Products of ‘transitive’ modal logics without the (abstract) finite model property

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Abstract

It is well known that many two-dimensional products of modal logics with at least one ‘transitive’ (but not ‘symmetric’) component lack the product finite model property. Here we show that products of two ‘transitive’ logics (such as, e.g., $\mathbf{K4} \times \mathbf{K4}$, $\mathbf{S4} \times \mathbf{S4}$, $\mathbf{Grz} \times \mathbf{Grz}$ and $\mathbf{GL} \times \mathbf{GL}$) do not have the (abstract) finite model property either. These are the first known examples of 2D modal product logics without the finite model property where both components are natural unimodal logics having the finite model property.

Keywords: multi-modal logic, finite model property, product logics.

1 Introduction

Products of modal (in particular, temporal, spatial, epistemic, etc.) logics is a very natural and clear construction arising in both pure logic and various applications; see, e.g., [14, 5, 2, 15, 9, 4, 19]. Introduced in the 1970s [17, 18], products have been intensively studied in the last decade (for a comprehensive exposition see [8]). The obtained results that are relevant to the decision problem for two-dimensional products can be briefly summarised as follows:

- We know that the product of two first-order definable and recursively enumerable logics is recursively enumerable [9].
- We know that 2D products with logics like **K** and **S5** are usually decidable (but of high—sometimes extremely high—computational complexity) [15, 9, 12, 8].
- We know that products of two ‘linear transitive’ logics like **K4.3** or **GL.3** are mostly undecidable [13, 16].
- Yet, despite all efforts, we still have *no clue* to the computational behaviour of products of two ‘transitive’ (but not ‘symmetric’) logics where at least one component logic has branching frames (say, $\mathbf{K4.3} \times \mathbf{K4}$ or $\mathbf{S4} \times \mathbf{S4}$). The only known results involve linear components that are either Noetherian or discrete. For example, it is known that $\mathsf{Log}(\mathbb{N}, <) \times \mathbf{K4}$ and $\mathsf{Log}(\mathbb{N}, <) \times \mathbf{S4}$ are not recursively enumerable [8] (and the available proof heavily uses the discreteness of $(\mathbb{N}, <)$).

Not only have we no idea about solutions to these decision problems, but—unlike the unimodal case [3]—very little is known about the frames for multimodal transitive logics with interacting modal operators in general. Without exaggeration one can say that the study of *multimodal transitive logics* in general, and *2D product logics* and *commutators* in particular, is one of the *most challenging and intriguing topics of modern modal logic.*

The aim of this note is twofold. First, we show that products (in fact, already the commutators) of two *standard* transitive modal logics are rather expressive: in particular, they can say that their models must be *infinite.* Although it is very tempting to try to show that finitely axiomatisable logics like **K4** × **K4** are decidable by proving first that they enjoy the finite model property (the authors have made several attempts in this direction), one should not yield to the temptation—*these logics do not have the finite model property.*

Second, we hope that the formulas below that require infinite transitive, commutative and Church-Rosser frames will either help in encoding some undecidable problem and showing that **K4** × **K4**-type logics are undecidable, or give a hint on how their infinite models can be represented by some finite means, say, using mosaics or quasimodels, in order to prove decidability.

2 Definitions

Given unimodal Kripke frames $\mathfrak{F}_1 = (W_1, R_1)$ and $\mathfrak{F}_2 = (W_2, R_2)$, their *product* is defined to be the bimodal frame

$$\mathfrak{F}_1 \times \mathfrak{F}_2 = (W_1 \times W_2, R_h, R_v),$$

where $W_1 \times W_2$ is the Cartesian product of W_1 and W_2 and, for all $u, u' \in W_1$, $v, v' \in W_2$,

$$\begin{aligned} (u, v)R_h(u', v') \quad &\text{iff} \quad uR_1u' \text{ and } v = v', \\ (u, v)R_v(u', v') \quad &\text{iff} \quad vR_2v' \text{ and } u = u'. \end{aligned}$$

Let L_1 be a normal (uni)modal logic in the language with the box $\boxminus$ and the diamond $\diamondminus$. Let L_2 be a normal (uni)modal logic in the language with the box $\boxbar$ and the diamond $\diamondbar$. Assume also that both L_1 and L_2 are Kripke complete. Then the *product* of the logics L_1 and L_2 is the normal (bi)modal logic $L_1 \times L_2$ in the language $\mathcal{ML}_2$ with the boxes $\boxminus$, $\boxbar$ and the diamonds $\diamondminus$, $\diamondbar$ which is characterised by the class of product frames $\mathfrak{F}_1 \times \mathfrak{F}_2$, where $\mathfrak{F}_i$ is a frame for L_i, $i = 1, 2$. (Here we assume that $\boxminus$ and $\diamondminus$ are interpreted by R_h, while $\boxbar$ and $\diamondbar$ are interpreted by R_v.)

Although product logics $L_1 \times L_2$ are Kripke complete by definition, of course there can be (and in general there *will* be) other, *non-product*, frames for $L_1 \times L_2$. This gives rise to two different types of the finite model property. As usual, a bimodal logic L (in particular, a product logic $L_1 \times L_2$) is said to have the (*abstract*) *finite model property* (*fmp*, for short) if, for every $\mathcal{ML}_2$-formula $\varphi \notin L$, there is a finite frame $\mathfrak{F}$ for L such that $\mathfrak{F} \not\models \varphi$. (By a standard argument, this means that $\mathfrak{M} \not\models \varphi$ for some *finite model* $\mathfrak{M}$ for L; see, e.g., [3].) And we say that $L_1 \times L_2$ has the *product finite model property* (*product fmp*, for short) if, for every $\mathcal{ML}_2$-formula $\varphi \notin L_1 \times L_2$, there is a finite *product* frame $\mathfrak{F}$ for $L_1 \times L_2$ such that $\mathfrak{F} \not\models \varphi$.

Clearly, the product fmp implies the fmp. Examples of 2D product logics having the product fmp (and so the fmp) are **K** × **K**, **K** × **S5**, and **S5** × **S5** (see [8] and references therein). On the other hand, there are product logics, such as **K4** × **S5** and **S4** × **S5**, that do enjoy the (abstract) fmp [9], but lack the product fmp [8]. In general, it is well known that many product logics with at least one 'transitive' (but not 'symmetric') component do not have the product fmp (see, e.g., Theorems 5.32, 5.33 and 7.10 in [8]). Here we recall an example of an $\mathcal{ML}_2$-formula that can be used to show the many such logics do not have the product fmp:

$$\Box_h^+ \Diamond_v p \wedge \Box_h^+ \Box_v (p \to \Diamond_h \Box_h^+ \neg p)$$

(here $\Box_h^+ \psi$ abbreviates $\psi \wedge \Box_h \psi$). Note that this formula (as well as the others known so far) is satisfiable in an appropriate finite (in fact, very small) non-product frame.

Product logics are defined in a semantical way: they are logics determined by classes of product frames. So a good starting point in understanding their behaviour is to find basic principles that hold for every product frame $(W_1 \times W_2, R_h, R_v)$:

- *left commutativity*: $\forall x \forall y \forall z \, \big(xR_v y \wedge yR_h z \to \exists u \, (xR_h u \wedge uR_v z)\big)$,
- *right commutativity*: $\forall x \forall y \forall z \, \big(xR_h y \wedge yR_v z \to \exists u \, (xR_v u \wedge uR_h z)\big)$,
- *Church–Rosser property*: $\forall x \forall y \forall z \, \big(xR_v y \wedge xR_h z \to \exists u \, (yR_h u \wedge zR_v u)\big)$.

These properties can also be expressed by the $\mathcal{ML}_2$-formulas

$$\Diamond_v \Diamond_h p \to \Diamond_h \Diamond_v p, \qquad \Diamond_h \Diamond_v p \to \Diamond_v \Diamond_h p, \qquad \Diamond_h \Box_v p \to \Box_v \Diamond_h p. \tag{1}$$

Given Kripke complete unimodal logics L_1 and L_2, their *commutator* $[L_1, L_2]$ is the smallest normal modal logic in the language $\mathcal{ML}_2$ which contains L_1, L_2 and the axioms (1).

As product frames satisfy the commutativity and Church-Rosser properties, we always have $[L_1, L_2] \subseteq L_1 \times L_2$. For some logics, in particular **K4** or **S4**, the converse also holds: for example, $\mathbf{K4} \times \mathbf{K4} = [\mathbf{K4}, \mathbf{K4}]$; see [9, 8]. On the other hand, e.g., $[\mathbf{K4.3}, \mathbf{K4}] \subsetneqq \mathbf{K4.3} \times \mathbf{K4}$; see Theorem 5.15 in [8]. In general, $[L_1, L_2]$ can even be Kripke incomplete.

3 Results

From now on we only consider products of '*transitive*' (uni)modal logics, that is, extensions of **K4**. Our aim is to show that products of two logics such as **K4**, **K4.3**, **S4**, **Grz** or **GL** do not have the (abstract) fmp. In fact, these are the first known examples of 2D modal product logics without the fmp where both components are standard (uni)modal logics having the fmp. A preceding example of such a product, where one of the components is bimodal ($\mathbf{Lin} \times \mathbf{S5}$), can be found in [15] (this result is generalised a bit in Theorem 5.30 of [8]).

We remind the reader that a frame (W, R) is called *Noetherian* if there is no infinite strictly ascending chain $x_0 R x_1 R x_2 R \ldots$ of points from W (i.e., no R-chain such that $x_i \neq x_{i+1}$, for all $i < \omega$).

Theorem 1. *Let L_1 and L_2 be Kripke complete normal (uni)modal logics containing* **K4** *and such that both L_1 and L_2 have among their frames a rooted Noetherian linear order with an infinite descending chain of distinct points. Then all bimodal logics L in the interval*

$$[L_1, L_2] \subseteq L \subseteq L_1 \times L_2$$

lack the (abstract) fmp.

Corollary 1.1. *Let L_1 and L_2 be any logics from the list*

$$\mathbf{K4},\ \mathbf{K4.1},\ \mathbf{K4.2},\ \mathbf{K4.3},\ \mathbf{S4},\ \mathbf{S4.1},\ \mathbf{S4.2},\ \mathbf{S4.3},\ \mathbf{GL},\ \mathbf{GL.3},\ \mathbf{Grz},\ \mathbf{Grz.3}.$$

Then neither $[L_1, L_2]$ nor $L_1 \times L_2$ have the (abstract) fmp.

Proof of Theorem 1. Let φ be the conjunction of the following bimodal formulas:

$$\Box_{-}\Box_{|}\big((h \vee \Diamond_{|} h \to \Box_{|} h) \wedge (\neg h \vee \Diamond_{|}\neg h \to \Box_{|}\neg h)\big), \tag{2}$$

$$\Box_{-}\Box_{|}\big((v \vee \Diamond_{-} v \to \Box_{-} v) \wedge (\neg v \vee \Diamond_{-}\neg v \to \Box_{-}\neg v)\big), \tag{3}$$

$$\neg h \wedge \neg v \wedge \Diamond_{|}\Diamond_{-}\big(h \wedge v \wedge \Box_{-}\Box_{|}(h \wedge v)\big), \tag{4}$$

$$\Box_{-}\Box_{|}(\blacksquare_{|}\bot \wedge \blacksquare_{-}\bot \to p), \tag{5}$$

$$\blacksquare_{|}\blacklozenge_{-}(\neg p \wedge \blacksquare_{-} p), \tag{6}$$

$$\blacksquare_{-}\blacklozenge_{|}(p \wedge \blacksquare_{|}\neg p), \tag{7}$$

$$\Box_{-}\Box_{|}\big(p \to \blacksquare_{-}(p \wedge \blacklozenge_{|} p)\big), \tag{8}$$

$$\Box_{-}\Box_{|}\big(\neg p \to \blacksquare_{|}(\neg p \wedge \blacklozenge_{-}\neg p)\big), \tag{9}$$

where

$$\begin{aligned}
\blacklozenge_{-}\psi &= \big[h \to \Diamond_{-}\big(\neg h \wedge (\psi \vee \Diamond_{-}\psi)\big)\big] \wedge \big[\neg h \to \Diamond_{-}\big(h \wedge (\psi \vee \Diamond_{-}\psi)\big)\big],\\
\blacklozenge_{|}\psi &= \big[v \to \Diamond_{|}\big(\neg v \wedge (\psi \vee \Diamond_{|}\psi)\big)\big] \wedge \big[\neg v \to \Diamond_{|}\big(v \wedge (\psi \vee \Diamond_{|}\psi)\big)\big],\\
\blacksquare_{-}\psi &= \neg\blacklozenge_{-}\neg\psi, \qquad \text{and} \qquad \blacksquare_{|}\psi = \neg\blacklozenge_{|}\neg\psi.
\end{aligned}$$

On the one hand, it is easy to see that φ is satisfiable in a product of two rooted Noetherian linear orders each of which contains an infinite descending chain of distinct points (such a product frame is a frame for L because $L \subseteq L_1 \times L_2$). Indeed, let $\mathfrak{F}_i = (W_i, <_i)$, $i = 1, 2$, be such frames with infinite descending chains

$$x_0 \gneqq_1 x_1 \gneqq_1 x_2 \gneqq_1 \dots \qquad \text{and} \qquad y_0 \gneqq_2 y_1 \gneqq_2 y_2 \gneqq_2 \dots$$

of points in W_1 and W_2, respectively. Define a valuation $\mathfrak{V}$ in $\mathfrak{F}_1 \times \mathfrak{F}_2$ by taking:

$$\begin{aligned}
\mathfrak{V}(h) &= \{(x, y) \mid x_0 <_1 x\} \cup \{(x, y) \mid x_{n+1} \lneqq_1 x \leq_1 x_n,\ n < \omega,\ n \text{ is even}\},\\
\mathfrak{V}(v) &= \{(x, y) \mid y_0 <_2 y\} \cup \{(x, y) \mid y_{n+1} \lneqq_2 y \leq_2 y_n,\ n < \omega,\ n \text{ is even}\},\\
\mathfrak{V}(p) &= \{(x, y) \mid x_1 \lneqq_1 x\} \cup \{(x, y) \mid x_{n+1} \lneqq_1 x,\ y \leq_2 y_n,\ n > 0\}
\end{aligned}$$

(see Fig. 1). Since $\mathfrak{F}_1$ is rooted and Noetherian, there is a $<_1$-greatest point z_1 in $\mathfrak{F}_1$ such that $z_1 <_1 x_n$ for all $n < \omega$. Similarly, there is a $<_2$-greatest point z_2 in $\mathfrak{F}_2$ such that $z_2 <_2 y_n$ for all $n < \omega$. The reader can easily check that, under the valuation $\mathfrak{V}$, we have $(z_1, z_2) \models \varphi$.

On the other hand, we will now show that φ is not satisfiable in any *finite frame* for L. To this end, suppose that

$$(\mathfrak{M}, r) \models \varphi$$

for a root r of a model $\mathfrak{M}$ based on a frame $\mathfrak{F} = (W, R_1, R_2)$ for L. Then, in view of $\mathbf{K4} \subseteq L_i$ and $[L_1, L_2] \subseteq L$, we know that

- both R_1 and R_2 are transitive,

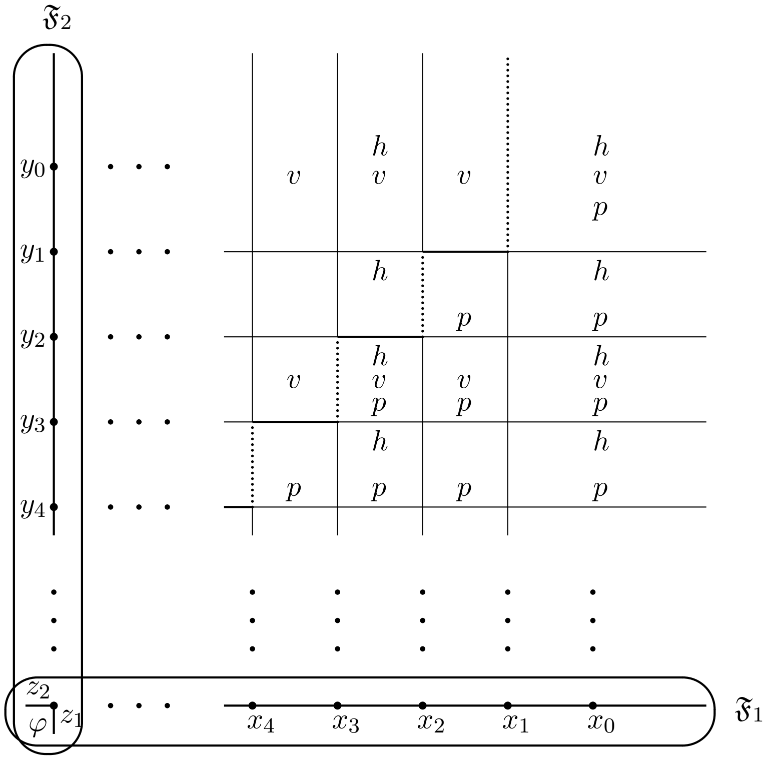

Figure 1: Satisfying φ in an infinite product frame.

- R_1 and R_2 commute, and
- R_1 and R_2 are Church–Rosser.

Define new relations $\bar{R}_i$, for $i = 1, 2$, by taking for all $x, y \in W$:

$$
\begin{aligned}
x\bar{R}_1 y \quad &\text{iff} \quad \exists z \in W \ \big[xR_1 z \text{ and } \big((\mathfrak{M}, x) \models h \iff (\mathfrak{M}, z) \models \neg h\big) \\
&\qquad\qquad \text{and (either } z = y \text{ or } zR_1 y)\big], \\
x\bar{R}_2 y \quad &\text{iff} \quad \exists z \in W \ \big[xR_2 z \text{ and } \big((\mathfrak{M}, x) \models v \iff (\mathfrak{M}, z) \models \neg v\big) \\
&\qquad\qquad \text{and (either } z = y \text{ or } zR_2 y)\big].
\end{aligned}
$$

Then, by the transitivity of R_i, we have $\bar{R}_i \subseteq R_i$ $(i = 1, 2)$. It is readily checked that, for all $x \in W$,

(𝔐, x) ⊨ ⬘ψ iff $\exists y \in W$ ($x\bar{R}_1 y$ and $(\mathfrak{M}, y) \models \psi$),

(𝔐, x) ⊨ ⬗ψ iff $\exists y \in W$ ($x\bar{R}_2 y$ and $(\mathfrak{M}, y) \models \psi$).

It is also straightforward to see that, in view of the properties of the R_i mentioned above and by (2)–(3), we have

both $\bar{R}_1$ and $\bar{R}_2$ are transitive, (tran)

$\bar{R}_1$ and $\bar{R}_2$ commute, and (com)

$\bar{R}_1$ and $\bar{R}_2$ are Church–Rosser. (chro)

We will be using the following notation. For every formula ψ, $\Diamond \in$ {⬘, ⬗} and $\Box \in$ {⬒, ◨}, let

$$\Diamond^0 \psi = \Box^0 \psi = \psi$$

and, for $n < \omega$, let

$$\Diamond^{n+1}\psi = \Diamond\Diamond^n\psi \qquad \text{and} \qquad \Box^{n+1}\psi = \Box\Box^n\psi.$$

Claim 1.1. *The following formulas are true in $\mathfrak{M}$, for all $n < \omega$:*

$$⊟⊡(\neg p \to ⬘\top), \tag{10}$$

$$⊟⊡(p \to ⬒^n ⬗^n p), \tag{11}$$

$$⊟⊡(\neg p \to ◨^n ⬘^n \neg p), \tag{12}$$

$$⊟⊡(◨\neg p \to ◨^{n+1} ⬘^n \neg p), \tag{13}$$

$$⊟⊡(⬒ p \to ⬒^{n+1} ⬗^n p). \tag{14}$$

Proof. Formula (10) is a straightforward consequence of (5), (9) and (com). We prove (11) and (12) by induction on n. The case $n = 0$ is obvious.

Suppose (11) holds for some n. Take some w with $(\mathfrak{M}, w) \models p$ and $z_1, \dots, z_n, z_{n+1}$ such that

$$w\bar{R}_1 z_1 \bar{R}_1 \dots \bar{R}_1 z_n \bar{R}_1 z_{n+1}.$$

Then $(\mathfrak{M}, z_n) \models p$ by (tran) and (8), and by IH there are $w_1, \dots, w_n$ such that

$$z_n \bar{R}_2 w_1 \bar{R}_2 \dots \bar{R}_2 w_n \qquad \text{and} \qquad (\mathfrak{M}, w_n) \models p.$$

By (chro), there are $s_1, \dots, s_n$ such that $w_i \bar{R}_1 s_i$, for $i = 1, \dots, n$, and $z_{n+1}\bar{R}_2 s_1 \bar{R}_2 \dots \bar{R}_2 s_n$. Since $w_n \bar{R}_1 s_n$, by (tran) and (8), $(\mathfrak{M}, s_n) \models p$ and there exists s_{n+1} such that

$$s_n \bar{R}_2 s_{n+1} \qquad \text{and} \qquad (\mathfrak{M}, s_{n+1}) \models p,$$

from which $(\mathfrak{M}, z_{n+1}) \models ⬗^{n+1} p$.

Now suppose that (12) holds for some n. Take some w with $(\mathfrak{M}, w) \models \neg p$ and $z_1, \dots, z_n, z_{n+1}$ such that

$$w\bar{R}_2 z_1 \bar{R}_2 \dots \bar{R}_2 z_n \bar{R}_2 z_{n+1}.$$

Then $(\mathfrak{M}, z_n) \models \neg p$ by (tran) and (9), and, by IH, there are $w_1, \dots, w_n$ such that

$$z_n \bar{R}_1 w_1 \bar{R}_1 \dots \bar{R}_1 w_n \quad \text{and} \quad (\mathfrak{M}, w_n) \models \neg p.$$

By (chro), there are $s_1, \dots s_n$ such that $w_i \bar{R}_2 s_i$, for $i = 1, \dots, n$, and $z_{n+1}\bar{R}_1 s_1 \bar{R}_1 \dots \bar{R}_1 s_n$. Since $w_n \bar{R}_2 s_n$, by (tran) and (9), $(\mathfrak{M}, s_n) \models \neg p$ and there exists s_{n+1} such that

$$s_n \bar{R}_1 s_{n+1} \qquad \text{and} \qquad (\mathfrak{M}, s_{n+1}) \models \neg p,$$

which shows that $(\mathfrak{M}, z_{n+1}) \models ⬘^{n+1} \neg p$.

Now, by $\bar{R}_2 \subseteq R_2$ and the transitivity of R_2, (12) actually implies

$$⊟⊡◨(\neg p \to ◨^n ⬘^n \neg p).$$

So (13) follows by the modal axiom K for ◨. (14) follows from (11) in a similar way. $\square$

We define inductively two infinite sequences

$$x_0, x_1, x_2, \dots \qquad \text{and} \qquad y_0, y_1, y_2, \dots$$

of points in W such that, for every $i < \omega$,

(i) $(\mathfrak{M}, x_i) \models p \wedge \boxminus \neg p$,

(ii) $(\mathfrak{M}, y_i) \models \neg p \wedge \boxminus p$,

(iii) there exists a point u_i such that $r\bar{R}_2 u_i$, $u_i \bar{R}_1 x_i$ and $u_i \bar{R}_1 y_i$, and

(iv) if $i > 0$ then there exists a point v_i such that $r\bar{R}_1 v_i$, $v_i \bar{R}_2 x_i$ and $v_i \bar{R}_2 y_{i-1}$.

(We do not claim at this point that, say, all the x_i are distinct.)

To begin with, by (2)–(4), there are u_0, x_0 such that $r\bar{R}_2 u_0 \bar{R}_1 x_0$ and

$$(\mathfrak{M}, x_0) \models \boxminus\bot \wedge \boxminus\bot. \tag{15}$$

By (5), $(\mathfrak{M}, x_0) \models p$. By (6), there is y_0 such that $u_0 \bar{R}_1 y_0$ and

$$(\mathfrak{M}, y_0) \models \neg p \wedge \boxminus p.$$

So (i)–(iii) hold for $i = 0$.

Now suppose that, for some $n < \omega$, x_i and y_i with (i)–(iv) have already been defined for all $i \leq n$. By (iii) for $i = n$ and by (com), there is v_{n+1} such that $r\bar{R}_1 v_{n+1} \bar{R}_2 y_n$. So by (7), there is x_{n+1} such that $v_{n+1} \bar{R}_2 x_{n+1}$ and

$$(\mathfrak{M}, x_{n+1}) \models p \wedge \boxminus \neg p.$$

Now again by (com), there is u_{n+1} such that $r\bar{R}_2 u_{n+1} \bar{R}_1 x_{n+1}$. So by (6), there is y_{n+1} such that $u_{n+1} \bar{R}_1 y_{n+1}$ and

$$(\mathfrak{M}, y_{n+1}) \models \neg p \wedge \boxminus p,$$

as required (see Fig. 2).

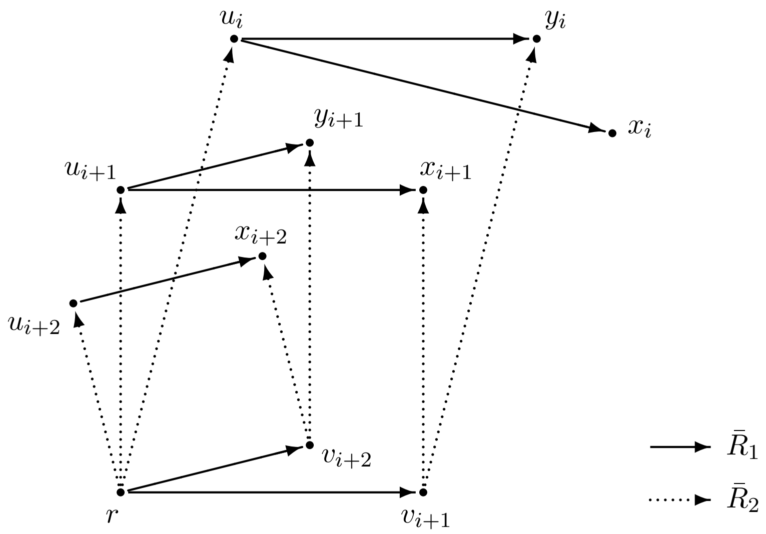

Figure 2: Generating the points x_i and y_i.

Next, we show that (i), (ii), and (10)–(14) imply the following claim:

Claim 1.2. *For all $i, n < \omega$,*

$$(\mathfrak{M}, x_i) \models ⬘^n \top \leftrightarrow ⬗^n \top, \tag{16}$$

$$(\mathfrak{M}, y_i) \models ⬘^{n+1} \top \leftrightarrow ⬗^n \top. \tag{17}$$

Proof. If $n = 0$ then (16) is obvious, and (17) follows from (ii) and (10). So we may assume that $n > 0$.

To prove (16), assume first that we have $x_i \models ⬘^n \top$. Then there is a point z such that $x_i \bar{R}_1^n z$. By (i), $x_i \models p$. So, by (11), $x_i \models ⬒^n ⬗^n p$, and therefore, $z \models ⬗^n p$. Thus we have a point u such that $z \bar{R}_2^n u$. Now, using (com), we find a point v such that $x_i \bar{R}_2^n v$ and $v \bar{R}_1^n u$, from which $x_i \models ⬗^n \top$. Conversely, suppose $x_i \models ⬗^n \top$, that is, there is a point z such that $x_i \bar{R}_2^n z$. By (i), $x_i \models ◨\neg p$, and so, by (13), $x_i \models ◨^n ⬘^{n-1} \neg p$. Therefore, $z \models ⬘^{n-1} \neg p$ and we have a point u such that $z \bar{R}_1^{n-1} u$ and $u \models \neg p$. So by (10), $u \models ⬘\top$ and we have a point v such that $u \bar{R}_1 v$, from which $z \bar{R}_1^n v$. Using (com), we find a point w such that $x_i \bar{R}_1^n w$ and $w \bar{R}_2^n v$. It follows that $x_i \models ⬘^n \top$.

To show (17), assume first that we have $y_i \models ⬘^{n+1} \top$. Then there is a point z such that $y_i \bar{R}_1^{n+1} z$. By (ii), $y_i \models ⬒p$. So, by (14), $y_i \models ⬒^{n+1} ⬗^n p$, and therefore, $z \models ⬗^n p$. Thus we have a point u such that $z \bar{R}_2^n u$. Now, using (com), we find a point v such that $y_i \bar{R}_2^n v$ and $v \bar{R}_1^{n+1} u$, from which $y_i \models ⬗^n \top$. Conversely, suppose $y_i \models ⬗^n \top$, that is, there is a point z such that $y_i \bar{R}_2^n z$. By (ii), $y_i \models \neg p$ and, by (12), $y_i \models ◨^n ⬘^n \neg p$. Therefore, $z \models ⬘^n \neg p$ and we have a point u such that $z \bar{R}_1^n u$ and $u \models \neg p$. So by (10), $u \models ⬘\top$ and we have a point v such that $u \bar{R}_1 v$, from which $z \bar{R}_1^{n+1} v$. Using (com), we find a point w such that $y_i \bar{R}_1^{n+1} w$ and $w \bar{R}_2^n v$. It follows that $y_i \models ⬘^{n+1} \top$. □

Finally, the following claim shows that the x_n are all distinct, and so the frame $\mathfrak{F}$ must be infinite:

Claim 1.3. *For every $n < \omega$,*

$$(\mathfrak{M}, x_n) \models ⬘^n \top \wedge ⬒^{n+1} \bot.$$

Proof. We proceed by induction on n. For $n = 0$ the claim holds by the definition of x_0 (see (15)).

Now suppose that the claim holds for some $n < \omega$. Then,

$$\begin{array}{rll}
(\mathfrak{M}, x_n) & \models ⬘^n \top & \text{(by IH)} \\
(\mathfrak{M}, x_n) & \models ⬗^n \top & \text{(by (16))} \\
(\mathfrak{M}, y_n) & \models ⬗^n \top & \text{(by (iii), (com) and (chro))} \\
(\mathfrak{M}, y_n) & \models ⬘^{n+1} \top & \text{(by (17))} \\
(\mathfrak{M}, x_{n+1}) & \models ⬘^{n+1} \top & \text{(by (iv), (com) and (chro)).}
\end{array}$$

On the other hand, we also have

$$\begin{array}{rll}
(\mathfrak{M}, x_n) & \models ⬒^{n+1} \top & \text{(by IH)} \\
(\mathfrak{M}, x_n) & \models ◨^{n+1} \bot & \text{(by (16))} \\
(\mathfrak{M}, y_n) & \models ◨^{n+1} \bot & \text{(by (iii), (com) and (chro))} \\
(\mathfrak{M}, y_n) & \models ⬒^{n+2} \bot & \text{(by (17))} \\
(\mathfrak{M}, x_{n+1}) & \models ⬒^{n+2} \bot & \text{(by (iv), (com) and (chro)).}
\end{array}$$

as required. $\square$

This completes the proof of Theorem 1. $\square$

It is worth noting that the proof above does not go through for 'products with *expanding domains*' where only one, say, the left commutativity principle holds. 'Expanding products' with **S4** are closely related to intuitionistic modal logics, e.g., to the transitive analogue of the Fischer Servi logic [6, 7] the decidability of which has remained open for a decade. They are also very close to dynamic topological logics interpreted in topological spaces with continuous functions; see, e.g., [1, 10]. On the one hand, it is not hard to see that 'expanding product logics' can always be reduced to product logics; see [11]. Thus, in principle 'expanding products' can be computationally simpler than the standard ones. However, no example is known so far where the product of two logics is undecidable, while their 'expanding product' is decidable.

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References

[1] S. Artemov, J. Davoren, and A. Nerode. Modal logics and topological semantics for hybrid systems. Technical Report MSI 97-05, Cornell University, 1997.

[2] F. Baader and H.-J. Ohlbach. A multi-dimensional terminological knowledge representation language. *Journal of Applied Non-Classical Logics*, 5:153–197, 1995.

[3] A. Chagrov and M. Zakharyaschev. *Modal Logic*, volume 35 of *Oxford Logic Guides*. Clarendon Press, Oxford, 1997.

[4] J. Davoren and R. Goré. Bimodal logics for reasoning about continuous dynamics. In F. Wolter, H. Wansing, M. de Rijke, and M. Zakharyaschev, editors, *Advances in Modal Logic, Volume 3*, pages 91–112. World Scientific, 2002.

[5] R. Fagin, J. Halpern, Y. Moses, and M. Vardi. *Reasoning about Knowledge*. MIT Press, 1995.

[6] G. Fischer Servi. Semantics for a class of intuitionistic modal calculi. In M. L. Dalla Chiara, editor, *Italian Studies in the Philosophy of Science*, pages 59–72. Reidel, Dordrecht, 1980.

[7] G. Fischer Servi. Axiomatizations for some intuitionistic modal logics. *Rendiconti di Matematica di Torino*, 42:179–194, 1984.

[8] D. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-Dimensional Modal Logics: Theory and Applications*, volume 148 of *Studies in Logic*. Elsevier, 2003.

[9] D. Gabbay and V. Shehtman. Products of modal logics. Part I. *Logic Journal of the IGPL*, 6:73–146, 1998.

[10] P. Kremer and G. Mints. Dynamic topological logic. Manuscript, 2004.

[11] A. Kurucz and M. Zakharyaschev. A note on relativised products of modal logics. In P. Balbiani, N.-Y. Suzuki, F. Wolter, and M. Zakharyaschev, editors, *Advances in Modal Logic, Volume 4*, pages 221–242. King's College Publications, 2003.

[12] M. Marx. Complexity of products of modal logics. *Journal of Logic and Computation*, 9:197–214, 1999.

[13] M. Marx and M. Reynolds. Undecidability of compass logic. *Journal of Logic and Computation*, 9:897–914, 1999.

[14] J. Reif and A. Sistla. A multiprocess network logic with temporal and spatial modalities. *Journal of Computer and System Sciences*, 30:41–53, 1985.

[15] M. Reynolds. A decidable temporal logic of parallelism. *Notre Dame Journal of Formal Logic*, 38:419–436, 1997.

[16] M. Reynolds and M. Zakharyaschev. On the products of linear modal logics. *Journal of Logic and Computation*, 11:909–931, 2001.

[17] K. Segerberg. Two-dimensional modal logic. *Journal of Philosophical Logic*, 2:77–96, 1973.

[18] V. Shehtman. Two-dimensional modal logics. *Mathematical Notices of the USSR Academy of Sciences*, 23:417–424, 1978. (Translated from Russian).

[19] F. Wolter and M. Zakharyaschev. Qualitative spatio-temporal representation and reasoning: a computational perspective. In G. Lakemeyer and B. Nebel, editors, *Exploring Artifitial Intelligence in the New Millenium*, pages 175–216. Morgan Kaufmann, 2002.