Well-quasi-ordering does not imply bounded clique-width

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Abstract. We present a hereditary class of graphs of unbounded clique-width which is well-quasi-ordered by the induced subgraph relation. This result provides the negative answer to a question asked by Daligault, Rao and Thomassé in [3].

1 Introduction

Well-quasi-ordering (WQO) is a highly desirable property and frequently discovered concept in mathematics and theoretical computer science [6, 11]. One of the most remarkable recent results in this area is the proof of Wagner's conjecture stating that the set of all finite graphs is well-quasi-ordered by the minor relation [14]. This is, however, not the case for the induced subgraph relation, since the set of cycles $\{C_n | n \geq 3\}$ forms an infinite antichain with respect to this relation. On the other hand, the induced subgraph relation may become a well-quasi-order when restricted to graphs in particular classes, such as cographs [4] or k-letter graphs [13]. It is interesting to observe that in both examples we deal with graphs of bounded clique-width, which is another property of great importance in mathematics and computer science. Moreover, the same is true for all available examples of hereditary graph classes which are well-quasi-ordered by the induced subgraph relation (see e.g. [9]). This raises an interesting question whether the clique-width is always bounded for graphs in well-quasi-ordered hereditary classes. This question was formally stated as an open problem (Problem 6) by Daligault, Rao and Thomassé in [3]. In the present paper, we answer former question negatively by exhibiting a hereditary class of graphs of unbounded clique-width which is well-quasi-ordered by the induced subgraph relation.

Remark. The proposed result *does not* resolve Conjecture 5 of [3] asserting that 2-well-quasi-orderability of a hereditary class implies its bounded cliquewidth.

Indeed, being 2-well-quasi-ordered is a stronger (more constrained) property than just being well-quasi-ordered. Therefore, the existence of a well-quasi-ordered class of unbounded cliquewidth says nothing about existence of such a 2-well-quasi-ordered class. In fact, we believe that the class considered in this paper is not 2-well-quasi-ordered.

Our result shows that it is generally non-trivial to determine whether a given problem definable in Monadic Second Order (MSO) logic is polynomially solvable on a WQO class, since unboundedness of clique-width does not allow a straightforward application of Courcelle et al.'s theorem [2]. This makes the WQO classes an interesting object to study from the algorithmic perspective.

Graphs in the class introduced in this paper are *dense* (in particular, they are P_7-free). The density is a necessary condition, because an earlier result [1] shows that for sparse graph classes (those where a large biclique is forbidden as a subgraph) well quasi-orderability by induced subgraphs imply bounded treewidth (and hence bounded clique-width). We believe that the result of [1] can be strengthened by showing that well quasi-orderability by induced subgraphs in sparse classes implies bounded pathwidth (and hence *linear* clique-width [7]). Our result proved in the present paper shows a stark contrast between dense and sparse graphs in this context.

The rest of the paper is structured as follows. In Section 2 we define the class of graphs studied in this paper and state the main result. The unboundedness of clique-width and well-quasi-orderability by induced subgraphs is proved in Sections 3 and 4, respectively. We use standard graph-theoretic notation as e.g. in [5]. The notions of clique-width and well-quasi-ordering are introduced in respective sections where they are actually used.

2   The main result

In this section, we define the class $\mathcal{D}$, which is the main object of the paper, and state the main result.

Let P be a path with vertex set $\{1, \ldots, n\}$ with two vertices i and j being adjacent if and only if $|i - j| = 1$. For vertex i, the largest 2^k that divides i is called the *power* of i and is denoted by $q(i)$. For example, $q(5) = 1, q(6) = 2, q(8) = 8, q(12) = 4$. Add edges to P that connect i and j whenever $q(i) = q(j)$. We denote the graph obtained in this way by D_n. Figure 1 illustrates graph D_{16}.

Clearly, the edges $E(D_n) \setminus E(P)$ form a set of disjoint cliques and we call them *power cliques*. If a power clique Q contains a vertex i with $q(i) = 2^k$ we say that Q *corresponds to* 2^k. We call P the *body* of D_n, the edges of $E(P)$ the *path edges*, and the edges of $E(D_n) \setminus E(P)$ the *clique edges*. The class $\mathcal{D}$ is the
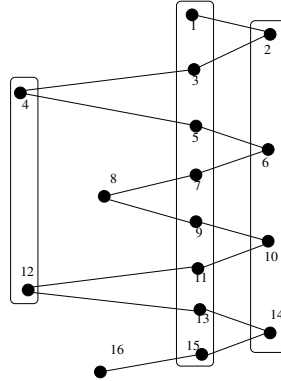
Fig. 1. Graph D_{16}. To avoid shading the picture with many edges, cliques are represented as rectangular boxes.

set of all graphs D_n and all their induced subgraphs. In what follows we prove that

- clique-width of graphs in $\mathcal{D}$ is unbounded (Section 3),
- graphs in $\mathcal{D}$ are well-quasi-ordered by the induced subgraph relation (Section 4).

These two facts imply the following conclusion, which is the main result of the paper.

Theorem 1. *Within the family of hereditary graph classes, there exist classes of unbounded clique-width which are well-quasi-ordered by the induced subgraph relation.*

3 Clique-width is unbounded in $\mathcal{D}$

The clique-width of a graph G, denoted $cwd(G)$, is the minimum number of labels needed to construct the graph by means of the four graph operations: creation of a new vertex, disjoint union of two labeled graphs, connecting vertices with specified labels i and j, and renaming label i to label j. Every graph G can be constructed by means of these four operations, and the process of the construction can be described either by an algebraic expression or by a rooted binary tree, whose leaves correspond to the vertices of G, the root corresponds to G and the internal nodes correspond to the union operations.

Given a graph G and a subset $U \subset V(G)$, we denote by $\overline{U}$ the set $V(G) - U$. We say that two vertices $x, y \in U$ are *U-similar* if $N(x) \cap \overline{U} = N(y) \cap \overline{U}$, i.e. if x and y have the same neighbourhood outside of U. Clearly, the U-similarity

is an equivalence relation and we denote the number of similarity classes of U by $\mu_G(U)$. Also, we denote

$$\mu(G) = \min_{\frac{1}{3}n \leq |U| \leq \frac{2}{3}n} \mu_G(U),$$

where $n = |V(G)|$. Our proof of the main result of this section is based on the following lemma.

Lemma 1. *For any graph G, $\mu(G) \leq cwd(G)$.*

Proof. Let T be an *optimal* decomposition tree, t a node of T and U_t the set of vertices of G that are leaves of the subtree of T rooted at t. It is known (see e.g. [12]) that $cwd(G) \geq \mu_G(U_t)$ for any node t of T. According to a well known folklore result, the binary tree T has a node t such that $\frac{1}{3}|V(G)| \leq |U_t| \leq \frac{2}{3}|V(G)|$, in which case $\mu_G(U_t) \geq \mu(G)$. Hence the lemma. $\square$

Let $U \subseteq V(D_n)$, and let P be the body of D_n. We denote by P^U the subgraph of P induced by U. In other words, P^U is obtained from $D_n[U]$ by removing the clique edges. Since P is a path, P^U is a graph every connected component of which is a path.

Lemma 2. *If P^U has $c + 1$ connected components, then $\mu_{D_n}(U) \geq c/2$.*

Proof. In the i-th connected component of P^U, $i \leq c$, we choose the last vertex (listed along the path P) and denote it by u_i. The next vertex of P, denoted $\overline{u}_i$, belongs to $\overline{U}$. This creates a matching of size c with edges $(u_i, \overline{u}_i)$. Note that none of $(u_i, \overline{u}_j)$ is a path edge for $i < j$. Among the chosen vertices of U at least half have the same parity. Their respective matched vertices of $\overline{U}$ have the opposite parity. Since the clique edges connect only the vertices of the *same* parity, we conclude that at least $c/2$ vertices of U have pairwise different neighbourhoods in $\overline{U}$, i.e. $\mu_{D_n}(U) \geq c/2$. $\square$

Note that if P^U has c connected components, then $P^{\overline{U}}$ has at least $c - 1$ connected components. Therefore, in light of Lemma 2, it remains to consider the case where both P^U and $P^{\overline{U}}$ have a limited number of connected components. Taking into account the definition of $\mu(G)$ and Lemma 1 we can assume that both U and $\overline{U}$ are 'large' (between one third and two thrid of $|V(G)|$), and hence each of P^U and $P^{\overline{U}}$ has a 'large' connected component. In order to address this case we use the following lemma which states that a large number of power cliques intersecting both U and $\overline{U}$ implies a large value of $\mu_{D_n}(U)$.

Lemma 3. *If there exist c different power cliques $Q_1, \ldots, Q_c$ each of which*

(1) corresponds to a power of 2 greater than 1 and
(2) intersects both U and $\overline{U}$

then $\mu_{D_n}(U) \geq c$.

Proof. Let u_i and $\overline{u}_i$ be some vertices in Q_i, which belong to U and $\overline{U}$, respectively. Since all the vertices in $M = \{u_1, \overline{u}_1, \ldots, u_c, \overline{u}_c\}$ are even and two even vertices are adjacent in D_n if and only if they belong to the same power clique, M induces a matching in D_n with edges $(u_i, \overline{u}_i)$, $i = 1, \ldots, c$. This implies that $u_1, \ldots, u_c$ have pairwise different neighbourhoods in $\overline{U}$, that is $\mu_{D_n}(U) \geq c$. $\square$

The only remaining ingredient to prove the main result of this section is the following lemma.

Lemma 4. *Let c be a constant and P' a subpath of P of length at least 2^{c+1}. Then P' intersects each of the power cliques corresponding to $2^1, \ldots, 2^c$.*

Proof. The statement easily follows from the fact that for a fixed k vertices v with $q(v) = 2^k$ are of the form $v = 2^k(2p + 1)$. That is, they occur in P with period 2^{k+1}. $\square$

Now we are ready to prove the main result of this section.

Theorem 2. *Let n and c be natural numbers such that $n \geq 3((2c + 1)(2^{c+1} - 1) + 1)$. Then $cwd(D_n) \geq c$ and hence the clique-width of graphs in $\mathcal{D}$ is unbounded.*

Proof. Let U be an arbitrary subset of vertices of D_n, such that $\frac{n}{3} \leq |U| \leq \frac{2n}{3}$. Note that the choice of U implies that the cardinalities of both U and $\overline{U}$ are at least $\frac{n}{3} \geq (2c + 1)(2^{c+1} - 1) + 1$.

If P^U has at least $2c+1$ connected components, then by Lemma 2 $\mu_{D_n}(U) \geq c$. Otherwise P^U has less than $2c + 1$ connected components and $P^{\overline{U}}$ has less than $2c + 2$ connected components. By the pigeonhole principle, both graphs have connected components of size at least 2^{c+1}. Clearly, these connected components are disjoint subpaths of P. By Lemma 4, the power cliques corresponding to $2^1, \ldots, 2^c$ intersect both U and $\overline{U}$, and hence, by Lemma 3, $\mu_{D_n}(U) \geq c$.

Since U has been chosen arbitrarily, we conclude that $\mu(D_n) \geq c$, and therefore, by Lemma 1, $cwd(D_n) \geq c$, as required. $\square$

4 $\mathcal{D}$ is WQO by induced subgraphs

A binary relation $\leq$ on a set W is a *quasi-order* (also known as *preorder*) if it is reflexive and transitive. Two elements $x, y \in W$ are said to be comparable with respect to $\leq$ if either $x \leq y$ or $y \leq x$. Otherwise, x and y are incomparable. A set of pairwise comparable elements is called a *chain* and a set of pairwise incomparable elements an *antichain*. A quasi-order $(W, \leq)$ is a *well-quasi-order* (WQO) if it contains neither infinite strictly decreasing chains nor infinite antichains.

In this section, we show that graphs in $\mathcal{D}$ are well-quasi-ordered by the induced subgraph relation. In the proof we apply the celebrated Higman's lemma [8] which can be stated as follows.

For an arbitrary set M, let M^* be the set of all finite sequences of elements of M. Any quasi-order $\leq$ on M defines a quasi-order $\preceq$ on M^* as follows: $(a_1, \ldots, a_m) \preceq (b_1, \ldots, b_n)$ if and only if there is an order-preserving injection $f : \{a_1, \ldots, a_m\} \to \{b_1, \ldots, b_n\}$ with $a_i \leq f(a_i)$ for each $i = 1, \ldots, m$.

Lemma 5. *[8] If $(M, \leq)$ is a* WQO, *then $(M^*, \preceq)$ is a* WQO.

Obviously, the induced subgraph relation contains no infinite strictly decreasing chains. Therefore, to prove that this relation is a WQO on $\mathcal{D}$ we need to show that for each infinite sequence $\mathcal{G} = G_1, G_2 \ldots$ of graphs in $\mathcal{D}$ there are i, j such that G_i is an induced subgraph of G_j.

We recall that $V(D_n)$ is the set of integers $1, 2, \ldots, n$ listed along the body of D_n and any graph in $\mathcal{D}$ is an induced subgraph of D_n with some n. Among all possible sets of integers inducing a graph (isomorphic to) $G \in \mathcal{D}$ we pick *one* (arbitrarily) and identify $V(G)$ with this set.

Any set of consecutive integers will be called an *interval* and any subgraph of D_n induced by an interval will be called a *factor*. The number of elements in an interval inducing a factor is called the *length* of the factor. If a graph $G \in \mathcal{D}$ is not a factor, its vertex set can be split into maximal intervals and we call the subgraphs of G induced by these intervals *factor-components* of G. The set of all factor-components of G will be denoted $\mathcal{F}(G)$.

Lemma 6. *If $\mathcal{G}$ contains graphs with arbitrarily long factor-components, then $\mathcal{G}$ is not an antichain.*

Proof. Pick an arbitrary G_i and let n be the smallest number such that G_i is an induced subgraph of D_n. By our assumption, there is G_j with factor-component F of length at least $5n$. Let us show that D_n is an induced subgraph of G_j. By the transitivity of the induced subgraph relation, this will imply that G_i is an induced subgraph of G_j.

Let 2^k be the smallest power of 2 larger than n. Clearly, $2^{k+1} \leq 4n$. Hence, by Lemma 4, there is a vertex y among the first $4n$ vertices of F with $q(y) = 2^k$. Let F' be the factor induced by the vertices of F starting at $y + 1$. Since F is of length at least $5n$ and y is among the first $4n$ vertices of F, the length of F' is at least n. Thus we can define an injective function $f : V(D_n) \to V(F')$ as follows: $f(z) = y + z$ for $1 \leq z \leq n$. We claim that f is an induced subgraph isomorphism from D_n to a subgraph of G_j. Clearly, $f(z + 1) = f(z) + 1$ for $1 \leq z < n$, hence it remains to verify that adjacencies and non-adjacencies are preserved for vertices z_1, z_2 of D_n such that $z_2 > z_1 + 1$. Clearly, in this case z_1 and z_2 are adjacent if and only if $q(z_1) = q(z_2)$. Moreover, since $f(z_2) > f(z_1) + 1$, $f(z_2)$ and $f(z_1)$ are adjacent if and only if $q(f(z_1)) = q(f(z_2))$. Below we prove that $q(f(z)) = q(z)$ for $1 \leq z \leq n$ and hence $q(z_1) = q(z_2)$ if and only if $q(f(z_1)) = q(f(z_2))$, implying the lemma.

Indeed, $f(z) = y + z = 2^k p + 2^{k_1} p_1$, where $2^{k_1} = q(z)$ and p, p_1 are odd numbers. Since $2^{k_1} \leq n < 2^k$, $k_1 < k$ and hence $y + z$ can be written as $2^{k_1}(2^{k-k_1}p + p_1)$. Since $k > k_1$, 2^{k-k_1} is even and hence $2^{k-k_1}p + p_1$ is odd. Consequently, $q(y + z) = 2^{k_1}$, as required. $\qquad\square$

From now on, we assume the length of factor-components of graphs in $\mathcal{G}$ is bounded by some constant $c = c(\mathcal{G})$. In what follows we prove that in this case $\mathcal{G}$ is not an antichain as well.

Let F be a factor. We say that a vertex u of F is *maximal* if $q(u) \geq q(v)$ for each vertex v of F different from u.

Lemma 7. *Every factor F of D_n contains precisely one maximal vertex.*

Proof. Suppose that F contains two maximal vertices $2^k p$ and $2^k(p + r)$ for some odd number p and even number $r \geq 2$. Then F also contains the vertex $2^k(p + 1)$. Clearly $p + 1$ is an even number and hence $q(2^k(p + 1)) \geq 2^{k+1}$, which contradicts the maximality of 2^k. $\qquad\square$

In light of Lemma 7, we denote the unique maximal vertex of F by $m(F)$. Also, let $s(F)$ be the smallest vertex of F.

Now we define two equivalence relations on the set of factor graphs as follows. We say that two factors F_1 and F_2 are

- t-equivalent if they are of the same length and $m(F_1) - s(F_1) = m(F_2) - s(F_2)$,
- ℓ-equivalent if $q(m(F_1)) = q(m(F_2))$.

We denote by L_i the ℓ-equivalence class such that $q(m(F)) = 2^i$ for every factor F in this class. We also order the t-equivalence classes (arbitrarily) and denote by T_j the j-th class in this order.

Lemma 8. *Let F be a factor of length at most c. Let v be a vertex of F different from its maximal vertex $m = m(F)$. Then $q(v) = q(|m - v|)$ and, in particular, $q(v) < c$.*

Proof. We can assume without loss of generality that $v > m$. Let k_1, p_1, k_2, p_2 be such that $m = 2^{k_1}p_1$ and $v - m = 2^{k_2}p_2$, with p_1, p_2 being odd numbers. Observe that $k_2 < k_1$. Indeed, otherwise $v = 2^{k_1}p_1 + 2^{k_2}p_2 = 2^{k_1}(p_1 + 2^{k_2-k_1}p_2)$, where $p_1 + 2^{k_2-k_1}p_2$ is a natural number. Therefore, $q(v) \geq 2^{k_1} = q(m)$ in contradiction either to the maximality of m or to Lemma 7.

Consequently, $v = 2^{k_1}p_1 + 2^{k_2}p_2 = 2^{k_2}(2^{k_1-k_2}p_1 + p_2)$, where $2^{k_1-k_2}p_1 + p_2$ is an odd number because of $2^{k_1-k_2}p_1$ being even. Hence, $q(v) = 2^{k_2} = q(v - m)$.

Finally, since the length of F is at most c, we conclude that $v - m < c$, and therefore $q(v) = q(v - m) < c$. $\square$

Corollary 1 *Let F be a factor of length at most c. Let m be a vertex of F with $q(m) \geq c$. Then m is the maximal vertex of F.*

Corollary 2 *Let F_1, F_2 be two t-equivalent factors. Then there exists an isomorphism f from F_1 to F_2 such that:*

(a) $f(m(F_1)) = m(F_2)$;
(b) $q(f(v)) = q(v)$ for all $v \in V(F_1)$ except possibly for $m(F_1)$.

Proof. We claim that the function f that maps the i-th vertex of factor F_1 (starting from the smallest) to the i-th vertex of factor F_2 is the desired isomorphism. Indeed, property (a) follows from the condition that the factors are t-equivalent. Now property (a) together with Lemma 8 implies property (b). Finally, since adjacency between vertices in a factor is completely determined by their adjacency in the body and by their powers, we conclude that f is, in fact, isomorphism. $\square$

For a graph $G \in \mathcal{D}$, we denote by $G_{i,j}$ the set of factor-components of G in $L_i \cap T_j$, and define a binary relation $\leq$ on graphs of $\mathcal{D}$ as follows: $G \leq H$ if and only if $|G_{i,j}| \leq |H_{i,j}|$ for all i and j (clearly in this definition one can be restricted to non-empty sets $G_{i,j}$).

Finally, for a constant $c = c(\mathcal{G})$ we slightly modify the definition of $\leq$ to $\leq_c$ as follows. We say that a mapping $h : \mathbb{N} \to \mathbb{N}$ is c-preserving if it is injective and

$h(i) = i$ for all $i \leq \lfloor \log c \rfloor$. Then $G \leq_c H$ if and only if there is a c-preserving mapping h such that $|G_{i,j}| \leq |H_{h(i),j}|$ for all i and j.

The importance of the binary relation $\leq_c$ is due to the following lemma.

Lemma 9. *Suppose the length of factor-components of G and H is bounded by c and $G \leq_c H$, then G is an induced subgraph of H.*

Proof. We say that a factor F is *low-powered* if $F \in L_i$, for some $i \leq \lfloor \log c \rfloor$, i.e. $q(m(F)) \leq c$.

It can be easily checked that the definition of $\leq_c$ implies the existence of an injective function $\phi : \mathcal{F}(G) \to \mathcal{F}(H)$ that possesses the following properties:

(1) ϕ maps each of the factors in $\mathcal{F}(G)$ to a t-equivalent factor in $\mathcal{F}(H)$;
(2) $F \in \mathcal{F}(G)$ is a low-powered factor if and only if $\phi(F)$ is;
(3) ϕ preserves power of the maximal vertex for each of the low-powered factors, i.e. $q(m(F)) = q(m(\phi(F)))$ for every low-powered factor $F \in \mathcal{F}(G)$;
(4) for any two factors $F_1, F_2 \in \mathcal{F}(G)$, $q(m(F_1)) = q(m(F_2))$ if and only if $q(m(\phi(F_1))) = q(m(\phi(F_2)))$.

To show that G is an induced subgraph of H we define a witnessing function that maps vertices of a factor $F \in \mathcal{F}(G)$ to vertices of $\phi(F) \in \mathcal{F}(H)$ according to an isomorphism described in Corollary 2. This mapping guarantees that a factor F of G is isomorphic to the factor $\phi(F)$ of H. Therefore it remains to check that adjacency relation between vertices in different factors is preserved under the defined mapping.

Note that adjacency between two vertices in different factors is determined entirely by powers of these vertices. Moreover, Corollary 2 and property (3) of ϕ imply that our mapping preserves powers of all vertices except possibly maximal vertices of power more that c. Therefore in order to complete the proof we need only to make sure that in graph G a maximal vertex m of a factor F with $q(m) > c$ is adjacent to a vertex v in a factor different from F if and only if the corresponding images of m and v are adjacent in H.

Taking into account Corollary 1 we derive that a maximal vertex with $q(m) > c$ is adjacent to a vertex v in a different factor if and only if v is maximal and $q(m) = q(v)$. Now the desired conclusion follows from Corollary 2 and properties (2) and (4) of function ϕ. $\qquad\square$

Lemma 10. *The set of graphs in $\mathcal{D}$ in which factor-components have size at most c is well-quasi-ordered by the $\leq_c$ relation.*

Proof. We associate with each graph $G \in \mathcal{D}$ containing no factor-component of size larger than c a matrix $M_G = m(i, j)$ with $m(i, j) = |G_{i,j}|$.

Each row of this matrix corresponds to an ℓ-equivalence class and we delete any row corresponding to L_i with $i > \lfloor \log c \rfloor$ which is empty (contains only 0s). This leaves a finite amount of rows (since G is finite).

Each column of M_G corresponds to a t-equivalence class and we delete all columns corresponding to t-equivalence classes containing factors of size larger than c (none of these classes has a factor-component of G). This leaves precisely $\binom{c+1}{2}$ columns in M_G.

We define the relation $\preceq_c$ on the set $\mathcal{M}$ of matrices constructed in this way as follows. For $M_1, M_2 \in \mathcal{M}$ we say that $M_1 \preceq_c M_2$ if and only if there is a c-preserving mapping β such that $m_1(i,j) \leq m_2(\beta(i),j)$ for all i and j.

It is not difficult to see that if $M_{G_1} \preceq_c M_{G_2}$, then $G_1 \leq_c G_2$. Therefore, if $\preceq_c$ is a well-quasi-order, then $\leq_c$ is a well-quasi-order too. The well-quasi-orderability of matrices follows by repeated applications of Higman's lemma. First, we split each matrix $M \in \mathcal{M}$ into two sub-matrices M' and M'' so that M' contains the first $\lfloor \log c \rfloor$ rows and M'' contains the remaining rows. Let $\mathcal{M}' = \{M' | M \in \mathcal{M}\}$ and $\mathcal{M}'' = \{M'' | M \in \mathcal{M}\}$

To see that the set of matrices $\mathcal{M}'$ is WQO we apply Higman's lemma twice. First, the set of rows is WQO since each of them is a finite word over the alphabet of non-negative integers (which is WQO by the ordinary arithmetic $\leq$ relation). Second, the set of matrices is WQO since each of them is a finite word over the alphabet of rows. Similarly, the set of matrices $\mathcal{M}''$ is WQO.

Note that in both applications of Lemma 5 to $\mathcal{M}'$ and in the first application to $\mathcal{M}''$, we considered sets of sequences of the *same length*. Hence, in this, case, Higman's lemma in fact implies the existence of two sequences one of them is *coordinate-wise* smaller than the other, exactly what we need in these cases.

Finally, the set of matrices $\mathcal{M}$ is WQO since each of them is a word of two letters over the alphabet $\mathcal{M}' \cup \mathcal{M}''$ which is WQO. $\qquad\square$

Combining Lemmas 6 and 10, we obtain the main result of this section.

Theorem 3. $\mathcal{D}$ is WQO *by the induced subgraph relation.*

References

1. Aistis Atminas, Vadim V. Lozin, and Igor Razgon. Well-quasi-ordering, tree-width and sub-quadratic properties of graphs. *CoRR*, abs/1410.3260, 2014.
2. Bruno Courcelle, Johann A. Makowsky, and Udi Rotics. Linear time solvable optimization problems on graphs of bounded clique-width. *Theory Comput. Syst.*, 33(2):125–150, 2000.
3. Jean Daligault, Michaël Rao, and Stéphan Thomassé. Well-quasi-order of relabel functions. *Order*, 27(3):301–315, 2010.
4. Peter Damaschke. Induced subgraphs and well-quasi-ordering. *Journal of Graph Theory*, 14(4):427–435, 1990.
5. Reinhard Diestel. *Graph Theory*. Springer-Verlag, third edition, 2005.

6. Alain Finkel and Ph. Schnoebelen. Well-structured transition systems everywhere! *Theor. Comput. Sci.*, 256(1-2):63–92, 2001.

7. Frank Gurski and Egon Wanke. On the relationship between nlc-width and linear nlc-width. *Theoretical Computer Science*, 347(1-2):76–89, 2005.

8. Graham Higman. Ordering by divisibility in abstract algebras. *Proceedings of the London Mathematical Society*, 2:326–336, 1952.

9. Nicholas Korpelainen and Vadim V. Lozin. Two forbidden induced subgraphs and well-quasi-ordering. *Discrete Mathematics*, 311(16):1813–1822, 2011.

10. Stephan Kreutzer and Siamak Tazari. Lower bounds for the complexity of monadic second-order logic. In *LICS*, pages 189–198, 2010.

11. Joseph B. Kruskal. The theory of well-quasi-ordering: A frequently discovered concept. *J. Comb. Theory, Ser. A*, 13(3):297–305, 1972.

12. Vadim V. Lozin and Dieter Rautenbach. The relative clique-width of a graph. *J. Comb. Theory, Ser. B*, 97(5):846–858, 2007.

13. Marko Petkovsek. Letter graphs and well-quasi-order by induced subgraphs. *Discrete Mathematics*, 244(1-3):375–388, 2002.

14. Neil Robertson and P. D. Seymour. Graph minors XX. Wagner's conjecture. *J. Comb. Theory Ser. B*, 92(2):325–357, 2004.