BOOLEAN ROLE INCLUSIONS IN DL-LITE WITH AND WITHOUT TIME

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Roman Kontchakov[1], **Vladislav Ryzhikov**[1], **Frank Wolter**[2], **and Michael Zakharyaschev**[1]

[1]Department of Computer Science and Inf. Systems, Birkbeck, University of London, UK
[2]Department of Computer Science, University of Liverpool, UK
{roman,vlad,michael}@dcs.bbk.ac.uk, wolter@liverpool.ac.uk

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ABSTRACT

Traditionally, description logic has focused on representing and reasoning about classes rather than relations, which has been justified by the deterioration of the computational properties if expressive role inclusions are added. The situation is even worse in the temporalised setting, where monodicity is viewed as an almost necessary condition for decidability. In this paper, we take a fresh look at the description logic *DL-Lite* with expressive role inclusions, both with and without a temporal dimension. While we confirm that full Boolean expressive power on roles leads to FO^2-like behaviour in the atemporal case and undecidability in the temporal case, we show that, rather surprisingly, the restriction to Krom and Horn role inclusions leads to much lower complexity in the atemporal case and to decidability (and EXPSPACE-completeness) in the temporal case, even if one admits full Booleans on concepts. The latter result is one of very few instances breaking the monodicity barrier in temporal FO. This is also reflected at the data complexity level, where we obtain new FO(RPR)- and FO$(<, +)$-rewritability results.

1 Introduction

Description logics (DLs) have often been described as decidable fragments of first-order logic (FO) that model a domain by introducing complex concept descriptions and subsumptions between them. In fact, the main syntactic difference between DLs and FO is that, in the former, one can construct new, complex, concept descriptions from atomic concepts using concept constructors without the explicit use of individual variables. The subsumption relationship between complex concepts is then expressed using concept inclusions (CIs). Interestingly, corresponding role (binary relation) constructors taking as input atomic roles and describing complex roles have never become mainstream except for role composition, thus admitting role inclusions (RIs) of the form $R_1 \circ \cdots \circ R_n \sqsubseteq R$, with some appropriate restrictions [6]. The advantages of even a very limited form of Boolean expressivity on roles is well known [21, 25, 28, 29], so one can only speculate about the reasons for them not becoming more popular. The main issue appears to be that, from a computational perspective, adding Boolean operators on roles leads to expressivity similar to that of the two-variable fragment FO^2 of FO [24, 26], which, while still decidable, is significantly more challenging for automated reasoning than typical DL fragments of FO with some form of the tree model property [18, 31]. In temporal DLs, the addition of expressivity for roles is even more problematic: just declaring a role to remain constant in time often leads to undecidability [23, 15]. Again, the reason is well understood: if one goes beyond the monodic fragment of first-order temporal logic and is thus able to represent how relations change in time, one typically can encode the halting problem for Turing machines by using the relations to represent the tape and time to encode the computation [15].

Our aim here is to revisit Boolean RIs in the context of (temporal) *DL-Lite* and introduce logics with new expressivity for roles, for which the *knowledge base* (*KB*) *satisfiability problem* is decidable in the temporal case and of significantly lower complexity than FO^2 in the atemporal one.

Recall that in *DL-Lite*$_{\mathcal{R}}$ [12], also denoted *DL-Lite*$^{\mathcal{H}}_{core}$ in the classification of [1], CIs and RIs take the form of binary Horn (aka core) formulas $\vartheta_1 \sqsubseteq \vartheta_2$ or $\vartheta_1 \sqcap \vartheta_2 \sqsubseteq \bot$, where the ϑ_i are either both concepts (that is, concept names or of the form $\exists R$) or roles. The *DL-Lite* languages we consider extend this schema by allowing CIs and RIs of the form

$$\vartheta_1 \sqcap \cdots \sqcap \vartheta_k \ \sqsubseteq \ \vartheta_{k+1} \sqcup \cdots \sqcup \vartheta_{k+m}, \tag{1}$$

where the ϑ_i are all concepts or, respectively, roles. We classify ontologies by the form of these inclusions. Let $\boldsymbol{c}, \boldsymbol{r} \in \{bool, g\text{-}bool, horn, krom, core\}$. Then *DL-Lite*$^{\boldsymbol{r}}_{\boldsymbol{c}}$ is the DL whose ontologies contain CIs and RIs of the form (1) satisfying the following for $\boldsymbol{c}$ and $\boldsymbol{r}$, respectively:

(core) $k + m \leq 2$ and $m \leq 1$,

(horn) $m \leq 1$,

(krom) $k + m \leq 2$,

(g-bool) any $k \geq 1$ and $m \geq 0$,

(bool) any $k \geq 0$ and $m \geq 0$.

It follows that *core* is included in both *krom* and *horn*, which are in *bool* (*g-bool* stands for *guarded bool*). The resulting languages provide a new way of classifying ontologies. While the languages *DL-Lite*$^{bool}_{\boldsymbol{c}}$ all have essentially the same expressivity as FO^2 and inherit NEXPTIME-completeness of KB satisfiability, the *DL-Lite*$^{krom}_{\boldsymbol{c}}$ provide a way of introducing 'covering' RIs $\top \sqsubseteq R_1 \sqcup R_2$ and also the complement of a role via disjointness and covering. Rather surprisingly, these disjunctions come for free as far as the complexity of KB satisfiability is concerned: even combined with Boolean CIs, satisfiability is still in NP, and combined with Krom CIs, it is even in NL. The full table of our complexity results is given below.

role inclusions	concept inclusions (*g*-)*bool*	*krom*	*horn*	*core*
bool	NEXPTIME			
g-bool	EXPTIME			
krom	NP	NL	NP	NL
horn	NP	P	P	P
core	NP	NL	P	NL

Our main aim in this paper is to investigate extensions of these *DL-Lite* languages with the standard linear temporal logic (*LTL*) operators $\Box_F/\Box_P$ (always in the future/past) and $\bigcirc_F/\bigcirc_P$ (in the next/previous moment) interpreted over the timeline $(\mathbb{Z}, <)$. The temporal DLs have an additional parameter $\boldsymbol{o} \in \{\Box, \bigcirc, \Box\bigcirc\}$: *DL-Lite*$^{\boldsymbol{o}}_{\boldsymbol{c}/\boldsymbol{r}}$ allows ontologies whose axioms (1) may contain operators from $\boldsymbol{o}$ (e.g., $\boldsymbol{o} = \Box$ permits $\Box_F/\Box_P$ only) and comply with $\boldsymbol{c}$ for CIs and $\boldsymbol{r}$ for RIs. A CI or RI is satisfied in a model if it holds globally, at all time points in $\mathbb{Z}$. Even in the minimal language *DL-Lite*$^{\bigcirc}_{core/core}$ we can state that a role R is expanding ($R \sqsubseteq \bigcirc_F R$) or constant (by adding $\bigcirc_F R \sqsubseteq R$). Using an auxiliary relation, we can also express $R \sqsubseteq \Box_F Q$ in *DL-Lite*$^{\bigcirc}_{core/core}$. Moving to *DL-Lite*$^{\Box\bigcirc}_{core/horn}$, we can express that R is convex or has a finite lifespan, and *DL-Lite*$^{\Box\bigcirc}_{core/krom}$ makes it possible to state that R causes Q to hold eventually; see Section 2 for more details and discussions.

Using temporalised RIs we can thus represent temporal knowledge about relations that goes significantly beyond the expressive power of languages in which only concepts and/or axioms are temporalised [5, 23, 15, 10, 19]. We show that, nevertheless, KB satisfiability is decidable (in fact, EXPSPACE-complete) for both *DL-Lite*$^{\bigcirc}_{bool/krom}$ and *DL-Lite*$^{\Box\bigcirc}_{bool/horn}$, that is, even with arbitrary Boolean concepts, neither Krom nor Horn RIs lead to undecidability. This is optimal, as we also show that satisfiability of *DL-Lite*$^{\bigcirc}_{g\text{-}bool/g\text{-}bool}$ KBs is undecidable.

role inclusions	concept inclusions (*g*-)*bool* *horn*
(*g*-)*bool*	undecidable
krom	? (EXPSPACE for $\bigcirc$-only RBox)
horn	EXPSPACE
core	PSPACE

We also investigate whether the satisfiability problem for KBs in our languages can be reduced to the query evaluation problem over the underlying temporal database, which clarifies the data complexity of the former. We show that *DL-Lite*$^{\Box\bigcirc}_{krom/core}$ ontologies are rewritable to $FO(<, +)$ over finite linear orders (with built-in predicates for order and plus), which corresponds to the data complexity in AC^0. On the other hand, we prove that *DL-Lite*$^{\Box\bigcirc}_{bool/horn}$ ontologies can only be rewritten to the extension of FO$(<,+)$ with $\times$ and relational primitive recursion, which entails NC^1-completeness for data complexity. The inevitable fly in the ointment is that there is a *DL-Lite*$^{\bigcirc}_{\textit{g-bool/g-bool}}$ ontology for which consistency with a given input data is undecidable.

2 Preliminaries

We use the standard DL syntax and semantics. Let a_i, $i < \omega$, be *individual names*, A_i *concept names* and P_i *role names*. We define *roles* S, *basic concepts* B, *temporalised roles* R and *temporalised concepts* C by the following grammar:

$$\begin{array}{rcl} S & ::= & P_i \mid P_i^-, \\ B & ::= & A_i \mid \exists S, \\ R & ::= & S \mid \Box_F R \mid \Box_P R \mid \bigcirc_F R \mid \bigcirc_P R, \\ C & ::= & B \mid \Box_F C \mid \Box_P C \mid \bigcirc_F C \mid \bigcirc_P C. \end{array}$$

A *concept* or *role inclusion* (CI or RI) takes the form (1), where the ϑ_i are all temporalised concepts or, respectively, all temporalised roles. (The empty $\sqcap$ is $\top$ and the empty $\sqcup$ is $\bot$.) A *TBox* $\mathcal{T}$ and an *RBox* $\mathcal{R}$ are finite sets of CIs and, respectively, RIs; their union $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$ is an *ontology*. The atemporal *DL-Lite*$^{\boldsymbol{r}}_{\boldsymbol{c}}$ and temporal *DL-Lite*$^{\boldsymbol{o}}_{\boldsymbol{c}/\boldsymbol{r}}$ were defined in the introduction. We also set *DL-Lite*$^{\boldsymbol{o}}_{\boldsymbol{c}}$ = *DL-Lite*$^{\boldsymbol{o}}_{\boldsymbol{c}/\boldsymbol{c}}$.

To illustrate, imagine an estate agency describing properties by their proximity to various amenities, using roles *wd* for 'walking distance' and *dd* for 'driving distance'. Then we can state in *DL-Lite*$^{\textit{krom}}_{\textit{core}}$ that $\top \sqsubseteq \textit{wd} \sqcup \textit{dd}$, that these roles are disjoint ($\textit{wd} \sqcap \textit{dd} \sqsubseteq \bot$) and symmetric ($\textit{wd} \sqsubseteq \textit{wd}^-$ and $\textit{dd} \sqsubseteq \textit{dd}^-$), and describe locations using CIs such as $\textit{FamilyLocation} \sqsubseteq \exists \textit{wd}.\textit{School} \sqcap \exists \textit{dd}.\textit{Pub}$, which requires fresh auxiliary role names, e.g., *wd.Pub* is replaced by $\exists \textit{wd}_P$ with $\exists \textit{wd}_P^- \sqsubseteq \textit{School}$ and $\textit{wd}_P \sqsubseteq \textit{wd}$. In *DL-Lite*$^{\textit{bool}}_{\textit{core}}$, we can say that $\textit{Station} \sqsubseteq \forall \textit{wd}.\textit{WellConnected}$ using fresh role names $\textit{wd}_C$ and $\textit{wd}'_C$ with CIs $\textit{Station} \sqcap \exists \textit{wd}'_C \sqsubseteq \bot$ and $\exists \textit{wd}_C^- \sqsubseteq \textit{WellConnected}$ with RI $\textit{wd} \sqsubseteq \textit{wd}_C \sqcup \textit{wd}'_C$ (see Theorem 1). In *DL-Lite*$^{\bigcirc}_{\textit{core/krom}}$, we can also express $\textit{SocialLocation} \sqsubseteq \exists \textit{wd}_P \sqcap \bigcirc_P \exists \textit{wd}_P \sqcap \bigcirc_P \bigcirc_P \exists \textit{wd}_P$ (over the past three years, there has been a pub within walking distance).

An *ABox*, $\mathcal{A}$, is a finite set of atoms of the form $A_i(a, \ell)$ and $P_i(a, b, \ell)$, where a, b are individual names and $\ell \in \mathbb{Z}$. We denote by $\mathsf{ind}(\mathcal{A})$ the set of individual names in $\mathcal{A}$, by $\min \mathcal{A}$ and $\max \mathcal{A}$ the minimal and maximal integers in $\mathcal{A}$, and set $\mathsf{tem}(\mathcal{A}) = \{ n \in \mathbb{Z} \mid \min \mathcal{A} \leq n \leq \max \mathcal{A} \}$. For simplicity, we assume that $\min \mathcal{A} = 0$. A *DL-Lite*$^{\boldsymbol{o}}_{\boldsymbol{c}/\boldsymbol{r}}$ *knowledge base* (KB) is a pair $(\mathcal{O}, \mathcal{A})$, where $\mathcal{O}$ is a *DL-Lite*$^{\boldsymbol{o}}_{\boldsymbol{c}/\boldsymbol{r}}$ ontology and $\mathcal{A}$ an ABox. The *size* $|\mathcal{O}|$ of $\mathcal{O}$ is the number of occurrences of symbols in it; the size of a TBox, RBox, ABox and KB is defined in the same way, with unary encoding of numbers in ABoxes.

A (*temporal*) *interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}(n)})$, where $\Delta^{\mathcal{I}} \neq \emptyset$ and, for each $n \in \mathbb{Z}$,

$$\mathcal{I}(n) = (\Delta^{\mathcal{I}}, a_0^{\mathcal{I}}, \dots, A_0^{\mathcal{I}(n)}, \dots, P_0^{\mathcal{I}(n)}, \dots)$$

is a standard DL interpretation with $a_i^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, $A_i^{\mathcal{I}(n)} \subseteq \Delta^{\mathcal{I}}$ and $P_i^{\mathcal{I}(n)} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The DL constructs and temporal operators are interpreted in $\mathcal{I}(n)$ as usual:

$$(P_i^-)^{\mathcal{I}(n)} = \{ (u,v) \mid (v,u) \in P_i^{\mathcal{I}(n)} \},$$
$$(\exists S)^{\mathcal{I}(n)} = \{ u \mid (u,v) \in S^{\mathcal{I}(n)}, \text{ for some } v \},$$
$$(\Box_F \vartheta)^{\mathcal{I}(n)} = \bigcap\nolimits_{k>n} \vartheta^{\mathcal{I}(k)}, \qquad (\Box_P \vartheta)^{\mathcal{I}(n)} = \bigcap\nolimits_{k<n} \vartheta^{\mathcal{I}(k)},$$
$$(\bigcirc_F \vartheta)^{\mathcal{I}(n)} = \vartheta^{\mathcal{I}(n+1)}, \qquad (\bigcirc_P \vartheta)^{\mathcal{I}(n)} = \vartheta^{\mathcal{I}(n-1)}.$$

CIs and RIs are interpreted in $\mathcal{I}$ *globally* in the sense that inclusion (1) is true in $\mathcal{I}$ if

$$\vartheta_1^{\mathcal{I}(n)} \cap \dots \cap \vartheta_k^{\mathcal{I}(n)} \subseteq \vartheta_{k+1}^{\mathcal{I}(n)} \cup \dots \cup \vartheta_{k+m}^{\mathcal{I}(n)}, \quad \text{for } \textit{all } n \in \mathbb{Z}.$$

For an inclusion α, we write $\mathcal{I} \models \alpha$ if α is true in $\mathcal{I}$. We call $\mathcal{I}$ a *model* of $(\mathcal{O}, \mathcal{A})$ and write $\mathcal{I} \models (\mathcal{O}, \mathcal{A})$ if $\mathcal{I} \models \alpha$ for all $\alpha \in \mathcal{O}$, $a^{\mathcal{I}} \in A^{\mathcal{I}(\ell)}$ for $A(a,\ell) \in \mathcal{A}$, and $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in P^{\mathcal{I}(\ell)}$ for $P(a,b,\ell) \in \mathcal{A}$. A KB is *satisfiable* if it has a model.

It is to be noted that the *LTL* operators $\Diamond_F$ (eventually), $\mathcal{U}$ (until) and their past counterparts can be expressed in *bool* using $\bigcirc_P$/$\bigcirc_F$ and $\Box_P$/$\Box_F$ [14, 2]. In many cases one does not need full Booleans: $\Diamond_P R \sqsubseteq Q$ is equivalent to $R \sqsubseteq \Box_F Q$, which can be expressed in *DL-Lite*$^{\bigcirc}_{\textit{core/core}}$ as $R \sqsubseteq \bigcirc_F S$, $S \sqsubseteq \bigcirc_F S$, $S \sqsubseteq Q$, where S is fresh. It immediately follows that convexity of R (that is, $\Diamond_P R \sqcap \Diamond_F R \sqsubseteq R$) can be expressed in *DL-Lite*$^{\bigcirc}_{\textit{core/horn}}$. Then, $R \sqsubseteq \Diamond_F Q$ can be simulated in *DL-Lite*$^{\Box\bigcirc}_{\textit{core/krom}}$ with $\top \sqsubseteq \bar{Q} \sqcup Q$ and $R \sqcap \Box_F \bar{Q} \sqsubseteq \bot$. That the lifespan of R is bounded can be expressed in *DL-Lite*$^{\Box\bigcirc}_{\textit{core/core}}$ using $\Box_P R \sqsubseteq \bot$ and $\Box_F R \sqsubseteq \bot$.

We are interested in the *combined* and *data* complexities of the *satisfiability problem for KBs*: for the former, both the ontology and the ABox of a KB are regarded as input, while for the latter, the ontology is fixed. We assume that $|\mathsf{tem}(\mathcal{A})| \geq |\mathsf{ind}(\mathcal{A})|$ in any input ABox $\mathcal{A}$ (if this is not so, we add the required number of dummies with the missing timestamps to $\mathcal{A}$). Let $\mathsf{ind}(\mathcal{A}) = \{a_1, \dots, a_m\}$. We encode $\mathcal{A}$ as a structure $\mathfrak{S}_\mathcal{A}$ with domain $\mathsf{tem}(\mathcal{A})$ ordered by $<$ such that $\mathfrak{S}_\mathcal{A} \models A(k, \ell)$ iff $A(a_k, \ell) \in \mathcal{A}$ and $\mathfrak{S}_\mathcal{A} \models P(k, k', \ell)$ iff $P(a_k, a_{k'}, \ell) \in \mathcal{A}$.

We establish our data complexity results by 'rewriting' ontologies to FO-sentences 'accepting' or 'rejecting' the input ABoxes. Let $\mathcal{L}$ be a class of FO-sentences interpreted over $\mathfrak{S}_\mathcal{A}$. Say that $\Phi \in \mathcal{L}$ is an *$\mathcal{L}$-rewriting of $\mathcal{O}$* if, for any ABox $\mathcal{A}$, the KB $(\mathcal{O}, \mathcal{A})$ is satisfiable iff $\mathfrak{S}_\mathcal{A} \models \Phi$. Here, we need three classes $\mathcal{L}$: (i) FO($<$) with binary and ternary predicates of the form $A_i(x, t)$ and $P_i(x, y, t)$ as well as $<$ and $=$; (ii) FO($<, +$) with an extra predicate PLUS: $\mathfrak{S}_\mathcal{A} \models \textsc{plus}(n, n_1, n_2)$ iff $n = n_1 + n_2$; and (iii) FO(RPR) that also has the predicate TIMES and *relational primitive recursion*, which allows us to construct formulas such as

$$\begin{bmatrix} Q_1(\boldsymbol{z}_1, t) \equiv \Theta_1\big(\boldsymbol{z}_1, t, Q_1(\boldsymbol{z}_1, t-1), \dots, Q_n(\boldsymbol{z}_n, t-1)\big) \\ \dots \\ Q_n(\boldsymbol{z}_n, t) \equiv \Theta_n\big(\boldsymbol{z}_n, t, Q_1(\boldsymbol{z}_1, t-1), \dots, Q_n(\boldsymbol{z}_n, t-1)\big) \end{bmatrix} \Psi,$$

where $[\dots]$ defines recursively, via the formulas Θ_i, the interpretations of the predicates Q_i in Ψ. It is known that evaluation of FO($<, +$)-sentences over $\mathfrak{S}_\mathcal{A}$ is in LOGTIME-uniform AC^0 for data complexity [22] and FO(RPR) $=$ NC^1 [13].

3 Reasoning with Atemporal *DL-Lite*

To begin with, we establish the complexity of reasoning with the plain DLs underlying the temporal $\textit{DL-Lite}^{\boldsymbol{o}}_{\boldsymbol{c}/\boldsymbol{r}}$ introduced above. We denote them by $\textit{DL-Lite}^{\boldsymbol{r}}_{\boldsymbol{c}}$, where as before $\boldsymbol{c}, \boldsymbol{r} \in \{\textit{bool}, \textit{g-bool}, \textit{horn}, \textit{krom}, \textit{core}\}$. The satisfiability problem for DLs of the form $\textit{DL-Lite}^{\textit{core}}_{\boldsymbol{c}}$ was studied by [12, 1]: it is NP-complete for $\textit{DL-Lite}^{\textit{core}}_{\textit{bool}}$, P-complete for $\textit{DL-Lite}^{\textit{core}}_{\textit{horn}}$, and NL-complete for $\textit{DL-Lite}^{\textit{core}}_{\textit{krom}}$ and $\textit{DL-Lite}^{\textit{core}}_{\textit{core}}$ KBs.

We show that $\textit{DL-Lite}^{\textit{bool}}_{\textit{bool}}$ can be regarded as a notational variant of the extension $\mathcal{ALCI}^{\sqcap,\neg}$ of $\mathcal{ALC}$ with inverse roles and Boolean operators on roles. This logic has, in turn, almost the same expressive power as FO^2, except that the identity role has to be added. In detail, let $\mathcal{ALCI}^{\sqcap,\neg}$ be the DL with roles S and concepts C defined by

$$\begin{aligned} S, S' \;&::=\; \top \;\mid\; P_i \;\mid\; S \sqcap S' \;\mid\; \neg S \;\mid\; S^-, \\ C, C' \;&::=\; \top \;\mid\; A_i \;\mid\; \exists S.C \;\mid\; C \sqcap C' \;\mid\; \neg C. \end{aligned}$$

An $\mathcal{ALCI}^{\sqcap,\neg}$-CI takes the form $C \sqsubseteq C'$ [26, 24, 16]. We say that a KB $\mathcal{K}$ is a *model conservative extension* of a KB $\mathcal{K}'$ if $\mathcal{K} \models \mathcal{K}'$, the signature of $\mathcal{K}$ contains the signature of $\mathcal{K}'$, and every model of $\mathcal{K}'$ can be extended to a model of $\mathcal{K}$ by providing interpretations of the fresh symbols of $\mathcal{K}$ and leaving the domain and the interpretation of the symbols in $\mathcal{K}'$ unchanged.

Theorem 1. (i) *For every $\textit{DL-Lite}^{\textit{bool}}_{\textit{bool}}$ KB, one can compute in logarithmic space an equivalent $\mathcal{ALCI}^{\sqcap,\neg}$ KB.*

(ii) *For every $\mathcal{ALCI}^{\sqcap,\neg}$ KB, one can compute in log-space a model conservative extension in DL-Lite$^{\textit{bool}}_{\textit{bool}}$.*

Proof. (i) Clearly, any CI in $\textit{DL-Lite}^{\textit{bool}}_{\textit{bool}}$ is an $\mathcal{ALCI}^{\sqcap,\neg}$-CI ($\exists R = \exists R.\top$). Any RI $S_1 \sqcap \dots \sqcap S_k \sqsubseteq S_{k+1} \sqcup \dots \sqcup S_{k+m}$ in $\textit{DL-Lite}^{\textit{bool}}_{\textit{bool}}$ is equivalent to the $\mathcal{ALCI}^{\sqcap,\neg}$-CI $\exists R.\top \sqsubseteq \bot$, where R abbreviates $S_1 \sqcap \dots \sqcap S_k \sqcap \neg S_{k+1} \sqcap \dots \sqcap \neg S_{k+m}$.

(ii) For any $\mathcal{ALCI}^{\sqcap,\neg}$ KB $\mathcal{K}$, we construct in linear time a model conservative extension of $\mathcal{K}$ in $\mathcal{ALCI}^{\sqcap,\neg}$ with CIs in normal form:

$$A \sqsubseteq \forall S.B, \qquad \forall S.B \sqsubseteq A, \qquad A_1 \sqcap A_2 \sqsubseteq B, \qquad A \sqsubseteq \neg B, \qquad \neg A \sqsubseteq B,$$

where A, B, A_1, A_2 range over concept names and $\top$. Next, we replace CIs $A \sqsubseteq \forall S.B$ and $\forall S.B \sqsubseteq A$ by $S \sqsubseteq Q \sqcup R$, $\exists Q^- \sqsubseteq B$, $\exists R \sqsubseteq \neg A$, and, respectively, $\neg A \sqsubseteq \exists R$, $R \sqsubseteq S$, $\exists R^- \sqsubseteq \neg B$, with fresh role names Q, R. Finally, RIs with a Boolean S are transformed into normal form (1), possibly using fresh role names for Boolean sub-roles, to obtain a model conservative extension of $\mathcal{K}$ in $\textit{DL-Lite}^{\textit{bool}}_{\textit{bool}}$. ❑

The NEXPTIME-completeness of $\mathcal{ALCI}^{\sqcap,\neg}$ KB satisfiability [26] implies that $\textit{DL-Lite}^{\textit{bool}}_{\textit{bool}}$ KB satisfiability is also NEXPTIME-complete. To bring down the complexity to EXPTIME, it suffices to avoid unguarded quantification by admitting only RIs with a non-empty left-hand side, as in the *g-bool* RIs. Then, for any $\textit{DL-Lite}^{\textit{g-bool}}_{\textit{bool}}$ KB, it is straightforward to compute in linear time an equivalent KB in the guarded two-variable fragment GF^2 of FO. Using the fact that KB satisfiability for the latter logic is in EXPTIME [17], we obtain the following:

Theorem 2. *KB satisfiability is* NEXPTIME*-complete for $\textit{DL-Lite}^{\textit{bool}}_{\textit{bool}}$ and* EXPTIME*-complete for $\textit{DL-Lite}^{\textit{g-bool}}_{\textit{bool}}$.*

We now show that the *DL-Lite* logics with Horn and Krom RIs are reducible to propositional logic. For an ontology $\mathcal{O}$, let $\mathit{role}^{\pm}(\mathcal{O}) = \{ P, P^- \mid P \text{ a role in } \mathcal{O} \}$. Let $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$. We assume that $\mathcal{R}$ is closed under taking the inverses of roles in RIs. Denote by $\mathsf{sub}_{\mathcal{T}}$ the set of concepts in $\mathcal{T}$ and their negations. A *concept type* τ for $\mathcal{T}$ is a maximal subset τ of $\mathsf{sub}_{\mathcal{T}}$ that is 'propositionally' consistent with $\mathcal{T}$: if $B_1, \dots, B_k \in \tau$ and $\mathcal{T}$ contains $B_1 \sqcap \dots \sqcap B_k \sqsubseteq B_{k+1} \sqcup \dots \sqcup B_{k+m}$, then one of $B_{k_1}, \dots, B_{k+m}$ is also in τ (note, however, that τ does not have to be consistent with $\mathcal{T}$ as it can contain $\exists P$ even if $\exists P^- \sqsubseteq \bot$ is in $\mathcal{T}$). Clearly, for an interpretation $\mathcal{J}$ and $u \in \Delta^{\mathcal{J}}$, the set comprising all $B \in \mathsf{sub}_{\mathcal{T}}$ with $u \in B^{\mathcal{J}}$ and all $\neg B \in \mathsf{sub}_{\mathcal{T}}$ with $u \notin B^{\mathcal{J}}$ is a concept type for $\mathcal{T}$; it is denoted $\tau_u^{\mathcal{J}}$ and called the *concept type of* u *in* $\mathcal{J}$. Similarly, let $\mathsf{sub}_{\mathcal{R}}$ be the set of roles in $\mathcal{R}$ and their negations. A *role type* ρ for $\mathcal{R}$ is a maximal subset of $\mathsf{sub}_{\mathcal{R}}$ propositionally consistent with $\mathcal{R}$. For $(u, v) \in \Delta^{\mathcal{J}} \times \Delta^{\mathcal{J}}$, the set comprising all $S \in \mathsf{sub}_{\mathcal{R}}$ with $(u, v) \in S^{\mathcal{J}}$ and all $\neg S \in \mathsf{sub}_{\mathcal{R}}$ with $(u, v) \notin S^{\mathcal{J}}$ is a role type for $\mathcal{R}$; it is denoted by $\rho_{u,v}^{\mathcal{J}}$ and called the *role type* of (u, v) in $\mathcal{J}$. For a set of role literals (roles and their negations) Ξ, let $\mathsf{cl}_{\mathcal{R}}(\Xi)$ be the set of all role literals L' such that $\mathcal{R} \models \bigsqcap_{L \in \Xi} L \sqsubseteq L'$. The following lemma plays a key role in the reduction.

Lemma 3. *For any satisfiable DL-Lite$_{bool}^{krom}$ KB $\mathcal{K} = (\mathcal{O}, \mathcal{A})$, $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$, there is a model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ of $\mathcal{K}$ such that*

$$\Delta^{\mathcal{I}} = \mathsf{ind}(\mathcal{A}) \cup \{ w_S^i \mid S \in \mathit{role}^{\pm}(\mathcal{O}) \text{ and } 0 \leq i < 3 \},$$

and $(u, v) \in S^{\mathcal{I}}$, for every $u \to_S v$ with $u \in (\exists S)^{\mathcal{I}}$, where

$$\to_S = \{(a, w_S^0) \mid a \in \mathsf{ind}(\mathcal{A})\} \cup \{(w_R^i, w_S^{i \oplus 1}) \mid w_R^i \in \Delta^{\mathcal{I}}\}$$

and $\oplus$ is addition modulo 3. In particular, DL-Lite$_{bool}^{krom}$ has the linear model property: $|\Delta^{\mathcal{I}}| = |\mathsf{ind}(\mathcal{A})| + 3|\mathit{role}^{\pm}(\mathcal{O})|$.

Proof. Given a model $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ of $\mathcal{K}$, we construct $\mathcal{I}$ as follows. For any $S \in \mathit{role}^{\pm}(\mathcal{O})$, if $S^{\mathcal{J}} \neq \emptyset$, then we pick $w_S \in (\exists S^-)^{\mathcal{J}}$; otherwise, we pick any $w_S \in \Delta^{\mathcal{J}}$. We assume that the w_S are distinct. Let $\Delta^{\mathcal{I}}$ comprise $\mathsf{ind}(\mathcal{A})$ and three copies w_S^0, w_S^1, w_S^2 of each w_S; cf. [9, Proposition 8.1.4]. This also fixes $\to_S$. Define $f \colon \Delta^{\mathcal{I}} \to \Delta^{\mathcal{J}}$ by taking $f(a) = a$, for all $a \in \mathsf{ind}(\mathcal{A})$, and $f(w_S^i) = w_S$, for all S and i. We then take concept types $\tau_u = \tau_{f(u)}^{\mathcal{J}}$, for all $u \in \Delta^{\mathcal{I}}$. To define $\rho_{u,v}$ for $u, v \in \Delta^{\mathcal{I}}$, we consider the following three cases.

If $u, v \in \mathsf{ind}(\mathcal{A})$, then we take $\Xi = \{S \mid S(a, b) \in \mathcal{A}\}$, assuming $P_i^-(a, b) \in \mathcal{A}$ whenever $P_i(b, a) \in \mathcal{A}$.
If $\exists S \in \tau_u$ and $u \to_S v$, then we take $\Xi = \{S\}$.
Otherwise, we take $\Xi = \emptyset$.

We begin with $\rho_{u,v} = \mathsf{cl}_{\mathcal{R}}(\Xi)$ and perform the following procedure for each RI $\top \sqsubseteq S_1 \sqcup S_2$ in $\mathcal{R}$ such that none of S_i and $\neg S_i$ is in $\rho_{u,v}$ yet. As $\mathcal{J} \models \mathcal{R}$, either S_1 or S_2 is in $\rho_{f(u),f(v)}^{\mathcal{J}}$. So, $\rho_{u,v}$ is extended with the respective $\mathsf{cl}_{\mathcal{R}}(\{S_i\})$. Since any contradiction derivable from Krom formulas is derivable from two literals, the resulting $\rho_{u,v}$ is consistent with $\mathcal{R}$ and both τ_u- and τ_v-compatible: that is, $\exists R \in \tau_u$ and $\exists R^- \in \tau_v$, for all $R \in \rho_{u,v}$. One can check that the constructed τ_u and $\rho_{u,v}$, for $u, v \in \Delta^{\mathcal{I}}$, are types for $\mathcal{T}$ and $\mathcal{R}$, respectively, and give rise to a model of $\mathcal{K}$. ❑

The existence of a model $\mathcal{I}$ from Lemma 3 can be encoded by a *propositional* formula $\varphi_{\mathcal{K}}$ whose propositional variables take the form $B^{\dagger}(u)$ and $P_i^{\dagger}(u, v)$, for $u, v \in \Delta^{\mathcal{I}}$, assuming that $(P_i^-)^{\dagger}(u, v) = P_i^{\dagger}(v, u)$. The formula $\varphi_{\mathcal{K}}$ is a conjunction of the following, for all $u, v \in \Delta^{\mathcal{I}}$:

$B_1^{\dagger}(u) \land \dots \land B_k^{\dagger}(u) \to B_{k+1}^{\dagger}(u) \lor \dots \lor B_{k+m}^{\dagger}(u)$,	for CI $B_1 \sqcap \dots \sqcap B_k \sqsubseteq B_{k+1} \sqcup \dots \sqcup B_{k+m}$ in $\mathcal{T}$,	(2)
$S_1^{\dagger}(u, v) \to S_2^{\dagger}(u, v)$,	for RI $S_1 \sqsubseteq S_2$ in $\mathcal{R}$,	(3)
$\neg S_1^{\dagger}(u, v) \lor \neg S_2^{\dagger}(u, v)$,	for RI $S_1 \sqcap S_2 \sqsubseteq \bot$ in $\mathcal{R}$,	(4)
$S_1^{\dagger}(u, v) \lor S_2^{\dagger}(u, v)$,	for RI $\top \sqsubseteq S_1 \sqcup S_2$ in $\mathcal{R}$,	(5)
$A^{\dagger}(a)$,	for $A(a) \in \mathcal{A}$,	(6)
$P^{\dagger}(a, b)$,	for $P(a, b) \in \mathcal{A}$,	(7)
$(\exists S)^{\dagger}(u) \to S^{\dagger}(u, v)$,	for each S with $u \to_S v$,	(8)
$S^{\dagger}(u, v) \to (\exists S)^{\dagger}(u)$,	for each S.	(9)

Clearly, $\mathcal{K}$ is satisfiable iff $\varphi_{\mathcal{K}}$ is satisfiable. Also, if $\mathcal{K}$ is in *DL-Lite*$_{krom}^{krom}$, then $\varphi_{\mathcal{K}}$ is a Krom formula constructed by a logspace transducer. Now, since *DL-Lite*$_{horn}^{krom}$ can express *DL-Lite*$_{bool}^{krom}$ (Krom RIs can simulate Krom CIs, and the latter can express the complement of concepts), we obtain the following:

Theorem 4. *Satisfiability is* NP*-complete for DL-Lite$_{bool}^{krom}$ and DL-Lite$_{horn}^{krom}$ KBs, and* NL*-complete for DL-Lite$_{krom}^{krom}$.*

Next, consider a *DL-Lite*$^{horn}_{bool}$ KB. It is readily seen that the proof of Lemma 3 goes through, except we need not perform the saturation step for RIs $\top \sqsubseteq S_1 \sqcup S_2$. Thus, the role types in $\mathcal{I}$ are minimal in the following sense:

Lemma A. *For any consistent DL-Lite*$^{horn}_{bool}$ *KB* $\mathcal{K} = (\mathcal{O}, \mathcal{A})$*, there is a model* $\mathcal{I}$ *as in Lemma 3 such that in addition*

$$\rho^{\mathcal{I}}_{a,b} = \mathsf{cl}_{\mathcal{R}}(\{S \mid S(a,b) \in \mathcal{A}\}), \qquad \text{for } a, b \in \mathsf{ind}(\mathcal{A});$$
$$\rho^{\mathcal{I}}_{u,v} = \mathsf{cl}_{\mathcal{R}}(\{S\}), \qquad \text{if } \exists S \in \tau^{\mathcal{I}}_u \text{ and } u \to_S v;$$
$$\rho^{\mathcal{I}}_{u,v} = \mathsf{cl}_{\mathcal{R}}(\emptyset), \qquad \text{otherwise}.$$

Observe now that the translation $\varphi_{\mathcal{K}}$ is a Horn formula for $\mathcal{K}$ in *DL-Lite*$^{horn}_{horn}$; but it is neither a Horn nor a Krom formula for *DL-Lite*$^{horn}_{krom}$. Nevertheless, we can compute in polynomial time the set $\varphi'_{\mathcal{R},\mathcal{A}}$ comprising all $Q^\dagger(a,b)$ for $a, b \in \mathsf{ind}(\mathcal{A})$ and $Q \in \mathsf{cl}_{\mathcal{R}}\{S \mid S(a,b) \in \mathcal{A}\}$; note that, if $\mathcal{R}, \mathcal{A}$ are inconsistent, then $\varphi'_{\mathcal{R},\mathcal{A}}$ will contain a role and its negation. We then replace (3)–(8) in $\varphi_{\mathcal{K}}$ by $\varphi'_{\mathcal{R},\mathcal{A}}$ with

$$S^\dagger(u,v), \qquad \text{for } S \in \mathsf{cl}_{\mathcal{R}}(\emptyset),$$
$$(\exists S)^\dagger(u) \to Q^\dagger(u,v), \qquad \text{for } Q \in \mathsf{cl}_{\mathcal{R}}(\{S\}) \text{ and } u \to_S v,$$

which are also computable in polynomial time. The result is a Krom formula of polynomial size in $|\mathcal{K}|$ that is equisatisfiable with $\mathcal{K}$. Thus, we obtain the following complexity results:

Theorem 5. *Satisfiability is* NP*-complete for DL-Lite*$^{horn}_{bool}$ *KBs, and* P*-complete for DL-Lite*$^{horn}_{horn}$ *and DL-Lite*$^{horn}_{krom}$ *KBs.*

4 Satisfiability of Temporal KBs

We now consider extensions *DL-Lite*$^{\boldsymbol{o}}_{\boldsymbol{c}/\boldsymbol{r}}$ of *DL-Lite*$^{\boldsymbol{r}}_{\boldsymbol{c}}$ with temporal operators in $\boldsymbol{o} \in \{\Box, \bigcirc, \Box\bigcirc\}$ that can be applied to concepts and roles. Our first observation is negative:

Theorem 6. *Satisfiability in DL-Lite*$^{\bigcirc}_{g\text{-}bool/g\text{-}bool}$ *is undecidable.*

Proof. The proof is by reduction of the undecidable $\mathbb{N} \times \mathbb{N}$-tiling problem [8]. Given a set $\mathfrak{T} = \{1, \dots, m\}$ of tile types, with the colours on the four edges of tile type i denoted by *top*(i), *bot*(i), *right*(i) and *left*(i), we define the following ontology $\mathcal{O}$, where R_i is a role name associated with the tile type $i \in \mathfrak{T}$:

$$I \sqsubseteq \bigsqcup_{i \in \mathfrak{T}} \exists R_i, \qquad R_i \sqsubseteq \bigsqcup_{\mathit{right}(i) = \mathit{left}(j)} \bigcirc_{\scriptscriptstyle F} R_j, \qquad \exists R_i^- \sqsubseteq \bigsqcup_{\mathit{top}(i) = \mathit{bot}(j)} \exists R_j, \qquad \exists R_i \sqcap \exists R_j \sqsubseteq \bot, \text{ for } i \neq j. \tag{10}$$

Then $(\mathcal{O}, \{I(a, 0)\})$ is satisfiable iff $\mathfrak{T}$ can tile $\mathbb{N} \times \mathbb{N}$. ❑

Fortunately, the temporal *DL-Lite* languages with Krom, Horn and core RIs turn out to be less naughty. In the remainder of this section, we develop reductions of these languages to propositional and first-order *LTL* with one variable.

4.1 Krom RIs

Given a *DL-Lite*$^{\bigcirc}_{bool/krom}$ KB $\mathcal{K} = (\mathcal{T} \cup \mathcal{R}, \mathcal{A})$, we construct a first-order temporal sentence $\Phi_{\mathcal{K}}$ with one free variable x. We assume that $\mathcal{K}$ has no nested temporal operators and that, in RIs of the form $\top \sqsubseteq R_1 \sqcup R_2$ from $\mathcal{R}$, both R_i are plain (atemporal) roles; also, $\mathcal{R}$ is closed under taking the inverses of roles in RIs. First, we set $\Phi_{\mathcal{K}} = \bot$ if $(\mathcal{R}, \mathcal{A})$ is unsatisfiable. Otherwise, we treat basic concepts in $\mathcal{K}$ as unary predicates and define $\Phi_{\mathcal{K}}$ as a conjunction of the following sentences, where $\Box = \Box_{\scriptscriptstyle F} \Box_{\scriptscriptstyle P}$:

$$\Box \forall x\, [C_1(x) \land \dots \land C_k(x) \to C_{k+1}(x) \lor \dots \lor C_{k+m}(x)], \qquad \text{for } C_1 \sqcap \dots \sqcap C_k \sqsubseteq C_{k+1} \sqcup \dots \sqcup C_{k+m} \text{ in } \mathcal{T}, \tag{11}$$
$$\Box \forall x\, [\exists S_1(x) \lor \exists S_2(x)], \qquad \text{for RI } \top \sqsubseteq S_1 \sqcup S_2 \text{ in } \mathcal{R}, \tag{12}$$
$$\Box [\forall x\, \exists S_1(x) \lor \forall x\, \exists S_2^-(x)], \qquad \text{for RI } \top \sqsubseteq S_1 \sqcup S_2 \text{ in } \mathcal{R}, \tag{13}$$
$$\bigcirc^{\ell}_{\scriptscriptstyle F} A(a), \qquad \text{for } A(a, \ell) \in \mathcal{A}, \tag{14}$$
$$\bigcirc^{\ell}_{\scriptscriptstyle F} \exists P(a) \text{ and } \bigcirc^{\ell}_{\scriptscriptstyle F} \exists P^-(b), \qquad \text{for } P(a, b, \ell) \in \mathcal{A}, \tag{15}$$
$$\Box [\exists x\, \exists P(x) \leftrightarrow \exists x\, \exists P^-(x)], \qquad \text{for role name } P \text{ in } \mathcal{T}, \tag{16}$$

and, for every RI $\bigcirc_1 S_1 \sqsubseteq \bigcirc_2 S_2$ with $\mathcal{R} \models \bigcirc_1 S_1 \sqsubseteq \bigcirc_2 S_2$, where each $\bigcirc_i$ is $\bigcirc_{\scriptscriptstyle F}$, $\bigcirc_{\scriptscriptstyle P}$ or blank, and $\bigcirc_1 S_1$ can be $\top$ and $\bigcirc_2 S_2$ can be $\bot$, the following:

$$\Box \forall x\, [\bigcirc_1 \exists S_1(x) \to \bigcirc_2 \exists S_2(x)]. \tag{17}$$

We observe that $\mathcal{R} \models \bigcirc_1 S_1 \sqsubseteq \bigcirc_2 S_2$ can be checked in P [4, Lemma 5.3], and so $\Phi_{\mathcal{K}}$ is constructed in polynomial time.

Lemma 7. *A DL-Lite$^{\bigcirc}_{bool/krom}$ KB is satisfiable iff $\Phi_\mathcal{K}$ is satisfiable.*

Proof. ($\Rightarrow$) Suppose $\mathcal{I} \models \mathcal{K}$. Treating $\mathcal{I}$ as a temporal FO interpretation $\mathfrak{M}$, we show that $\mathfrak{M} \models \Phi_\mathcal{K}$. The only non-standard axioms are (13). Suppose $\mathfrak{M} \models \top \sqsubseteq S_1 \sqcup S_2$ and $\mathfrak{M}, n \not\models \exists S_1(d)$, for some d and n. Then, for every $e \in \Delta^\mathfrak{M}$, we have $\mathfrak{M}, n \models S_2(d, e)$, and so $\mathfrak{M}, n \models \exists S_2^-(e)$.

($\Leftarrow$) Suppose $\mathfrak{M} \models \Phi_\mathcal{K}$. We require the following property of $\mathfrak{M}$, which follows from (12)–(13): for any $n \in \mathbb{Z}$, $d, e \in \Delta^\mathfrak{M}$, and any RI $\top \sqsubseteq S_1 \sqcup S_2$ in $\mathcal{R}$,

$$\mathfrak{M}, n \models \exists S_1(d) \text{ and } \mathfrak{M}, n \models \exists S_1^-(e) \quad \text{or} \quad \mathfrak{M}, n \models \exists S_2(d) \text{ and } \mathfrak{M}, n \models \exists S_2^-(e). \tag{18}$$

We construct a model $\mathcal{I}$ of $\mathcal{K}$ in a step-by-step manner, regarding $\mathcal{I}$ as an ABox (a set of ground atoms). To begin with, we put in $\mathcal{I}$ all $P(a, b, n) \in \mathcal{A}$. We then proceed in three steps.

Step 1: If $\top \sqsubseteq S_1 \sqcup S_2$ is in $\mathcal{R}$ with $\mathfrak{M}, n \models \exists S_1(a)$ and either $\mathfrak{M}, n \not\models \exists S_2(a)$ or $\mathfrak{M}, n \not\models \exists S_2^-(b)$, for $n \in \mathbb{Z}$ and $a, b \in \mathsf{ind}(\mathcal{A})$, then, by (18), $\mathfrak{M}, n \models \exists S_1^-(b)$, and we add $S_1(a, b, n)$ to $\mathcal{I}$. We do the same for $\exists S_1^-$, $\exists S_2$ and $\exists S_2^-$ (recall that $\mathcal{R}$ is closed under role inverses).

We now show that the constructed ABox is consistent with $\mathcal{R}$. Suppose otherwise. Then there are some $P(a, b, n), R(a, b, n) \in \mathcal{I}$ and $P \sqcap R \sqsubseteq \bot$ in $\mathcal{R}$. Two cases need consideration. (i) Suppose that both of these atoms were added at Step 1 because of some RIs $\top \sqsubseteq P \sqcup Q$ and $\top \sqsubseteq R \sqcup S$. In this case, $\mathcal{R} \models P \sqsubseteq S$. But then, by (17), $\mathfrak{M}, n \models \exists S(a)$ and $\mathfrak{M}, n \models \exists S^-(b)$, which contradicts the definition of Step 1. (ii) As $\mathcal{A}$ is consistent with $\mathcal{R}$, the only other possibility is that $R(a, b, n) \in \mathcal{A}$ and $P(a, b, n)$ was added at Step 1 because of some RI $\top \sqsubseteq P \sqcup Q$. In this case, $\mathcal{R} \models R \sqsubseteq Q$, whence, by (15), $\mathfrak{M}, n \models \exists Q(a)$ and $\mathfrak{M}, n \models \exists Q^-(b)$, contrary to the fact that $P(a, b, n)$ was added at Step 1.

Step 2: If $P(a, b, n) \in \mathcal{I}$ and $\mathcal{R} \models P \sqsubseteq \bigcirc^k Q$, for some $k \in \mathbb{Z}$, then we add $Q(a, b, n + k)$ to $\mathcal{I}$. We do this for all roles P and R and all $n, k \in \mathbb{Z}$. We show that the resulting $\mathcal{I}$ is consistent with $\mathcal{R}$. Suppose otherwise, and we have $Q(a, b, n), Q'(a, b, n) \in \mathcal{I}$ with $Q \sqcap Q' \sqsubseteq \bot$ in $\mathcal{R}$. Suppose $Q(a, b, n)$ was added to $\mathcal{I}$ because of $\mathcal{R} \models P \sqsubseteq \bigcirc^k Q$ and $Q'(a, b, n)$ because of $\mathcal{R} \models P' \sqsubseteq \bigcirc^m Q'$, with both initiating P- and P'-atoms constructed at Step 1. Since $\mathcal{R} \models Q \sqsubseteq \neg Q'$ and $\mathcal{R} \models \neg Q' \sqsubseteq \neg\bigcirc^{-m} P'$, we then arrive to a contradiction with the consistency with $\mathcal{R}$ of the ABox resulting from Step 1.

Step 3: Suppose $\top \sqsubseteq P \sqcup Q$ is in $\mathcal{R}$, $\mathfrak{M}, n \models \exists P(a)$, $\mathfrak{M}, n \models \exists P^-(b)$, $\mathfrak{M}, n \models \exists Q(a)$, $\mathfrak{M}, n \models \exists Q^-(b)$, but neither $P(a, b, n)$ nor $Q(a, b, n)$ are in $\mathcal{I}$. Then we add one of them, say $P(a, b, n)$, to $\mathcal{I}$, which cannot lead to inconsistency with $\mathcal{R}$. Indeed, if we had $P \sqcap S \sqsubseteq \bot$ in $\mathcal{R}$ and $S(a, b, n)$ in $\mathcal{I}$, then $\mathcal{R} \models S \sqsubseteq Q$, and so $Q(a, b, n)$ would have been added to $\mathcal{I}$ at Step 2. We take the closure of $P(a, b, n)$ as at Step 2, and do the same with other RIs of that form and time instants n.

We conclude the first stage of constructing $\mathcal{I}$ by adding to it the atoms $B(a, n)$ with $\mathfrak{M}, n \models B(a)$, for all $n \in \mathbb{Z}$ and $a \in \mathsf{ind}(\mathcal{A})$. By construction, $P(a, b, n) \in \mathcal{I}$ implies $\exists P(a, n), \exists P^-(b, n) \in \mathcal{I}$, but not necessarily the other way round.

So suppose $\exists P(a, n) \in \mathcal{I}$ but there is no $P(a, b, n)$ in $\mathcal{I}$. Take a fresh individual w_P and add it to the domain of $\mathcal{I}$. By (16), we have $\mathfrak{M}, n \models \exists P^-(d)$, for some $d \in \Delta^\mathfrak{M}$. Now, whenever $\mathfrak{M}, m \models B(d)$, for a basic concept B and $m \in \mathbb{Z}$, we add $B(w_P, m)$ to $\mathcal{I}$. We also add $P(a, w_P, n)$ to $\mathcal{I}$. Clearly, the result is consistent with $\mathcal{R}$. We then apply to $\mathcal{I}$ the three-step procedure described above, and repeat this *ad infinitum*. It is readily seen that the obtained interpretation $\mathcal{I}$ is a model of $\mathcal{K}$. ❑

Theorem 8. *The satisfiability problem for DL-Lite$^{\bigcirc}_{bool/krom}$ KBs is* EXPSPACE*-complete.*

Proof. The upper bound follows from Lemma 7 since the one-variable fragment of first-order *LTL* is known to be EXPSPACE-complete [20, 15]; the lower one is proved by reduction of the $\mathbb{N} \times (2^n - 1)$ corridor tiling problem [30]: given a finite set $\mathfrak{T}$ of tile types $\{1, \dots, m\}$ with four colours *up*(i), *down*(i), *left*(i) and *right*(i) and a distinguished colour W, decide whether $\mathfrak{T}$ can tile the grid $\{(t, s) \mid t \in \mathbb{N}, 1 \leq s < 2^n\}$ so that **(b$_1$)** tile 0 is placed at $(0, 1)$, **(b$_2$)** every tile i placed at every $(c, 1)$ has *down*$(i) = W$, and **(b$_3$)** every tile i placed at every $(c, 2^n - 1)$ has *up*$(i) = W$.

Let $\mathcal{A} = \{A(a, 0)\}$ and $\mathcal{O}$ contain the following:

$$A \sqsubseteq \bigcirc_F^{2^n} D, \qquad D \sqsubseteq \bigcirc_F^{2^n} D,$$

$$A \sqsubseteq \bigsqcap_{1 \leq s < 2^n} \bigcirc_F^s \exists P, \qquad \exists P^- \sqsubseteq \bigsqcup_{i \in \mathfrak{T}} T_i,$$

$$
\begin{array}{ll}
T_i \sqsubseteq \bigcirc_F^{2^n} \bigsqcup_{right(i)=left(j)} T_j, & \text{for } i \in \mathfrak{T},\\
T_i \sqcap \exists S_i^- \sqsubseteq \bot, & \text{for } i \in \mathfrak{T},\\
\top \sqsubseteq S_i \sqcup Q_i, & \text{for } i \in \mathfrak{T},\\
\exists Q_i \sqcap \bigcirc_F \exists Q_j \sqsubseteq \bot, & \text{for } i, j \in \mathfrak{T} \text{ with } up(i) \neq down(j).
\end{array}
$$

Observe that $(\mathcal{O}, \mathcal{A})$ is satisfiable iff there is a placement of tiles on the $\mathbb{N} \times (2^n - 1)$ grid: each of the $(2^n - 1)$ P-successors of a created at moments $1, \dots, 2^n$ represents a column of the corridor. Note, however, that the size of the CIs is exponential in n. We now describe how they can be replaced by polynomial-size CIs.

Consider a CI $A \sqsubseteq \bigcirc_F^{2^n} D$. We express it using the following CIs:

$$
\begin{array}{ll}
A \sqsubseteq \bigcirc_F(\neg B_{n-1} \sqcap \dots \sqcap \neg B_0), & \\
B_{n-1} \sqcap \dots \sqcap B_0 \sqsubseteq D, & \\
\neg B_k \sqcap B_{k-1} \sqcap \dots \sqcap B_0 \sqsubseteq \bigcirc_F(B_k \sqcap \neg B_{k-1} \sqcap \dots \sqcap \neg B_0), & \text{for } 0 \leq k < n,\\
\neg B_j \sqcap \neg B_k \sqsubseteq \bigcirc_F \neg B_j \quad \text{and} \quad B_j \sqcap \neg B_k \sqsubseteq \bigcirc_F B_j, & \text{for } 0 \leq k < j < n,
\end{array}
$$

which have to be converted into normal form (1). Intuitively, they encode a binary counter from 0 to $2^n - 1$, where $\neg B_i$ and B_i stand for 'the ith bit of the counter is 0 and, respectively, 1'. Other CIs of the form $C_1 \sqsubseteq \bigcirc_F^{2^n} C_2$ are handled similarly. For $A \sqsubseteq \bigsqcap_{1 \leq s < 2^n} \bigcirc_F^s \exists P$, we use the $B_k \sqsubseteq \exists P$, for $0 \leq k < n$, instead of $B_{n-1} \sqcap \dots \sqcap B_0 \sqsubseteq \exists P$.

To ensure that **($\mathbf{b}_1$)–($\mathbf{b}_3$)** are satisfied, we add to $\mathcal{O}$ the CIs

$$
\begin{array}{ll}
A \sqcap \bigcirc_F \exists Q_i \sqsubseteq \bot, & \text{for } i \in \mathfrak{T} \setminus \{0\},\\
D \sqcap \bigcirc_F \exists Q_i \sqsubseteq \bot, & \text{for } down(i) \neq W,\\
\bigcirc_F D \sqcap \exists Q_i \sqsubseteq \bot, & \text{for } up(i) \neq W.
\end{array}
$$

One can show that $(\mathcal{O}, \mathcal{A})$ is as required. ❑

4.2 Horn RIs

Let $\mathcal{K} = (\mathcal{T} \cup \mathcal{R}, \mathcal{A})$ be a *DL-Lite*$^{\Box\bigcirc}_{bool/horn}$ KB. We assume that $\mathcal{R}$ is closed under taking the inverses of roles in RIs and contains all roles in $\mathcal{T}$. A *beam* $\boldsymbol{b}$ *for* $\mathcal{T}$ is a function from $\mathbb{Z}$ to the set of all concept types for $\mathcal{T}$ such that, for all $n \in \mathbb{Z}$,

$$
\bigcirc_F C \in \boldsymbol{b}(n) \text{ iff } C \in \boldsymbol{b}(n+1), \qquad \bigcirc_P C \in \boldsymbol{b}(n) \text{ iff } C \in \boldsymbol{b}(n-1), \tag{19}
$$

$$
\Box_F C \in \boldsymbol{b}(n) \text{ iff } C \in \boldsymbol{b}(k), \text{ for all } k > n, \qquad \Box_P C \in \boldsymbol{b}(n) \text{ iff } C \in \boldsymbol{b}(k), \text{ for all } k < n. \tag{20}
$$

The function $\boldsymbol{b}_u^{\mathcal{I}} \colon n \mapsto \{ C \in \mathsf{sub}_{\mathcal{T}} \mid u \in C^{\mathcal{I}(n)} \}$ (we specify only the positive component of types) is a beam, for any $\mathcal{I}$ and $u \in \Delta^{\mathcal{I}}$; we will refer to it as *the beam of* u *in* $\mathcal{I}$.

A *rod* $\boldsymbol{r}$ *for* $\mathcal{R}$ is a function from $\mathbb{Z}$ to the set of all role types for $\mathcal{R}$ such that (19)–(20) and their past-time counterparts hold for all $n \in \mathbb{Z}$ with $\boldsymbol{b}$ replaced by $\boldsymbol{r}$ and C by temporalised roles S. For any $\mathcal{I}$ and $u, v \in \Delta^{\mathcal{I}}$, the function $\boldsymbol{r}_{u,v}^{\mathcal{I}} \colon n \mapsto \{ R \in \mathsf{sub}_{\mathcal{R}} \mid (u, v) \in R^{\mathcal{I}(n)} \}$ is a rod for $\mathcal{R}$. Fix individual names d, e. Since the RIs in $\mathcal{R}$ are Horn, given any ABox $\mathcal{A}$ with of atoms of the form $S(d, e, \ell)$, define the $\mathcal{R}$-*canonical rod* $\boldsymbol{r}_{\mathcal{A}}$ for $\mathcal{A}$ (consistent with $\mathcal{R}$): $\boldsymbol{r}_{\mathcal{A}} \colon n \mapsto \{ R \in \mathsf{sub}_{\mathcal{R}} \mid \mathcal{R}, \mathcal{A} \models R(d, e, n) \}$. In other words, $\mathcal{R}$-canonical rods are the minimal rods for $\mathcal{R}$ 'containing' all atoms of $\mathcal{A}$: for any R and $n \in \mathbb{Z}$,

$$
R \in \boldsymbol{r}_{\mathcal{A}}(n) \quad \text{iff} \quad R \in \boldsymbol{r}(n), \text{ for all rods } \boldsymbol{r} \text{ for } \mathcal{R} \text{ such that } S \in \boldsymbol{r}(\ell), \text{ for each } S(d, e, \ell) \in \mathcal{A}. \tag{21}
$$

Finally, given a beam $\boldsymbol{b}$, we say a rod $\boldsymbol{r}$ is $\boldsymbol{b}$-*compatible* if $\exists S \in \boldsymbol{b}(n)$ whenever $S \in \boldsymbol{r}(n)$, for all $n \in \mathbb{Z}$ and basic concepts $\exists S$. We are now fully equipped to prove the following characterisation of *DL-Lite*$^{\Box\bigcirc}_{bool/horn}$ KBs satisfiability, where beams can be 'shifted' in (23) to achieve a finite representation.

Lemma 9. *Let $\mathcal{K} = (\mathcal{T} \cup \mathcal{R}, \mathcal{A})$ be a DL-Lite$^{\Box\bigcirc}_{bool/horn}$ KB. Let $\Delta = \mathsf{ind}(\mathcal{A}) \cup \{ w_S \mid S \in role^{\pm}(\mathcal{R}) \}$. Then $\mathcal{K}$ is satisfiable iff there are beams $\boldsymbol{b}_u$, $u \in \Delta$, for $\mathcal{T}$ such that*

$$
A \in \boldsymbol{b}_a(\ell), \textit{ for all } A(a, \ell) \in \mathcal{A}, \tag{22}
$$

$$
\textit{if } \exists S \in \boldsymbol{b}_u(n), \textit{ then } \exists S^- \in \boldsymbol{b}_{w_{S^-}}(k), \textit{ for some } k \in \mathbb{Z}, \tag{23}
$$

$$
\textit{for any } a, b \in \mathsf{ind}(\mathcal{A}), \textit{ there is a } \boldsymbol{b}_a\textit{-compatible rod } \boldsymbol{r} \textit{ for } \mathcal{R} \textit{ with } S \in \boldsymbol{r}(\ell), \textit{ for all } S(a, b, \ell) \in \mathcal{A}, \tag{24}
$$

$$
\exists S \in \boldsymbol{b}_u(n) \textit{ iff there is a } \boldsymbol{b}_u\textit{-compatible rod } \boldsymbol{r} \textit{ for } \mathcal{R} \textit{ with } S \in \boldsymbol{r}(n). \tag{25}
$$

Moreover, for any beams $\boldsymbol{b}_w$, $w \in \Xi$, for $\mathcal{T}$ as above, there is a model $\mathcal{I}$ of $\mathcal{K}$ such that

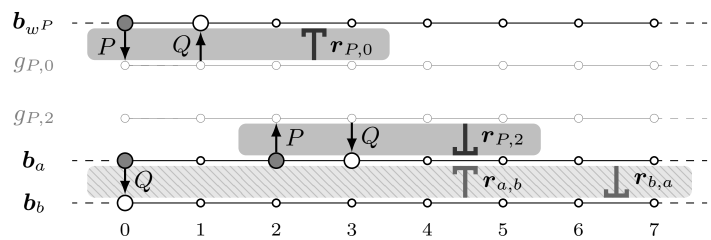
for any $a \in \mathsf{ind}(\mathcal{A})$, the beam $\boldsymbol{b}^{\mathcal{I}}_{a^{\mathcal{I}}}$ coincides with $\boldsymbol{b}_a$,

for any $u \in \Delta^{\mathcal{I}} \setminus \{a^{\mathcal{I}} \mid a \in \mathsf{ind}(\mathcal{A})\}$, there is S and $n \in \mathbb{Z}$ such that $\boldsymbol{b}^{\mathcal{I}}_u(k) = \boldsymbol{b}_{w_S}(k+n)$, for all $k \in \mathbb{Z}$, and

for any $a, b \in \mathsf{ind}(\mathcal{A})$, the rod $\boldsymbol{r}^{\mathcal{I}}_{a^{\mathcal{I}},b^{\mathcal{I}}}$ is the $\mathcal{R}$-canonical rod for $\mathcal{A}_{a,b} = \{\, S(d,e,\ell) \mid S(a,b,\ell) \in \mathcal{A}\,\}$.

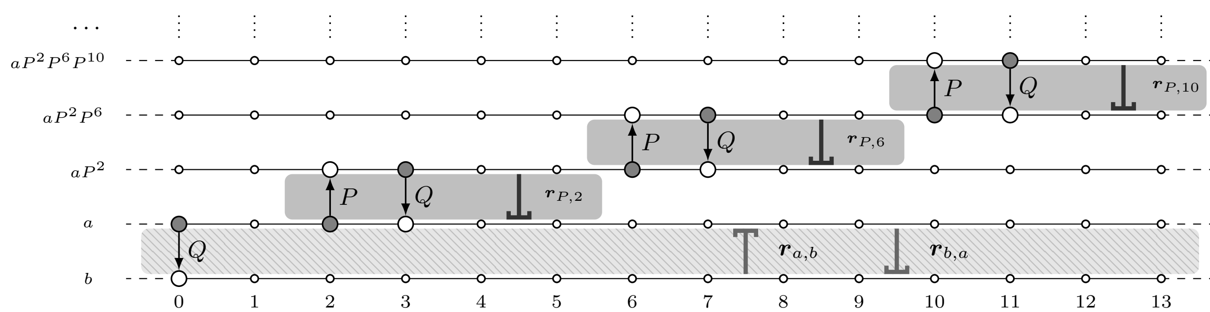
We illustrate the construction by the following example.

Example 1. Let $\mathcal{K} = (\mathcal{O}, \{Q(a,b,0)\})$, where $\mathcal{O}$ consists of $\exists Q \sqcap \Box_F A \sqsubseteq \bot$, $\top \sqsubseteq A \sqcup \exists P$ and $P^- \sqsubseteq \bigcirc_F Q$, obtained by converting $\exists Q \sqsubseteq \Diamond_F \exists P$ and $P^- \sqsubseteq \bigcirc_F Q$ into normal form (1). Beams and rods in Lemma 9 are depicted below:



Beams $\boldsymbol{b}_a$, $\boldsymbol{b}_b$ and $\boldsymbol{b}_{w^P}$ are shown by horizontal lines: the concept type contains $\exists P$ or $\exists Q$ whenever the large node is grey; similarly, the type contains $\exists P^-$ or $\exists Q^-$ whenever the large node is white (the label of the arrow specifies the role); we omit A to avoid clutter. The rods are the arrows between the pairs of horizontal lines. For example, the rod in (24) for a and b is labelled $\boldsymbol{r}_{a,b}$: it contains only Q at 0 (only the positive components of types are given); the rod in (24) for b and a is labelled $\boldsymbol{r}_{b,a}$, and in this case, it is the mirror image of $\boldsymbol{r}_{a,b}$. In fact, if we choose $\mathcal{R}$-canonical rods in (24), then the rod for any b, a will be the mirror image of the rod for a, b. The rod $\boldsymbol{r}_{P,2}$ required by (25) for $\exists P$ on $\boldsymbol{b}_a$ at moment 2 is depicted between $\boldsymbol{b}_a$ and $g_{P,2}$: it contains P at 2 and Q at 3. In fact, it should be clear that, if we choose canonical $\mathcal{R}$-rods in (25), then they will all be isomorphic copies of at most $|\mathcal{R}|$ rods: more precisely, they will be of the form $\boldsymbol{r}_{\{S(d,e,n)\}}$, for a role S from $\mathcal{R}$.

In the proof of Lemma 9, we show how this collection of beams and $\mathcal{R}$-canonical rods can be used to obtain a model $\mathcal{I}$ of $\mathcal{K}$ shown below (again, with A omitted):



Proof. ($\Leftarrow$) Suppose that we have the required collection of beams $\boldsymbol{b}_u$ for $\mathcal{T}$. We construct by induction a sequence $\mathcal{I}_m = (\Delta^{\mathcal{I}_m}, \cdot^{\mathcal{I}_m(n)})$, for $m < \omega$, of temporal interpretations in the following way. To begin with, we set $\Delta^{\mathcal{I}_0} = \mathsf{ind}(\mathcal{A})$, $f_0(a,n) = (a,n)$, for all $a \in \mathsf{ind}(\mathcal{A})$ and $n \in \mathbb{Z}$, $A^{\mathcal{I}_0(n)} = \{a \mid A \in \boldsymbol{b}_a(n)\}$ and $P^{\mathcal{I}_0(n)} = \{(a,b) \mid P \in \boldsymbol{r}_{a,b}(n)\}$, where $\boldsymbol{r}_{a,b}$ is the $\mathcal{R}$-canonical rod for $\mathcal{A}_{a,b}$, which exists by (24), and which, by (21), is compatible with $\boldsymbol{b}_a$ (its inverse is compatible with $\boldsymbol{b}_b$). Suppose next that $\mathcal{I}_m$, for $m \geq 0$, has already been defined, that the elements of $\Delta^{\mathcal{I}_m}$ are words of the form $\lambda = aS_1^{n_1} \dots S_l^{n_l}$, for $a \in \mathsf{ind}(\mathcal{A})$, $n_i \in \mathbb{Z}$ and $l \geq 0$, and that we have a map $f_m \colon \Delta^{\mathcal{I}_m} \times \mathbb{Z} \to \Delta \times \mathbb{Z}$. We call a pair (λ, n) an *S-defect* in $\mathcal{I}_m$ if (i) $f_m(\lambda, n) = (w, n')$, (ii) $\exists S \in \boldsymbol{b}_w(n')$ and (iii) $(\lambda, \lambda') \notin S^{\mathcal{I}_m(n)}$ for any $\lambda' \in \Delta^{\mathcal{I}_m}$. For any role S and any such *S-defect* (λ, n) in $\mathcal{I}_m$, we add the word λS^n to $\Delta^{\mathcal{I}_m}$ and denote the result by $\Delta^{\mathcal{I}_{m+1}}$. By (23), we have $\exists S^- \in \boldsymbol{b}_{w_{S^-}}(l)$, for some $l \in \mathbb{Z}$. We fix one such l and extend f_m to f_{m+1} by setting $f_{m+1}(\lambda S^n, k) = (w_{S^-}, k - n + l)$, for any $k \in \mathbb{Z}$. We also define $A^{\mathcal{I}_{m+1}(k)}$ by extending $A^{\mathcal{I}_m(k)}$ with those λS^n for which $A \in \boldsymbol{b}_{w_{S^-}}(k - n + l)$, and we define $P^{\mathcal{I}_{m+1}(k)}$ by extending $P^{\mathcal{I}_m(k)}$ with $(\lambda, \lambda S^n)$ for which $P \in \boldsymbol{r}_{S,n}(k)$ and with $(\lambda S^n, \lambda)$ for which $P^- \in \boldsymbol{r}_{S,n}(k)$, where $\boldsymbol{r}_{S,n}$ is the $\mathcal{R}$-canonical rod for $\{S^\ddagger(n)\}$, which exists by (25) and which, by (21), is compatible with the beams.

Finally, let $\mathcal{I}$ and f be the unions of all $\mathcal{I}_m$ and f_m, for $m < \omega$, respectively. We show that $\mathcal{I}$ is a model of $\mathcal{K}$. It follows immediately from the construction that $\mathcal{I}$ is a model of $\mathcal{R}$ and $\mathcal{A}$. To show that $\mathcal{I}$ is also a model of $\mathcal{T}$, it suffices to prove that, for any $\lambda \in \Delta^{\mathcal{I}}$ and any role Q, we have $\lambda \in (\exists Q)^{\mathcal{I}(k)}$ iff $\exists Q \in \boldsymbol{b}_w(l)$, where $f(\lambda, k) = (w, l)$. The implication ($\Leftarrow$) follows directly from the procedure of 'curing defects'. Let $\lambda \in (\exists Q)^{\mathcal{I}(k)}$, and so $(\lambda, \lambda') \in Q^{\mathcal{I}(k)}$, for some $\lambda' \in \Delta^{\mathcal{I}}$. Two cases are possible now.

– If $\lambda, \lambda' \in \mathsf{ind}(\mathcal{A})$, then $Q \in \boldsymbol{r}_{\lambda,\lambda'}(k)$. Then, by (24), $\exists Q \in \boldsymbol{b}_\lambda(k)$. It remains to recall that $f(\lambda, k) = (\lambda, k)$.

– If $\lambda' \notin \mathsf{ind}(\mathcal{A})$, then $\lambda' = \lambda S^n$, for some S and n, and $Q \in \boldsymbol{r}_{S,n}(k)$. We also have $\exists S \in \boldsymbol{b}_u(n')$, where $f(\lambda, n) = (u, n')$. By (25), there is a rod $\boldsymbol{r}$ for $\mathcal{R}$ such that $S \in \boldsymbol{r}(n')$, and so, we must have $Q \in \boldsymbol{r}(l)$. Since $\boldsymbol{r}$ is compatible with $\boldsymbol{b}_u$, we obtain $\exists Q \in \boldsymbol{b}_u(l)$, as required.

($\Rightarrow$) Given a model $\mathcal{I}$ of $\mathcal{K}$, we construct beams $\boldsymbol{b}_u$ for $\mathcal{T}$ as follows. Set $\boldsymbol{b}_a = \boldsymbol{b}^{\mathcal{I}}_{a^{\mathcal{I}}}$, for all $a \in \mathsf{ind}(\mathcal{A})$. For each S, if $S^{\mathcal{I}(n)} \neq \emptyset$, for some $n \in \mathbb{Z}$, then set $\boldsymbol{b}_{w_S} = \boldsymbol{b}^{\mathcal{I}}_u$, for $u \in (\exists S)^{\mathcal{I}(n)}$; otherwise, set $\boldsymbol{b}_{w_S} = \boldsymbol{b}^{\mathcal{I}}_{a^{\mathcal{I}}}$, for an arbitrary $a \in \mathsf{ind}(\mathcal{A})$. It is straightforward to check that these beams are as required. ❑

We now reduce existence of the required collection of beams to the satisfiability problem for the one-variable fragment of first-order *LTL*, which is known to be EXPSPACE-complete [20, 15] and thus establish decidability and the upper complexity bound for *DL-Lite*$^{\Box\bigcirc}_{bool/horn}$, which turns out to be tight.

Theorem 10. *The satisfiability problem for DL-Lite*$^{\Box\bigcirc}_{bool/horn}$ *KBs is* EXPSPACE*-complete.*

Proof. We first show decidability and the upper complexity bound. Let $\mathcal{K} = (\mathcal{O}, \mathcal{A})$ be a *DL-Lite*$^{\Box\bigcirc}_{bool/horn}$ KB with $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$. We assume that $\mathcal{R}$ is closed under taking the inverses of roles in RIs.

We define a translation $\psi_{\mathcal{K}}$ of $\mathcal{K}$ into first-order *LTL* with a single individual variable x. We treat elements of Δ as constants in the first-order language, basic concepts B as unary predicates and roles P as binary predicates, assuming that $P_i^-(u, x) = P_i(x, u)$, and let $\psi_{\mathcal{K}}$ be a conjunction of the following sentences, for all constants $u \in \Delta$:

$$\Box\big(C_1(u) \land \dots \land C_k(u) \to C_{k+1}(u) \lor \dots \lor C_{k+m}(u)\big), \quad \text{for CI } C_1 \sqcap \dots \sqcap C_k \sqsubseteq C_{k+1} \sqcup \dots \sqcup C_{k+m} \text{ in } \mathcal{T}, \tag{26}$$

$$\Box\forall x\, \big(R_1(u, x) \land \dots \land R_k(u, x) \to R(u, x)\big), \quad \text{for RI } R_1 \sqcap \dots \sqcap R_k \sqsubseteq R \text{ in } \mathcal{R}, \tag{27}$$

$$\Box\forall x\, \big(R_1(u, x) \land \dots \land R_k(u, x) \to \bot\big), \quad \text{for RI } R_1 \sqcap \dots \sqcap R_k \sqsubseteq \bot \text{ in } \mathcal{R}, \tag{28}$$

$$\bigcirc_F^\ell A(a), \quad \text{for } A(a, \ell) \in \mathcal{A}, \tag{29}$$

$$\bigcirc_F^\ell P(a, b), \quad \text{for } P(a, b, \ell) \in \mathcal{A}, \tag{30}$$

$$\Box\big[(\exists S)(u) \to \Diamond_F \Diamond_P (\exists S^-)(w_{S^-})\big], \quad \text{for } S \in \mathit{role}^{\pm}(\mathcal{O}), \tag{31}$$

$$\Box\big((\exists S)(u) \leftrightarrow \exists x\, S(u, x)\big), \quad \text{for } S \in \mathit{role}^{\pm}(\mathcal{O}). \tag{32}$$

The 'interesting' conjuncts in $\psi_{\mathcal{K}}$ are (31) and (32), which reflect the interaction between $\mathcal{T}$ and $\mathcal{R}$. It can be shown that $\psi_{\mathcal{K}}$ is satisfiable iff there are beams as required by Lemma 9. It can be seen that each collection of beams $\boldsymbol{b}_u$, $u \in \Delta$, for $\mathcal{T}$ gives rise to a model $\mathfrak{M}$ of $\psi_{\mathcal{K}}$: the domain of $\mathfrak{M}$ comprises Δ and elements $g_{S,m}$, for a role S and $m \in \mathbb{Z}$. Then, we fix $\mathcal{R}$-canonical rods $\boldsymbol{r}_{a,b}$ for $\mathcal{A}_{a,b}$, which exist by (24), and $\mathcal{R}$-canonical rods $\boldsymbol{r}_{S,m}$ for $\{S(d, e, m)\}$ for every S and every $m \in \mathbb{Z}$ with $\exists S \in \boldsymbol{b}_u(m)$, for some $u \in \Delta$, which exist by (25), and set, for all $n \in \mathbb{Z}$, basic concepts B, role names P and roles S',

$$\mathfrak{M}, n \models B(u) \ \text{ iff } \ B \in \boldsymbol{b}_u(n), \ \text{for } u \in \Delta,$$
$$\mathfrak{M}, n \models P(a, b) \ \text{ iff } \ P \in \boldsymbol{r}_{a,b}(n), \ \text{for } a, b \in \mathsf{ind}(\mathcal{A}),$$
$$\mathfrak{M}, n \models S'(u, g_{S,m}) \ \text{ iff } \ S' \in \boldsymbol{r}_{S,m}(n), \ \text{for } u \in \Delta, m \in \mathbb{Z} \text{ and roles } S \text{ with } \exists S \in \boldsymbol{b}_u(m).$$

It is readily checked that $\mathfrak{M}$ is as required (in Example 1, the $g_{S,m}$ are represented explicitly by grey horizontal lines). Conversely, it can be verified that every model $\mathfrak{M}$ of $\psi_{\mathcal{K}}$ gives rise to the required collection of beams for $\mathcal{T}$.

The lower bound is established by reduction of the non-halting problem for deterministic Turing machines with exponential tape. More precisely, we assume that the head of a given machine M never runs beyond the first 2^n cells of its tape on an input word $\boldsymbol{a}$ of length m, where $n = p(m)$ for some polynomial p (we will also assume that it never attempts to access cells before the start of the tape). We construct a *DL-Lite*$^{\bigcirc}_{horn}$ ontology that encodes the computation of M on $\boldsymbol{a}$ using a single individual o. The initial configuration is spread over the time instants $1, \dots, 2^n$, from which the first m instants represent $\boldsymbol{a}$ and the remaining ones encode the blank symbol $\#$. The second configuration uses the next 2^n instants $2^n + 1, \dots, 2^n + 2^n$, etc. The configurations are encoded with the following concept names:

– $H_{q,a}$ contains o at the moment $i2^n + j$ whenever the machine head scans the jth cell of the ith configuration and sees symbol a, with q being the current state of the machine;

– S_a contains o at $i2^n + j$ whenever the jth cell of the ith configuration contains a but is *not scanned* by the head.

We can encode computations of the Turing machine M with tape alphabet Γ and transition function δ by means of the following CIs, for $a', a'' \in \Gamma$:

$$\begin{aligned}
\bigcirc_P S_{a'} \sqcap H_{q,a} \sqcap \bigcirc_F S_{a''} &\sqsubseteq \bigcirc_F^{2^n}(\bigcirc_P S_{a'} \sqcap S_b \sqcap \bigcirc_F H_{q',a''}), && \text{for } \delta(q,a) = (q',b,R),\\
\bigcirc_P S_{a'} \sqcap H_{q,a} \sqcap \bigcirc_F S_{a''} &\sqsubseteq \bigcirc_F^{2^n}(\bigcirc_P H_{q',a'} \sqcap S_b \sqcap \bigcirc_F S_{a''}), && \text{for } \delta(q,a) = (q',b,L),\\
\bigcirc_P S_{a'} \sqcap S_a \sqcap \bigcirc_F S_{a''} &\sqsubseteq \bigcirc_F^{2^n} S_a, && \text{for } a' \in \Gamma,
\end{aligned}$$

where $\vartheta \sqsubseteq \bigcirc_F^n(\vartheta_1 \sqcap \dots \sqcap \vartheta_k)$ abbreviates the $\vartheta \sqsubseteq \bigcirc_F^n \vartheta_i$, $1 \le i \le k$. Concepts I and E mark the start and the end of the input $\boldsymbol{a}$, respectively, and F marks the cells of the first configuration, which is filled with blanks $\#$ by using CIs

$$I \sqsubseteq \bigcirc_F^{2^n-2} F, \qquad \bigcirc_F F \sqsubseteq F, \qquad E \sqsubseteq \bigcirc_F E, \qquad E \sqcap F \sqsubseteq \bigcirc_F S_\#;$$

we also assume that $S_\#$ holds at moment 0. Finally,

$$H_{q,a} \sqsubseteq \bot, \text{ for all accepting / rejecting states } q \text{ and } a \in \Gamma,$$

ensure non-termination. These CIs are, however, of the exponential size. We show now how to convert them into a *DL-Lite*$^{\bigcirc}_{horn}$ ontology of polynomial size. Consider a CI of the form $A \sqsubseteq \bigcirc_F^{2^n} B$. First, we replace the concept $\bigcirc_F^{2^n} B$ by $\exists P$ and add the CI $\exists Q \sqsubseteq B$ to the TBox, where P and Q are fresh role names. Then, similarly to the reduction in the proof of Theorem 8, we use RIs with fresh role names $P_0, \dots, P_{n-1}, \bar{P}_0, \dots, \bar{P}_{n-1}$ to encode a binary counter from 0 to $2^n - 1$, where roles $\bar{P}_i$ and P_i stand for 'the ith bit of the counter is 0 and, respectively, 1', and ensure that $\mathcal{R} \models P \sqsubseteq \bigcirc_F^{2^n} Q$ but $\mathcal{R} \not\models P \sqsubseteq \bigcirc_F^{i} Q$ for any $i \neq 2^n$. (Note that $\exists P$ requires a different P-successor at each time point.) Further details are left to the reader. ❑

4.3 Core RIs

We now modify the technique developed above to reduce *DL-Lite*$^{\Box\bigcirc}_{bool/core}$ to *LTL*. The reduction is based on the following observation. Let $\mathcal{R}$ be a *DL-Lite*$^{\Box\bigcirc}_{bool/core}$ RBox and consider the $\mathcal{R}$-canonical rod $\boldsymbol{r}$ for some $\mathcal{A}_R = \{R(d,e,0)\}$. Then $S \in \boldsymbol{r}(n)$ iff one of the following conditions holds:

– $\mathcal{R}', \mathcal{A}_R \models S(d,e,n)$, where $\mathcal{R}'$ is obtained from $\mathcal{R}$ by removing the RIs with $\Box$,
– there is $m > n$ with $|m| \le 2^{|\mathcal{R}|}$ and $\Box_P S \in \boldsymbol{r}(m)$,
– there is $m < n$ with $|m| \le 2^{|\mathcal{R}|}$ and $\Box_F S \in \boldsymbol{r}(m)$.

Let $\min_{R,S}$ be the minimal integer with $\Box_F S \in \boldsymbol{r}(m)$; if it exists, then $|\min_{R,S}| \le 2^{|\mathcal{R}|}$. The maximal integer $\max_{R,S}$ with $\Box_P S \in \boldsymbol{r}(m)$ has the same bound (if exists). The following example shows that these integers can indeed be exponential in $|\mathcal{R}|$.

Example 2. Let $\mathcal{R}$ be the following *DL-Lite*$^{\Box\bigcirc}_{bool/core}$ RBox:

$$\begin{array}{lllll}
P \sqsubseteq R_0, & R_i \sqsubseteq \bigcirc_F R_{(i+1) \bmod 2}, & \text{for } 0 \le i < 2, & R_1 \sqsubseteq Q, & \\
P \sqsubseteq Q_0, & Q_i \sqsubseteq \bigcirc_F Q_{(i+1) \bmod 3}, & \text{for } 0 \le i < 3, & Q_1 \sqsubseteq Q, & Q_2 \sqsubseteq Q,\\
 & & & P \sqsubseteq Q, & P \sqsubseteq \Box_P Q.
\end{array}$$

Clearly, $\mathcal{R} \models P \sqsubseteq \bigcirc_F^6 \Box_P Q$. If instead of the 2- and 3-cycles we use p_i-cycles, where p_i is the ith prime number and $1 \le i \le n$, then $\mathcal{R} \models P \sqsubseteq \bigcirc_F^{p_1 \times \dots \times p_n} \Box_P Q$.

In any case, the existence and binary representations of $\min_{R,S}$ and $\max_{R,S}$ can be computed in PSPACE.

Theorem 11. *For DL-Lite*$^{\Box\bigcirc}_{bool/core}$ *and DL-Lite*$^{\Box\bigcirc}_{horn/core}$ *KBs, the satisfiability problem is* PSPACE*-complete.*

Proof. We encode $\mathcal{K}$ in *LTL* following the proof of Theorem 10 and representing (26)–(31) as *LTL*-formulas with variables of the form $C^\dagger(u)$, $R^\dagger(u,v)$, for $u, v \in \Delta$. Sentences (32), however, require a different treatment. First, take

$$\Box\big(\bigcirc_1(\exists S_1)^\dagger(u) \to \bigcirc_2(\exists S_2)^\dagger(u)\big), \tag{33}$$

for every $\bigcirc_1 S_1 \sqsubseteq \bigcirc_2 S_2$ in $\mathcal{R}$, where each $\bigcirc_i$ is $\bigcirc_F$, $\bigcirc_P$ or blank. Then, we need CIs of the form $\exists R \sqsubseteq \bigcirc^{\max_{R,S}} \Box_P \exists S$ and $\exists R \sqsubseteq \bigcirc^{\min_{R,S}} \Box_F \exists S$, for all R and S with defined $\max_{R,S}$ and $\min_{R,S}$, which are not entailed by (33). These integers can be represented in binary using n bits, where n is polynomial in $|\mathcal{R}|$. Assuming that $\max_{R,S} \geq 0$, we encode, for example, $\exists R \sqsubseteq \bigcirc^{\max_{R,S}} \Box_P \exists S$ by

$$\Box\big(\Box_F \Diamond_F (\exists R)^\dagger(u) \to \Box_F (\exists S)^\dagger(u)\big), \quad (34)$$

$$\Box\big((\exists R)^\dagger(u) \wedge \neg\Diamond_F (\exists R)^\dagger(u) \to \bigcirc_F^{\max_{R,S}} D_u^{R,S}\big), \quad (35)$$

$$\Box\big(D_u^{R,S} \to \Box_P (\exists S)^\dagger(u)\big), \quad (36)$$

where (35) is expressed by $O(n^2)$ formulas encoding the binary counter (similar to those in the proof of Theorem 8). To explain the meaning of (34)–(36), consider any $w \in \Delta^\mathcal{I}$ in a model $\mathcal{I}$ of $\mathcal{K}$. If $w \in (\exists R)^{\mathcal{I}(n)}$ for infinitely many $n > 0$, then $w \in (\exists S)^{\mathcal{I}(n)}$ for all n, which is captured by (34). Otherwise, there is n such that $w \in (\exists R)^{\mathcal{I}(n)}$ and $w \notin (\exists R)^{\mathcal{I}(m)}$, for $m > n$, whence $w \in (\exists S)^{\mathcal{I}(k)}$, for any $k < n + \max_{R,S}$, which is captured by (35) and (36).

The *LTL* translation $\Psi_\mathcal{K}$ of $\mathcal{K}$ is a conjunction of (26)–(31), (33) and (34)–(36) for all R and S with defined $\max_{R,S}$, and their counterparts for $\exists R \sqsubseteq \bigcirc^{\min_{R,S}} \Box_F \exists S$. One can show that $\mathcal{K}$ is satisfiable iff $\Psi_\mathcal{K}$ is satisfiable. The PSPACE lower bound follows from the fact that every *LTL*-formula is equisatisfiable with some $LTL_{core}^{\Box\bigcirc}$ KB. ❑

5 FO(RPR)-Rewritability of $DL\text{-}Lite_{bool/horn}^{\Box\bigcirc}$

We next investigate the data complexity of the satisfiability problem for temporal *DL-Lite* KBs. Again, out first result is negative:

Theorem 12. *There is a $DL\text{-}Lite_{g\text{-}bool/g\text{-}bool}^{\bigcirc}$ ontology $\mathcal{O}$ for which the satisfiability of $(\mathcal{O}, \mathcal{A})$, for a given $\mathcal{A}$, is undecidable.*

Proof. Using the representation of the universal Turing machine by means of tiles (see, e.g., [9]), we obtain a set $\mathfrak{U}$ of tile types for which the following problem is undecidable: given a finite sequence of tile types $i_0, \dots, i_n$, decide whether $\mathfrak{U}$ can tile the $\mathbb{N} \times \mathbb{N}$ grid so that tiles of types $i_0, \dots, i_n$ are placed on $(0, 0), \dots, (n, 0)$, respectively. Given such $i_0, \dots, i_n$, we take the ABox $\mathcal{A} = \{ I(a, 0), R_{i_0}(a, b, 0), \dots, R_{i_n}(a, b, n) \}$. Then $\mathfrak{U}$ can tile $\mathbb{N} \times \mathbb{N}$ with $i_0, \dots, i_n$ on the first row iff $\mathcal{A}$ is consistent with the ontology $\mathcal{O}_\mathfrak{U}$, which is defined by (10) for the set $\mathfrak{U}$, iff $A(a, 0)$ is *not* a certain answer to OMAQ $(\mathcal{O}_\mathfrak{U}, A)$ over $\mathcal{A}$, where A is a fresh concept name. Thus, $\mathcal{O}_\mathfrak{U}$ is as required. ❑

We obtain our positive results by means of FO-rewritability. Let $\mathcal{L} \in \{ \text{FO}(<), \text{FO}(<, +), \text{FO(RPR)} \}$. Our first aim is to show that $\mathcal{L}$-rewritability of $DL\text{-}Lite_{bool/horn}^{\Box\bigcirc}$ ontologies can be reduced to $\mathcal{L}$-rewritability of *ontology-mediated atomic queries* (or OMAQs) with *LTL* ontologies.

An *OMAQ* is a pair of the form $(\mathcal{O}, A)$ or $(\mathcal{O}, P)$, where $\mathcal{O}$ is an ontology, A a concept and P a role name. A *certain answer* to $(\mathcal{O}, A)$ over an ABox $\mathcal{A}$ is any $(a, \ell) \in \mathsf{ind}(\mathcal{A}) \times \mathsf{tem}(\mathcal{A})$ such that $a^\mathcal{I} \in A^{\mathcal{I}(\ell)}$ for every model $\mathcal{I}$ of $(\mathcal{O}, \mathcal{A})$; a *certain answer* to $(\mathcal{O}, P)$ over $\mathcal{A}$ is any $(a, b, \ell) \in \mathsf{ind}(\mathcal{A}) \times \mathsf{ind}(\mathcal{A}) \times \mathsf{tem}(\mathcal{A})$ with $(a^\mathcal{I}, b^\mathcal{I}) \in P^{\mathcal{I}(\ell)}$ for every $\mathcal{I} \models (\mathcal{O}, \mathcal{A})$. The set of all certain answers to $(\mathcal{O}, A)$ over $\mathcal{A}$ is denoted by $\mathsf{ans}(\mathcal{O}, A, \mathcal{A})$. As a technical tool in our constructions, we also require 'certain answers' in which ℓ ranges over the whole $\mathbb{Z}$ rather than only the *active temporal domain* $\mathsf{tem}(\mathcal{A})$; we denote the set of such certain answers *over $\mathcal{A}$ and $\mathbb{Z}$* by $\mathsf{ans}^\mathbb{Z}(\mathcal{O}, A, \mathcal{A})$. An *$\mathcal{L}$-rewriting* of $(\mathcal{O}, A)$ is an $\mathcal{L}$-formula $\Phi(x, t)$ such that (a, ℓ) is a certain answer to $(\mathcal{O}, A)$ over any ABox $\mathcal{A}$ iff $\mathfrak{S}_\mathcal{A} \models \Phi(a, \ell)$; an $\mathcal{L}$-rewriting of $(\mathcal{O}, P)$ is defined similarly.

First, we show how to reduce the satisfiability problem for $DL\text{-}Lite_{bool/horn}^{\Box\bigcirc}$ ontologies $\mathcal{O}$ to answering OMAQs $(\mathcal{O}', A_\bot)$ with a $\bot$-free ontology $\mathcal{O}'$ and a concept name $A_\bot$. More precisely, for any ABox $\mathcal{A}$, the KB $(\mathcal{O}, \mathcal{A})$ is satisfiable iff $(\mathcal{O}', A_\bot)$ has no certain answers over $\mathcal{A}$.

Let $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$. We define $\mathcal{O}' = \mathcal{T}' \cup \mathcal{R}'$ as follows. The RBox $\mathcal{R}'$ is obtained by replacing every occurrence of $\bot$ in $\mathcal{R}$ with a fresh role name $P_\bot$ and adding the RI $P \sqsubseteq P_\bot$, for any P *inconsistent with* $\mathcal{O}$ in the sense that $(\mathcal{O}, \{P(a, b, 0)\})$ has no models. The TBox $\mathcal{T}'$ results from replacing every $\bot$ in $\mathcal{T}$ with a fresh concept $A_\bot$ and adding the CIs $\exists P_\bot \sqsubseteq A_\bot$, $\exists P_\bot^- \sqsubseteq A_\bot$ together with $A_\bot \sqsubseteq \Box_F A_\bot$ and $A_\bot \sqsubseteq \Box_P A_\bot$ saying that $A_\bot$ is global: if $u \in A_\bot^{\mathcal{I}(n)}$ for *some* $n \in \mathbb{Z}$, then $u \in A_\bot^{\mathcal{I}(n)}$ for *all* $n \in \mathbb{Z}$. By Theorem 10, $\mathcal{O}'$ can be constructed in exponential space.

Theorem 13. *If $\Phi_\bot(x, t)$ is an $\mathcal{L}$-rewriting of the OMAQ $(\mathcal{O}', A_\bot)$, then $\exists x, t\, \Phi_\bot(x, t)$ is an $\mathcal{L}$-rewriting of $\mathcal{O}$.*

Proof. It suffices to show that, for any ABox $\mathcal{A}$, the KB $(\mathcal{O}, \mathcal{A})$ is satisfiable iff $(\mathcal{O}', A_\bot)$ has no certain answers over $\mathcal{A}$. Suppose $\mathcal{O}$ and $\mathcal{A}$ are consistent. Given any model $\mathcal{I}$ of $\mathcal{O}$ and $\mathcal{A}$, we extend it to an interpretation $\mathcal{I}'$ by setting $A_\bot^{\mathcal{I}'(n)} = \emptyset$ and $P_\bot^{\mathcal{I}'(n)} = \emptyset$, for all $n \in \mathbb{Z}$. Clearly, $\mathcal{I}'$ is a model of $\mathcal{O}'$, and so there are no certain answers to $(\mathcal{O}', A_\bot)$ over $\mathcal{A}$.

Conversely, suppose that $(\mathcal{O}', A_\bot)$ has no certain answers over $\mathcal{A}$. We show how to construct a model $\mathcal{I}$ of $(\mathcal{O}, \mathcal{A})$. Obviously, $\mathcal{O}'$ and $\mathcal{A}$ are consistent. For each $a \in \mathsf{ind}(\mathcal{A})$, there is a model $\mathcal{I}_a$ of $(\mathcal{O}', \mathcal{A})$ such that $a^{\mathcal{I}_a} \notin A_\bot^{\mathcal{I}_a(n)}$ for all $n \in \mathbb{Z}$ (recall that $A_\bot$ is global). Also, for each role S consistent with $\mathcal{O}$, there is a model $\mathcal{I}_S$ of $(\mathcal{O}', \{S(w, u, 0)\})$ such that $w^{\mathcal{I}_S} \notin A_\bot^{\mathcal{I}_S(n)}$ for all $n \in \mathbb{Z}$ (again, $A_\bot$ is global). We take, for each $a \in \mathsf{ind}(\mathcal{A})$, the beam $\boldsymbol{b}_a$ for $a^{\mathcal{I}_a}$ in $\mathcal{I}_a$, and for each role S consistent with $\mathcal{O}$, the beam $\boldsymbol{b}_{w_S}$ of $w^{\mathcal{I}_S}$ in $\mathcal{I}_S$, and apply Lemma 9 to obtain a model $\mathcal{I}$ of $(\mathcal{O}', \mathcal{A})$ such that $A_\bot^{\mathcal{I}} = \emptyset$ and $P_\bot^{\mathcal{I}} = \emptyset$. By construction, $\mathcal{I}$ is also a model of $(\mathcal{O}, \mathcal{A})$. ❑

Next, we show that $\mathcal{L}$-rewritability of a $\bot$-free OMAQ with an *DL-Lite*$^{\Box\bigcirc}_{bool/horn}$ ontology is reducible to $\mathcal{L}$-rewritability of a role-free OMAQ. Ontologies without roles are clearly a notational variant of *LTL* ontologies; hence, in this case we prefer to write '*LTL*$^{\Box\bigcirc}_{bool}$ ontologies'. We first explain the reduction by instructive examples. The first two examples illustrate the interaction between the DL and temporal dimensions in *DL-Lite*$^{\Box\bigcirc}_{bool/horn}$ that we need to take into account when constructing the *LTL* OMAQs to which the rewritability of $\bot$-free *DL-Lite*$^{\Box\bigcirc}_{bool/horn}$ OMAQs is reduced.

Example 3. Let $\mathcal{T} = \{\, B \sqsubseteq \exists P,\ \exists Q \sqsubseteq A \,\}$ and $\mathcal{R} = \{\, P \sqsubseteq \bigcirc_{\!F} Q \,\}$. An obvious idea of constructing a rewriting for the OMAQ $\boldsymbol{q} = (\mathcal{T} \cup \mathcal{R},\ A)$ would be to find first a rewriting of the *LTL* OMAQ $(\mathcal{T}^\dagger, A^\dagger)$ obtained from $(\mathcal{T}, A)$ by replacing the basic concepts $\exists P$ and $\exists Q$ with surrogate concept names $(\exists P)^\dagger = E_P$ and $(\exists Q)^\dagger = E_Q$, respectively. This would give us the first-order query $A(t) \lor E_Q(t)$. By restoring the intended meaning of A and E_Q, we would then obtain $A(x,t) \lor \exists y\, Q(x, y, t)$. The second step would be to rewrite, using the RBox $\mathcal{R}$, the atom $Q(x, y, t)$ into $Q(x, y, t) \lor P(x, y, t-1)$. Alas, the resulting formula

$$A(x,t) \lor \exists y \left(Q(x,y,t) \lor P(x,y,t-1)\right)$$

falls short of being a rewriting of $\boldsymbol{q}$ as it does not return the certain answer $(a, 1)$ over $\mathcal{A} = \{\, B(a, 0) \,\}$. The reason is that, in our construction, we did not take into account the concept inclusion $\exists P \sqsubseteq \bigcirc_{\!F} \exists Q$, which is a consequence of $\mathcal{R}$. If we now add the 'connecting axiom' $(\exists P)^\dagger \sqsubseteq \bigcirc_{\!F} (\exists Q)^\dagger$ to $\mathcal{T}^\dagger$, then in the first step we obtain $A(t) \lor E_Q(t) \lor E_P(t-1) \lor B(t-1)$, which gives us the correct FO($<$)-rewriting

$$A(x,t) \lor \exists y \left(Q(x,y,t) \lor P(x,y,t-1)\right) \lor \exists y\, P(x,y,t-1) \lor B(x,t-1)$$

of $\boldsymbol{q}$, where the third disjunct is obviously redundant and can be omitted.

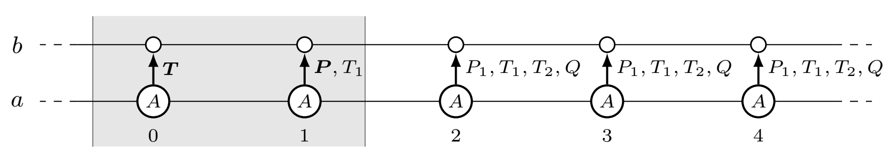
Example 4. Consider now $\mathcal{T} = \{\, \exists Q \sqsubseteq \Box_{\!P} A \,\}$, $\mathcal{R} = \{\, P \sqsubseteq \Box_{\!F} P_1,\ T \sqsubseteq \Box_{\!F} T_1,\ T_1 \sqsubseteq \Box_{\!F} T_2,\ P_1 \sqcap T_2 \sqsubseteq Q \,\}$ and $\boldsymbol{q} = (\mathcal{T} \cup \mathcal{R},\ A)$. The two-step construction outlined in Example 3 would give us first the formula

$$\Phi(x,t) \ = \ A(x,t) \lor \exists t' \left((t < t') \land \exists y\, Q(x,y,t')\right)$$

as the rewriting of $(\mathcal{T}, A)$. The reader can also readily check that the following formula is a rewriting of $(\mathcal{R}, Q)$:

$$\begin{aligned}\Psi(x,y,t') \ = \ Q(x,y,t') \ \lor \ \big(&\left[P_1(x,y,t') \lor \exists t'' \left((t'' < t') \land P(x,y,t'')\right)\right] \land \\ &\left[T_2(x,y,t') \lor \exists t'' \left((t'' < t') \land \left(T_1(x,y,t'') \lor \exists t''' \left((t''' < t'') \land T(x,y,t''')\right)\right)\right)\right]\big).\end{aligned}$$

However, the result of replacing $Q(x, y, t')$ in $\Phi(x, t)$ with $\Psi(x, y, t')$ is not an FO-rewriting of $(\mathcal{O}, A)$: when evaluated over $\mathcal{A} = \{\, T(a, b, 0),\ P(a, b, 1) \,\}$, it does not return the certain answers $(a, 0)$ and $(a, 1)$; see below, where the active temporal domain is shaded:



(Note that these answers would be found had we evaluated the obtained 'rewriting' over $\mathbb{Z}$ rather than $\{0, 1\}$.) This time, in the two-step construction of the rewriting, we are missing the 'consequence' $\exists(\Box_{\!F} P_1 \sqcap \Box_{\!F} T_2) \sqsubseteq \Box_{\!F} \exists Q$ of $\mathcal{R}$ and $\mathcal{T}$. To fix the problem, we can take a fresh role name G_ρ, for $\rho = \{\, \Box_{\!F} P_1, \Box_{\!F} T_2 \,\}$ (the pair (a, b) in the picture above would belong to G_ρ at moment $1 = \max \mathcal{A}$), and add the 'connecting axiom' $\exists G_\rho \sqsubseteq \Box_{\!F} \exists Q$ to $\mathcal{T}$. Then, in the first step, we rewrite the extended TBox and A into the formula

$$\Phi'(x,t) \ = \ A(x,t) \lor \exists t' \left((t < t') \land \exists y\, Q(x,y,t')\right) \lor \exists y\, G_\rho(x,y,t),$$

where we replace $Q(x, y, t')$ with $\Psi(x, y, t')$ as before, and restore the intended meaning of $G_\rho(x, y, t)$ by rewriting $(\mathcal{R}, \Box_{\!F} P_1 \sqcap \Box_{\!F} T_2)$ into $P(x, y, t) \land \left(T_1(x, y, t) \lor \exists t' \left((t' < t) \land T(x, y, t')\right)\right)$ and substituting it for $G_\rho(x, y, t)$ in $\Phi'(x, t)$.

We now formally define the connecting axioms for a given $\textit{DL-Lite}^{\Box\bigcirc}_{\textit{bool/horn}}$ ($\bot$-free) ontology $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$. We assume that $\mathcal{R}$ contains all the role names from $\mathcal{T}$. Recall that a role type ρ for $\mathcal{R}$ is a maximal subset of $\mathsf{sub}_\mathcal{R}$ consistent with $\mathcal{R}$. In this section, we use only the positive part of role types (ignoring all the negated roles): in particular, we say that a type is non-empty if it contains a role R. Given a role type ρ, we consider the $\mathcal{R}$-canonical rod $\boldsymbol{r}_\rho$ for $\{R(0,d,e) \mid R \in \rho\}$. Note that, by definition, we have $\boldsymbol{r}_\rho(0) = \rho$. By the well-known properties of *LTL*, we can find positive integers $s^\rho \leq |\mathcal{R}|$ and $p^\rho \leq 2^{2|\mathcal{R}|}$ such that

$$\boldsymbol{r}_\rho(n) = \boldsymbol{r}_\rho(n - p^\rho), \quad \text{for } n \leq -s^\rho, \qquad \text{and} \qquad \boldsymbol{r}_\rho(n) = \boldsymbol{r}_\rho(n + p^\rho), \quad \text{for } n \geq s^\rho.$$

For a role type ρ for $\mathcal{R}$, we take a fresh role name G_ρ and fresh concept names D^n_ρ, for $-s^\rho - p^\rho < n < s^\rho + p^\rho$, and define the following CIs:

$$\exists G_\rho \sqsubseteq D^0_\rho, \qquad D^n_\rho \sqsubseteq \bigcirc_F D^{n+1}_\rho, \ \text{ for } 0 \leq n < s^\rho + p^\rho - 1, \qquad D^{s^\rho + p^\rho - 1}_\rho \sqsubseteq \bigcirc_F D^{s^\rho}_\rho,$$
$$\text{and} \qquad D^n_\rho \sqsubseteq \exists S, \ \text{ for roles } S \in \boldsymbol{r}_\rho(n) \text{ and } 0 \leq n < s^\rho + p^\rho,$$

together with symmetrical CIs for $-s^\rho - p^\rho \leq n \leq 0$ for the past-time 'loop'. Let **(con)** be the set of all such CIs for all possible role types ρ for $\mathcal{R}$, and let $\mathcal{T}_\mathcal{R} = \mathcal{T} \cup$ **(con)**.

Example 5. In Example 3, for role type $\rho = \{P, \bigcirc_F Q\}$, we have $s^\rho = 2$, $p^\rho = 1$, and so $\mathcal{T}_\mathcal{R}$ contains the following:

$$\exists P \sqsubseteq D^0_\rho, \quad D^0_\rho \sqsubseteq \bigcirc_F D^1_\rho, \quad D^1_\rho \sqsubseteq \bigcirc_F D^2_\rho, \quad D^2_\rho \sqsubseteq \bigcirc_F D^2_\rho, \quad \text{and} \quad D^0_\rho \sqsubseteq \exists P, \quad D^1_\rho \sqsubseteq \exists Q,$$

which imply $\exists P \sqsubseteq \bigcirc_F \exists Q$. In the context of Example 4, for role type $\rho = \{\Box_F P_1, \Box_F T_2\}$, we have $s^\rho = 1$, $p^\rho = 1$, and so $\mathcal{T}_\mathcal{R}$ contains the following CIs:

$$\exists G_\rho \sqsubseteq D^0_\rho, \quad D^0_\rho \sqsubseteq \bigcirc_F D^1_\rho, \quad D^1_\rho \sqsubseteq \bigcirc_F D^1_\rho, \quad \text{and} \quad D^1_\rho \sqsubseteq \exists P_1, \quad D^1_\rho \sqsubseteq \exists T_2, \quad D^1_\rho \sqsubseteq \exists Q.$$

Note that, in this case, instead of two CIs $D^0_\rho \sqsubseteq \bigcirc_F D^1_\rho$ and $D^1_\rho \sqsubseteq \bigcirc_F D^1_\rho$, we could use a single $D^0_\rho \sqsubseteq \Box_F D^1_\rho$.

We denote by $\mathcal{T}^\ddagger_\mathcal{R}$ be the $\textit{LTL}^{\Box\bigcirc}_{\textit{bool}}$ TBox obtained from $\mathcal{T}_\mathcal{R}$ by replacing every basic concept B in it with the surrogate $B^\dagger$. Now, consider an ABox $\mathcal{A}$. For any $a, b \in \mathsf{ind}(\mathcal{A})$, let $\boldsymbol{r}_{a,b}$ be the $\mathcal{R}$-canonical rod for $\mathcal{A}_{a,b}$. We split $\mathcal{A}$ into the concept and role components, $\mathcal{U}$ and $\mathcal{B}$, as follows:

$$\mathcal{U} = \{ A(a,\ell) \mid A(a,\ell) \in \mathcal{A} \},$$
$$\mathcal{B} = \{ \exists G_\rho(a,\ell) \mid a \in \mathsf{ind}(\mathcal{A}),\ \ell \in \mathsf{tem}(\mathcal{A}) \text{ and } \rho = \boldsymbol{r}_{a,b}(\ell) \text{ is non-empty, for some } b \in \mathsf{ind}(\mathcal{A}) \}.$$

We denote by $\mathcal{U}^\dagger_a$ and $\mathcal{B}^\dagger_a$ the sets of all atoms $A^\dagger(\ell)$, for $A(a,\ell) \in \mathcal{U}$, and $(\exists G_\rho)^\dagger(\ell)$, for $\exists G_\rho(a,\ell) \in \mathcal{B}$, respectively. Observe now that the connecting axioms are such that $(\mathbf{con})^\dagger$ is an $\textit{LTL}^{\bigcirc}_{\textit{core}}$ ontology, and the ABox $\mathcal{B}$ is defined so that, for any $a \in \mathsf{ind}(\mathcal{A})$ and $n \in \mathbb{Z}$,

$$S \in \boldsymbol{r}_{a,b}(n), \text{ for some } b \in \mathsf{ind}(\mathcal{A}), \quad \text{iff} \quad (\exists S)^\dagger(n) \in \mathcal{C}_{(\mathbf{con})^\dagger, \mathcal{B}^\dagger_a}, \qquad \text{for any role } S \text{ in } \mathcal{R}. \tag{37}$$

Indeed, suppose that $S \in \boldsymbol{r}_{a,b}(n)$. If $n \in \mathsf{tem}(\mathcal{A})$, then, for $\rho = \boldsymbol{r}_{a,b}(n)$, we have $S \in \rho$ and $\exists G_\rho(a,n) \in \mathcal{B}$, whence $(\exists G_\rho)^\dagger(n) \in \mathcal{B}^\dagger_a$, and so $(\exists S)^\dagger(n) \in \mathcal{C}_{(\mathbf{con})^\dagger, \mathcal{B}^\dagger_a}$. If $n > \max \mathcal{A}$, then we consider $\rho = \boldsymbol{r}_{a,b}(\max \mathcal{A})$. It should be clear that the $\mathcal{R}$-canonical rod for $\{R(\max \mathcal{A}, d, e) \mid R \in \rho\}$ contains $S(n)$. So, ρ is non-empty and $\exists G_\rho(a, \max \mathcal{A}) \in \mathcal{B}$, whence $(\exists G_\rho)^\dagger(\max \mathcal{A}) \in \mathcal{B}^\dagger_a$, and so $(\exists S)^\dagger(n) \in \mathcal{C}_{(\mathbf{con})^\dagger, \mathcal{B}^\dagger_a}$ The case $n < \min \mathcal{A}$ is symmetric. The converse implication follows directly from the definition of **(con)**. We use (37) to establish the following key technical result:

Lemma B. *Let $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$ and A a concept name from $\mathcal{O}$. Then, for any ABox $\mathcal{A}$, we have*

$$\mathsf{ans}^\mathbb{Z}(\mathcal{O}, A, \mathcal{A}) = \{ (a,n) \mid a \in \mathsf{ind}(\mathcal{A}) \textit{ and } n \in \mathsf{ans}^\mathbb{Z}(\mathcal{T}^\ddagger_\mathcal{R}, A, \mathcal{U}^\dagger_a \cup \mathcal{B}^\dagger_a) \}.$$

Proof. ($\subseteq$) Suppose $n \notin \mathsf{ans}^\mathbb{Z}(\mathcal{T}^\ddagger_\mathcal{R}, B^\dagger, \mathcal{U}^\dagger_a \cup \mathcal{B}^\dagger_a)$. Then there is an *LTL* model $\mathcal{I}_a$ of $(\mathcal{T}^\ddagger_\mathcal{R}, \mathcal{U}^\dagger_a \cup \mathcal{B}^\dagger_a)$ with $\mathcal{I}_a, n \not\models A$. We define a model $\mathcal{I}$ of $(\mathcal{O}, \mathcal{A})$ with $a^\mathcal{I} \notin A^{\mathcal{I}(n)}$ using unravelling (Lemma 9). To begin with, we take the beam $\boldsymbol{b}_a \colon n \mapsto \{ C \in \mathsf{sub}_\mathcal{T} \mid \mathcal{I}_a, n \models C^\dagger \}$; note that $\boldsymbol{b}_a$ is a beam for $\mathcal{T}$ because $\mathcal{T}^\ddagger_\mathcal{R}$ extends $\mathcal{T}^\dagger$. By (37), the $\mathcal{R}$-canonical rod $\boldsymbol{r}_{a,b}$ for $\mathcal{A}_{a,b}$ is $\boldsymbol{b}_a$-compatible, for each $b \in \mathsf{ind}(\mathcal{A})$. Next, we fix a model $\mathcal{J}$ of $(\mathcal{O}, \mathcal{A})$ and, for every $b \in \mathsf{ind}(\mathcal{A}) \setminus \{a\}$, take the beam $\boldsymbol{b}_b$ of $b^\mathcal{J}$ in $\mathcal{J}$. By Lemma 9, we obtain a model $\mathcal{I}$ of $(\mathcal{O}, \mathcal{A})$ with $a^\mathcal{I} \notin A^{\mathcal{I}(n)}$.

The implication ($\supseteq$) is straightforward. ❑

We now use this technical result to construct rewritings for OMAQs $(\mathcal{O}, B)$ from rewritings of suitable *LTL* OMAQs, where we identify a role type ρ with the intersection of all $R \in \rho$ and $\rho^\ddagger$:

Theorem 14. *A DL-Lite$^{\Box\bigcirc}_{bool/horn}$ OMAQ $(\mathcal{O}, A)$ with a $\bot$-free $\mathcal{O} = \mathcal{T} \cup \mathcal{R}$ is $\mathcal{L}$-rewritable whenever*

- *the LTL$^{\Box\bigcirc}_{bool}$ OMAQ $(\mathcal{T}^\dagger_\mathcal{R}, A)$ is $\mathcal{L}$-rewritable and*
- *the LTL$^{\Box\bigcirc}_{horn}$ OMAQ $(\mathcal{R}, R)$ is $\mathcal{L}$-rewritable, for every temporalised role in $\mathcal{R}$.*

Proof. We obviously have $\mathcal{L}$-rewritings $\boldsymbol{Q}_\rho(x, y, t)$ of $\boldsymbol{q}_\rho = (\mathcal{R}, \rho)$. Then the $\mathcal{L}$-formula $\boldsymbol{Q}(x, t)$ obtained from an $\mathcal{L}$-rewriting $\boldsymbol{Q}^\dagger(t)$ of $\boldsymbol{q}^\dagger$ by replacing every $A'(s)$ in it with $A'(x, s)$, every $(\exists P)^\dagger(s)$ with $\exists y\, P(x, y, s)$, every $(\exists P^-)^\dagger(s)$ with $\exists y\, P(y, x, s)$, every $(\exists G_\rho)^\dagger(s)$ with $\exists y\, \boldsymbol{Q}_\rho(x, y, s)$ and, in the case of FO(RPR), by replacing every $Q(t_1, \dots, t_k)$, for a relation variable Q, with $R(x, t_1, \dots, t_k)$ is an $\mathcal{L}$-rewriting of $\boldsymbol{q}$.

Indeed, we show that $\mathcal{O}, \mathcal{A} \models A(a, \ell)$ iff $\mathfrak{S}_\mathcal{A} \models \boldsymbol{Q}(a, \ell)$, for any ABox $\mathcal{A}$, any $\ell \in \mathsf{tem}(\mathcal{A})$ and any $a \in \mathsf{ind}(\mathcal{A})$. If A is a concept name not in $\mathcal{O}$, then the claim is trivial. Otherwise, by Lemma B, $\mathcal{O}, \mathcal{A} \models A(a, \ell)$ iff $\mathcal{T}^\dagger_\mathcal{R}, \mathcal{U}^\dagger_a \cup \mathcal{B}^\dagger_a \models A(\ell)$. As $\boldsymbol{Q}^\dagger(t)$ is an $\mathcal{L}$-rewriting of $(\mathcal{T}^\dagger_\mathcal{R}, A)$, the latter is equivalent to $\mathfrak{S}_{\mathcal{U}^\dagger_a \cup \mathcal{B}^\dagger_a} \models \boldsymbol{Q}^\dagger(\ell)$. Now, since $\boldsymbol{Q}_\rho(x, y, t)$ is an $\mathcal{L}$-rewriting of $\boldsymbol{q}_\rho$, for all $b \in \mathsf{ind}(\mathcal{A})$ and $n \in \mathsf{tem}(\mathcal{A})$, we have $(\exists G_\rho)^\dagger(n) \in \mathcal{B}^\dagger_b$ iff $\rho = \boldsymbol{r}_{b,c}(n)$ for the canonical rod for $\mathcal{A}_{b,c}$, for some $c \in \mathsf{ind}(\mathcal{A})$, iff $\mathfrak{S}_\mathcal{A} \models \exists y\, \boldsymbol{Q}_\rho(b, y, n)$. Then, $\mathfrak{S}_{\mathcal{U}^\dagger_a \cup \mathcal{B}^\dagger_a} \models \boldsymbol{Q}^\dagger(\ell)$ iff $\mathfrak{S}_\mathcal{A} \models \boldsymbol{Q}(a, \ell)$, as required. ❑

As a first consequence of Theorems 13 and 14, we immediately obtain:

Theorem 15. *Every DL-Lite$^{\Box\bigcirc}_{bool/horn}$ ontology is* FO(RPR)*-rewritable.*

Indeed, we can obviously rewrite any *LTL*$^{\Box\bigcirc}_{bool}$ OMAQ to a *monadic second-order* (or MSO($<$)-) formula, mimicking the *LTL* semantics, and then use [13], according to which MSO($<$)-rewritability implies FO(RPR)-rewritability. Note that, as follows from [3, Theorem 9], satisfiability of *LTL*$^{\bigcirc}_{horn}$ KBs is NC^1-hard for data complexity, and so satisfiability of *DL-Lite*$^{\Box\bigcirc}_{bool/horn}$ ontologies is NC^1-complete.

6 FO($<, +$)-Rewritability of *DL-Lite*$^{\Box\bigcirc}_{krom/core}$

If $\mathcal{O} = (\mathcal{T}, \mathcal{R})$ is a *DL-Lite*$^{\Box\bigcirc}_{krom/core}$ ontology, then the TBox $\mathcal{T}_\mathcal{R}$ constructed above is in *DL-Lite*$^{\Box\bigcirc}_{krom/core}$, and so, by Theorem 14, we can show $\mathcal{L}$-rewritability of $\mathcal{O}$ by establishing $\mathcal{L}$-rewritability of every *LTL*$^{\Box\bigcirc}_{krom}$ OMAQ. It is known from [3] that *LTL*$^{\bigcirc}_{krom}$ OMAQs are FO($<, +$)-rewritable, while *LTL*$^{\Box}_{bool}$ OMAQs are FO($<$)-rewritable. Here we establish FO($<, +$)-rewritability of all *LTL*$^{\Box\bigcirc}_{krom}$ OMAQs. The proof utilises the monotonicity of the $\Box$ operators, similarly to the proof of [3, Theorem 11]. However, the latter relies on partially-ordered NFAs accepting the models of $(\mathcal{O}, \mathcal{A})$, which do not work in the presence of $\bigcirc$. Our key observation here is that every model of $(\mathcal{O}, \mathcal{A})$ has at most $O(|\mathcal{O}|)$ timestamps such that the same $\Box$-concepts hold between any two nearest of them. The placement of these timestamps and their concept-types can be described by an FO($<$)-formula. However, to check whether these types are compatible (i.e., satisfiable in some model), we require FO($<, +$)-formulas similar to those in the proof of [3, Theorem 10].

Theorem 16. *Any LTL$^{\Box\bigcirc}_{krom}$ OMAQ is* FO($<, +$)*-rewritable.*

Proof. Let $\boldsymbol{q} = (\mathcal{O}, A)$ be an *LTL*$^{\Box\bigcirc}_{krom}$ OMAQ. We can assume that A occurs in $\mathcal{O}$, which has no nested occurrences of temporal operators and contains CIs $\bigcirc B \equiv A_{\bigcirc B}$, for every $\bigcirc B$ in $\mathcal{O}$ with $\bigcirc \in \{\bigcirc_F, \bigcirc_P\}$. Define an NFA $\mathfrak{A}_\mathcal{O}$ that recognises ABoxes $\mathcal{A}$ consistent with $\mathcal{O}$, represented as words $X_{\min \mathcal{A}}, \dots, X_{\max \mathcal{A}}$, where

$$X_i = \{\, B \mid B(i) \in \mathcal{A} \text{ and } B \text{ occurs in } \mathcal{O} \,\}, \quad i \in \mathsf{tem}(\mathcal{A}).$$

The set $\mathfrak{T}$ of states in $\mathfrak{A}_\mathcal{O}$ comprises maximal sets τ of concepts of $\mathcal{O}$ consistent with $\mathcal{O}$; we refer to such τ as *types for $\mathcal{O}$*. Now, for any $\tau, \tau' \in \mathfrak{T}$ and an alphabet symbol X, the NFA $\mathfrak{A}_\mathcal{O}$ has a transition $\tau \to_X \tau'$ just in case the following conditions hold:

$$X \subseteq \tau', \tag{38}$$

$$\bigcirc_F C \in \tau \text{ iff } C \in \tau', \qquad \bigcirc_P C \in \tau' \text{ iff } C \in \tau, \tag{39}$$

$$\Box_F C \in \tau \text{ iff } C, \Box_F C \in \tau', \qquad \Box_P C \in \tau' \text{ iff } C, \Box_P C \in \tau; \tag{40}$$

As $\tau \to_X \tau'$ implies $\tau \to_\emptyset \tau'$, for any X, we omit $\emptyset$ from $\to_\emptyset$. Since all τ in $\mathfrak{T}$ are consistent with $\mathcal{O}$, every state in $\mathfrak{A}_\mathcal{O}$ has a $\to$-predecessor and a $\to$-successor. Thus, for any ABox $\mathcal{A}$ represented as $X_0, X_1, \dots, X_m$, a timestamp ℓ $(0 \le \ell \le m)$ is not a certain answer to $\boldsymbol{q}$ over $\mathcal{A}$ iff there is a path

$$\pi \;=\; \tau_{-1} \to_{X_0} \tau_0 \to_{X_1} \tau_1 \to_{X_2} \dots \to_{X_m} \tau_m,$$

in $\mathfrak{A}_\mathcal{O}$ with $A \notin \tau_\ell$. This criterion can be encoded by an *infinite* FO-expression $\Psi(t)$ of the form

$$\neg\Big[\bigvee_{\substack{\tau_0\to\ldots\to\tau_m\\ \text{is a path in } \mathfrak{A}_\mathcal{O}}} \Big(\bigwedge_{0\le i\le m} \mathsf{type}_{\tau_i}(i) \;\wedge \bigvee_{0\le i\le m \text{ with } A\notin\tau_i} (t=i)\Big)\Big],$$

where the disjunction is over all (possibly infinitely many) paths and $\mathsf{type}_\tau(t)$ is a conjunction of all $\neg B(t)$ with $B \notin \tau$, for concept names B in $\mathcal{O}$: the first conjunct ensures, by contraposition, that any B from X_i also belongs to τ_i, while the second conjunct guarantees that $A \notin \tau_\ell$ in case $\ell = t$.

We write $\tau \to^\Box \tau'$ if τ and τ' satisfy (40), but not necessarily (39). One can show that any path $\tau_0 \to \ldots \to \tau_m$ in $\mathfrak{A}_\mathcal{O}$ contains a subsequence

$$\tau_{s_0} \to^\Box \tau_{s_1} \to^\Box \ldots \to^\Box \tau_{s_{d-1}} \to^\Box \tau_{s_d}$$

such that $0 = s_0 < s_1 < \cdots < s_{d-1} < s_d = m$ for $d \le 2|\mathcal{O}| + 1$ and, for all $i < d$, either $\Box C, C \in \tau_{s_i}, \tau_j$ or $\Box C \notin \tau_{s_i}, \tau_j$, for all $\Box C$ in $\mathcal{O}$, $\Box \in \{\Box_P, \Box_F\}$, and all $j \in (s_i, s_{i+1})$.

To deal with the $\bigcirc$-operators, we consider the $\mathit{LTL}^{\bigcirc}_{\mathit{krom}}$ ontology $\tilde{\mathcal{O}}$ obtained from $\mathcal{O}$ by first extending it with the CIs

$$\Box_P C \sqsubseteq \bigcirc_P \Box_P C \;\text{ and }\; \Box_P C \sqsubseteq \bigcirc_P C, \quad \text{for all } \Box_P C \text{ in } \mathcal{O}, \tag{41}$$
$$\Box_F C \sqsubseteq \bigcirc_F \Box_F C \;\text{ and }\; \Box_F C \sqsubseteq \bigcirc_F C, \quad \text{for all } \Box_F C \text{ in } \mathcal{O}, \tag{42}$$

which are obvious $\mathit{LTL}^{\Box\bigcirc}_{\mathit{krom}}$ tautologies, and then replacing every $\Box_P C$ and $\Box_F C$ with its *surrogate*, a fresh concept name. Let $G_{\tilde{\mathcal{O}}}$ be the infinite directed graph whose vertices are pairs (L, n), for a simple literal L (a concept name or its negation) in $\tilde{\mathcal{O}}$ and $n \in \mathbb{Z}$. It contains an edge from (L, n) to $(L', n + k)$, for $k \in \{-1, 0, 1\}$, iff $\tilde{\mathcal{O}} \models L \sqsubseteq \bigcirc^k L'$. We write $(L_1, n_1) \rightsquigarrow (L_2, n_2)$ if $G_{\tilde{\mathcal{O}}}$ has a path from (L_1, n_1) to (L_2, n_2), which means that $\tilde{\mathcal{O}} \models \bigcirc^{n_1} L_1 \sqsubseteq \bigcirc^{n_2} L_2$. We slightly abuse notation and write, for example, $L \in \tau$ for a type τ in case L is the surrogate for $\Box_P C$ and τ contains $\Box_P C$.

Lemma 17. *For any ABox $\mathcal{A}$, a timestamp $\ell \in \mathsf{tem}(\mathcal{A})$ is not a certain answer to $\boldsymbol{Q}$ over $\mathcal{A}$ iff there are $d \le 2|\mathcal{O}| + 2$, a sequence $\tau_0 \to^\Box \ldots \to^\Box \tau_d$ of types for $\mathcal{O}$ and a sequence $\min \mathcal{A} = s_0 < \cdots < s_d = \max \mathcal{A}$ satisfying the following conditions:*

$$B \in \tau_i, \textit{ for each } B(s_i) \in \mathcal{A}, \qquad \textit{for all } 1 \le i \le d; \tag{43}$$
$$(B, n) \not\rightsquigarrow (\neg B', n'), \textit{ for } s_i < n, n' < s_{i+1} \textit{ with } B(n), B'(n') \in \mathcal{A}, \qquad \textit{for all } 1 \le i < d; \tag{44}$$
$$(L, s_i) \not\rightsquigarrow (\neg B', n'), \textit{ for } L \in \tau_i \textit{ and } s_i < n' < s_{i+1} \textit{ with } B'(n') \in \mathcal{A}, \qquad \textit{for all } 1 \le i < d; \tag{45}$$
$$(B, n) \not\rightsquigarrow (\neg L', s_{i+1}), \textit{ for } s_i < n < s_{i+1} \textit{ with } B(n) \in \mathcal{A} \textit{ and } L' \in \tau_{i+1}, \qquad \textit{for all } 1 \le i < d; \tag{46}$$
$$(L, s_i) \not\rightsquigarrow (\neg L', s_{i+1}), \textit{ for } L \in \tau_i \textit{ and } L' \in \tau_{i+1}, \qquad \textit{for all } 1 \le i < d; \tag{47}$$
$$\ell = s_i, \textit{ for some } 0 \le i \le d \textit{ such that } A \notin \tau_i. \tag{48}$$

Proof. ($\Leftarrow$) Suppose $\ell \notin \mathsf{ans}(\boldsymbol{q}, \mathcal{A})$. Then there is a model $\mathcal{I}$ of $\mathcal{O}$ and $\mathcal{A}$ such that $\ell \notin A^\mathcal{I}$. We consider the sequence $\bar{\tau}_0 \to \bar{\tau}_1 \to \ldots \to \bar{\tau}_{m-1} \to \bar{\tau}_m$ of types for $\mathcal{O}$ given by $\mathcal{I}$, where $0 = \min \mathcal{A}$ and $m = \max \mathcal{A}$. As argued above, we can find subsequences of types and respective indexes between $\min \mathcal{A}$ and $\max \mathcal{A}$, whose length does not exceed $2|\mathcal{O}| + 2$. We add $\bar{\tau}_\ell$ to obtain the sequences satisfying conditions (43)–(48).

($\Rightarrow$) Suppose there sequence $\tau_0 \to^\Box \tau_1 \to^\Box \cdots \to^\Box \tau_d$ of types for $\mathcal{O}$ and $\min \mathcal{A} = s_0 < s_1 < \cdots < s_d = \max \mathcal{A}$ of indexes satisfying conditions (43)–(48). We use these sequences to construct a model $\mathcal{I}$ of $\mathcal{O}$ and $\mathcal{A}$ with $\ell \notin A^\mathcal{I}$. The model is defined as a sequence of types $\bar{\tau}_n$, for $n \in \mathbb{Z}$. We begin by setting $\bar{\tau}_{s_i} = \tau_i$, for $0 \le i \le d$. Then, since $\bar{\tau}_{\min \mathcal{A}} = \tau_0$ is consistent with $\mathcal{O}$, there is a model $\mathcal{I}_{\min}$ of $\mathcal{O}$ with type τ_0 at $\min \mathcal{A}$, and so, we take the types $\bar{\tau}_n$ given by $\mathcal{I}_{\min}$ for $n < \min \mathcal{A}$. Similarly, the types $\bar{\tau}_n$, for $n > \max \mathcal{A}$, are provided by a model of $\mathcal{O}$ with $\bar{\tau}_{\max} = \tau_d$ at $\max \mathcal{A}$. Now, let $1 \le i < d$. We show how to construct the $\bar{\tau}_j$, for $s_i < j < s_{i+1}$, in a step-by-step manner.

Step 0: for all j with $s_i < j < s_{i+1}$, set $\bar{\tau}_j = \{ B \mid B(j) \in \mathcal{A} \}$.

Step 1: for all k with $s_i \le k \le s_{i+1}$ and all j with $s_i < j < s_{i+1}$, if $L \in \bar{\tau}_k$ and $(L, k) \rightsquigarrow (L', j)$, then add L' to $\bar{\tau}_j$.

Step $m > 1$: pick $\bar{\tau}_k$, for $s_i < k < s_{i+1}$, and a literal L with $L, \neg L \notin \bar{\tau}_k$, terminating the construction if there are none. Add L to $\bar{\tau}_k$, and, for all j with $s_i < j < s_{i+1}$, if $(L, k) \rightsquigarrow (L', j)$, then add L' to $\bar{\tau}_j$.

Note that $\bar{\tau}_k$ could also be extended with $\neg L$—either choice is consistent with the previously constructed types $\bar{\tau}_j$.

By induction on m, we show that the $\bar{\tau}_k$ constructed in Step m is *conflict-free* in the sense that there is no k, $s_i < k < s_{i+1}$, and no literal L_0 with $L_0, \neg L_0 \in \bar{\tau}_k$, which is obvious for $m = 0$. Suppose that $\bar{\tau}_k$ is not conflict-free after Step 1. Then one of the following six cases has happened in Step 1 (we assume $s_i < n, n' < s_{i+1}$, if relevant, below):

– if $L, L' \in \bar{\tau}_{s_i}$, $(L, s_i) \rightsquigarrow (L_0, k)$, $(L', s_i) \rightsquigarrow (\neg L_0, k)$, then $(L, s_i) \rightsquigarrow (\neg L', s_i)$, contrary to consistency of τ_i with $\mathcal{O}$;

– if $L, L' \in \bar{\tau}_{s_{i+1}}$, $(L, s_{i+1}) \rightsquigarrow (L_0, k)$, $(L', s_{i+1}) \rightsquigarrow (\neg L_0, k)$, then $(L, s_{i+1}) \rightsquigarrow (\neg L', s_{i+1})$, which is also impossible;

– if $B(n), B'(n') \in \mathcal{A}$ with $(B, n) \rightsquigarrow (L_0, k)$, $(B', n') \rightsquigarrow (\neg L_0, k)$, then $(B, n) \rightsquigarrow (\neg B', n')$, contrary to (44);

– if $L \in \bar{\tau}_{s_i}$, $(L, s_i) \rightsquigarrow (L_0, k)$ and $B'(n') \in \mathcal{A}$, $(B', n') \rightsquigarrow (\neg L_0, k)$, then $(L, s_i) \rightsquigarrow (\neg B', n')$, contrary to (45);

– if $B(n) \in \mathcal{A}$, $(B, n) \rightsquigarrow (L_0, k)$ and $L' \in \bar{\tau}_{s_{i+1}}$, $(L', s_{i+1}) \rightsquigarrow (\neg L_0, k)$, then $(B, n) \rightsquigarrow (\neg L', s_{i+1})$, contrary to (46);

– if $L \in \bar{\tau}_{s_i}$, $(L, s_i) \rightsquigarrow (L_0, k)$ and $L' \in \bar{\tau}_{s_{i+1}}$, $(L', s_{i+1}) \rightsquigarrow (\neg L_0, k)$, then $(L, s_i) \rightsquigarrow (\neg L', s_{i+1})$, contrary to (47).

Thus, the $\bar{\tau}_k$ constructed in Step 1 are conflict-free. Suppose now that all of the $\bar{\tau}_k$ are conflict-free after Step m, $m \geq 1$, while some $\bar{\tau}_j$ after Step $m + 1$ is not. It follows that some $\bar{\tau}_k$ is extended with L in Step $m + 1$, $(L, k) \rightsquigarrow (L', j)$, but $\bar{\tau}_j$ contained $\neg L'$ (at least) since Step m. Now, as $(\neg L', j) \rightsquigarrow (\neg L, k)$, the type $\bar{\tau}_k$ already contained $\neg L$ in Step m, and so L could not be added in Step $m + 1$.

Let $\bar{\tau}_n$, $n \in \mathbb{Z}$, be the resulting sequence of conflict-free types. Define an interpretation $\mathcal{I}$ by taking $n \in B^{\mathcal{I}}$ iff $B \in \bar{\tau}_n$, for every concept name B in $\mathcal{O}$. In view of (43) and Step 0, we have $\mathcal{I} \models \mathcal{A}$ but $\mathcal{I} \not\models A(\ell)$. We show $\mathcal{I} \models \mathcal{O}$. Since the $\bar{\tau}_n$ are conflict-free, and thus consistent with $\tilde{\mathcal{O}}$, and since $L_1 \in \bar{\tau}_n$ implies $L_2 \in \bar{\tau}_n$ if $\tilde{\mathcal{O}}$ contains $L_1 \sqsubseteq L_2$, and $L_1 \in \bar{\tau}_n$ implies $L_2 \notin \bar{\tau}_n$ if $\tilde{\mathcal{O}}$ contains $L_1 \sqcap L_2 \sqsubseteq \bot$, it is sufficient to prove that

$$A_{\bigcirc_F B} \in \bar{\tau}_n \quad \text{iff} \quad B \in \bar{\tau}_{n+1} \qquad \text{and} \qquad \Box_F C \in \bar{\tau}_n \quad \text{iff} \quad C \in \bar{\tau}_k \text{ for all } k > n,$$

and the past counterparts of these equivalences. We readily obtain the first equivalence since $(A_{\bigcirc_F B}, n) \rightsquigarrow (B, n + 1)$ and $(B, n{+}1) \rightsquigarrow (A_{\bigcirc_F B}, n)$, and similarly for $\bigcirc_P$. It thus remains to show the second equivalence. For all $n \geq \max \mathcal{A}$, the claim is immediate from the choice of the $\bar{\tau}_k$ for $k \geq s_d = \max \mathcal{A}$. We then proceed by induction on i from $d - 1$ to 0 assuming that the claim holds for all $n \geq s_{i+1}$. We consider the following three options. If $\Box_F C \in \bar{\tau}_{s_i}$, then, by (40), we have $\Box_F C, C \in \bar{\tau}_{s_{i+1}}$, and the claim for all $n \geq s_i$ follows from the fact that $\tilde{\mathcal{O}}$ contains CIs (41). If $\Box_F C \notin \bar{\tau}_{s_i}$, then, by (40), either $\Box_F C \notin \bar{\tau}_{s_{i+1}}$ or $C \notin \bar{\tau}_{s_{i+1}}$; in either case, the claim for all $n \geq s_i$ is immediate from the induction hypothesis and the fact that $\tilde{\mathcal{O}}$ contains CIs (41). This finishes the inductive argument, and the claim for $n < s_0$ then follows from the choice of the $\bar{\tau}_k$ for $k \leq s_0 = \min \mathcal{A}$. A symmetric argument shows that $\Box_P C \in \bar{\tau}_n$ iff $L \in \bar{\tau}_k$ for all $k < n$. This completes the proof of Lemma 17. ❑

We can now define an FO$(<, +)$-rewriting $\boldsymbol{Q}(t)$ of $\boldsymbol{q}$ by encoding the conditions of Lemma 17 as follows:

$$\boldsymbol{Q}(t) = \neg\Big[\bigvee_{d \leq 2|\mathcal{O}|+2} \; \bigvee_{\tau_0 \to^{\Box} \dots \to^{\Box} \tau_d} \exists t_0, \dots, t_d \, \Big(\mathsf{path}_{\tau_0 \to^{\Box} \dots \to^{\Box} \tau_d}(t_0, \dots, t_d) \land \bigvee_{0 \leq i \leq d \text{ with } A \notin \tau_i} (t = t_i)\Big)\Big],$$

where $\mathsf{path}_{\tau_0 \to^{\Box} \dots \to^{\Box} \tau_d}(t_0, \dots, t_d)$ is the formula

$$\begin{aligned}
&(t_0 = \min) \land (t_d = \max) \land \bigwedge_{0 \leq i < d} (t_i < t_{i+1}) \land \bigwedge_{0 \leq i \leq d} \mathsf{type}_{\tau_i}(t_i) \\
\land &\bigwedge_{0 \leq i < d} \Big[\bigwedge_{L \in \tau_i,\, L' \in \tau_{i+1}} \neg \mathsf{entails}_{L, \neg L'}(t_i, t_{i+1}) \\
&\qquad \land \bigwedge_{L \in \tau_i} \forall t' \in (t_i, t_{i+1}) \, \big(B'(t') \to \neg \mathsf{entails}_{L, \neg B'}(t_i, t')\big)
\end{aligned}$$

$$\wedge \bigwedge_{L' \in \tau_{i+1}} \forall t \in (t_i, t_{i+1}) \big(B(t) \to \neg \mathsf{entails}_{B,\neg L'}(t, t_{i+1})\big)$$

$$\wedge \bigwedge_{B,B' \text{ in } \tilde{\mathcal{O}}} \forall t, t' \in (t_i, t_{i+1}) \big(B(t) \wedge B'(t') \to \neg \mathsf{entails}_{B,\neg B'}(t, t')\big)\Big]$$

and where $\mathsf{entails}_{L_1,L_2}$ is such that $\mathfrak{S}_\mathcal{A} \models \mathsf{entails}_{L_1,L_2}(n_1, n_2)$ iff $\tilde{\mathcal{O}} \models \bigcirc^{n_1} L \sqsubseteq \bigcirc^{n_2} L_2$, for any $n_1, n_2 \in \mathsf{tem}(\mathcal{A})$; see [3, Theorem 10]. Note that the outermost disjunction in $\boldsymbol{Q}(t)$ can be empty, in particular when $\mathcal{O}$ is inconsistent, in which case the rewriting $\boldsymbol{Q}(t)$ is simply $\top$. ❑

As a consequence of Theorems 13, 14 and 16, we obtain:

Theorem 18. *DL-Lite*$^{\Box\bigcirc}_{krom/core}$ *ontologies are all* FO$(<,+)$*-rewritable.*

7 Conclusions

We extended the *DL-Lite* family of description logics by languages with Krom, Horn and arbitrary Boolean role inclusions and identified their computational complexity. We observed, in particular, that Boolean RIs make *DL-Lite* as expressive as FO^2, while covering Krom RIs $\top \sqsubseteq R_1 \sqcup R_2$ come for free as far as satisfiability is concerned.

We used those languages as a basis for defining a new type of temporal DLs. So far the main approach to designing well-behaved fragments of first-order temporal logic has been the monodicity principle, which disallows temporal operators before a formula with two or more free variables. The main contribution of this paper is to show that by restricting the use of classical connectives one can obtain natural and decidable fragments whose expressivity for binary relations is not captured by the monodicity principle.

Interesting directions of future work include establishing the tight combined complexity of *DL-Lite*$^{\Box\bigcirc}_{horn/krom}$ and the data complexity of *DL-Lite* with Krom RIs. We also plan to investigate the problem of answering queries mediated by ontologies in our temporal languages. Answering unions of conjunctive queries (UCQs) is undecidable with *DL-Lite*$^{krom}_{krom}$ ontologies [27] and 2EXPTIME-complete for *DL-Lite*$^{g\text{-}bool}_{bool}$ [7, 11]. UCQs with *DL-Lite*$^{horn}_{horn}$ ontologies are FO$(<)$-rewritable; with *DL-Lite*$^{g\text{-}bool}_{bool}$ ontologies they are CONP-complete for data complexity. Temporal *instance* queries are FO$(<)$-rewritable for *DL-Lite*$^{\Box}_{core}$ and FO$(<,+)$-rewritable for *DL-Lite*$^{\bigcirc}_{core}$ [3].

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