb{R}^2$: **(Schaefer, Sedgwick & Štefankovič 03):** **Sat**$(\mathcal{RCC}\text{-}8, \mathcal{D}(\mathbb{R}^2))$ is **NP**-complete

($\mathcal{D}(\mathbb{R}^2)$ = closed disc-homeomorphs in $\mathbb{R}^2$)

Theorem. **Sat**$(\mathcal{RCC}\text{-}8c, \mathbf{RC}(\mathbb{R}^2))$ and **Sat**$(\mathcal{RCC}\text{-}8c, \mathbf{RCP}(\mathbb{R}^2))$ coincide,
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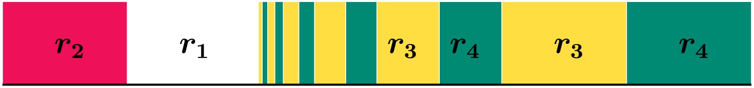

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language	**RC**$(\mathbb{R})$	**RCP**$(\mathbb{R})$	**RC**$(\mathbb{R}^2)$	**RCP**$(\mathbb{R}^2)$
$\mathcal{RCC}$-8c	$\leq$PSPACE,$\geq$NP	NP	NP	
$\mathcal{B}c$	NP		$\geq$PSPACE	$\geq$PSPACE
$\mathcal{C}c$	PSPACE	PSPACE	$\geq$EXPTIME	$\geq$EXPTIME

	$\mathbb{R}$	$\mathcal{S}(\mathbb{R})$	$\mathbb{R}^2$	$\mathcal{S}(\mathbb{R}^2)$
$\mathcal{S}4_u c$	PSPACE	PSPACE	$\geq$EXPTIME	$\geq$EXPTIME

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