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Boundary Properties of Well-Quasi-Ordered Sets of Graphs

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Abstract Let $\mathcal{Y}_k$ be the family of hereditary classes of graphs defined by k forbidden induced subgraphs. In Korpelainen and Lozin (Discrete Math 311:1813–1822, 2011), it was shown that $\mathcal{Y}_2$ contains only finitely many minimal classes that are not well-quasi-ordered (wqo) by the induced subgraph relation. This implies, in particular, that the problem of deciding whether a class from $\mathcal{Y}_2$ is wqo or not admits an efficient solution. Unfortunately, this idea does not work for $k \geq 3$, as we show in the present paper. To overcome this difficulty, we introduce the notion of boundary properties of well-quasi-ordered sets of graphs. The importance of this notion is due to the fact that for each k, a class from $\mathcal{Y}_k$ is wqo if and only if it contains none of the boundary properties. We show that the number of boundary properties is generally infinite. On the other hand, we prove that for each fixed k, there is a finite collection of boundary properties that allow to determine whether a class from $\mathcal{Y}_k$ is wqo or not.

Keywords Well-quasi-order · Induced subgraphs

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1 Introduction

A *graph property*, also known as a *class of graphs*, is any set of graphs closed under isomorphism. A property is *hereditary* if it is closed under the induced subgraph relation. Two prominent families of hereditary properties are monotone properties (closed under the subgraph relation) and minor-closed properties (closed under the minor relation).

According to the celebrated result of Robertson and Seymour [15], the minor relation is a *well-quasi-order* (wqo) on the set of all simple finite graphs. This means that there are no infinite antichains with respect to the minor relation. In other words, in any infinite set of graphs, there is a pair of graphs one of which is a minor of the other. This implies, in particular, that any minor-closed graph property can be characterized by finitely many "forbidden" minors and, as a result, can be recognized in polynomial time.

When we switch to the subgraph or induced subgraph relation, the situation becomes more complicated. The set of all chordless cycles gives an example of an infinite antichain with respect to both relations, i.e. neither of these relations is a well-quasi-order. On the other hand, each of them may become a well-quasi-order when restricted to graphs in special classes. The question of well-quasi-orderability of monotone classes with respect to each relation (subgraphs and induced subgraphs) was completely solved by Ding in [9] as follows: a monotone property is well-quasi-ordered by the induced subgraph relation (and thus also by the subgraph relation) if and only if it contains only finitely many cycles and finitely many graphs of the form H_i represented in Fig. 1.

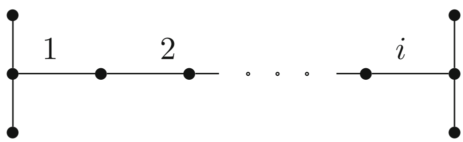
In the present paper, we study well-quasi-orderability of hereditary classes with respect to the induced subgraph relation. It is well known that a class is hereditary if and only if it can be characterized in terms of forbidden induced subgraphs. More precisely, for a set M of graphs, let us denote by $Free(M)$ the class of graphs containing no induced subgraphs isomorphic to graphs in M. Then a class X of graphs is hereditary if and only if $X = Free(M)$ for some set M, in which case M is called a set of forbidden induced subgraphs for X.

Following [7], we restrict the question of well-quasi-orderability to finitely defined hereditary properties, i.e. those defined by finitely many forbidden induced subgraphs. Let us denote by $\mathcal{Y}_k$ the family of hereditary classes of graphs defined by k forbidden induced subgraphs. We call classes in $\mathcal{Y}_1$ *monogenic* and classes in $\mathcal{Y}_2$ *bigenic*.

For monogenic classes, the question of well-quasi-orderability is simple: there is only one maximal class in this family which is well-quasi-ordered by the induced subgraph relation, the class of P_4-free graphs [8].

For bigenic classes, the situation is more complicated, but still manageable. It was shown in [11] that in this family there are finitely many minimal classes that are not

Fig. 1 Graph H_i



well-quasi-ordered. This implies, in particular, that the problem of deciding whether a bigenic class is well-quasi-ordered admits an efficient solution.

In the present paper, we show that for more than two forbidden induced subgraphs, the situation becomes uncontrollable in the sense that for each $k \geq 3$, the family $\mathcal{Y}_k$ contains infinitely many minimal classes that are not well-quasi-ordered.

In order to study the problem under these circumstances, we employ the notion of boundary properties of graphs, which was previously applied to some algorithmic graph problems. The importance of this notion is due to the fact that for each k, a class from $\mathcal{Y}_k$ is well-quasi-ordered if and only if it contains none of the boundary properties. We show that the number of boundary properties is generally infinite. On the other hand, we prove that for each fixed k, there is a finite collection of boundary properties that allow to determine whether a class from $\mathcal{Y}_k$ is well-quasi-ordered or not.

The organization of the paper is as follows. In the rest of this section, we introduce some notation and terminology. In Section 2, we show that for each $k \geq 3$, the family $\mathcal{Y}_k$ contains infinitely many minimal classes that are not well-quasi-ordered. In Section 3, we introduce the notion of a boundary class of graphs and prove several results related to this notion. In Section 4, we show that the number of boundary properties is infinite, while in Section 5, we prove that for each fixed k, there is a finite collection of boundary properties that allow us to determine whether a class from $\mathcal{Y}_k$ is wqo or not. Finally, in Section 6, we conclude the paper with a few open questions.

All graphs in this paper are undirected, without loops or multiple edges. As usual, by P_n, C_n and K_n we denote a chordless path, a chordless cycle and a complete graph on n vertices, respectively. Also, $K_{n,m}$ is the complete bipartite graph with parts of size n and m.

A graph G is an *induced subgraph* of a graph H if G can be obtained from H by deletion of vertices. Also, G is a *subgraph* of H if G can be obtained from H by deletion of vertices and/or edges.

To simplify the discussion, we use the term *bad* to refer to classes of graphs that are not well-quasi-ordered by the induced subgraph relation and the term *good* to refer to those classes that are well-quasi-ordered.

2 On the Number of Minimal Bad Classes in $\mathcal{Y}_k$

The following theorem shows that the number of minimal bad classes in $\mathcal{Y}_1$ is finite.

Theorem 1 *A monogenic class is wqo if and only if it contains none of the following as subclasses: $Free(C_3)$, $Free(C_4)$, $Free(C_5)$, $Free(\overline{C}_3)$, $Free(\overline{C}_4)$.*

Proof None of the listed classes is good, since each of them contains either infinitely many cycles or infinitely many complements of cycles. Therefore, if a monogenic class contains one of them, then it is bad.

On the other hand, if a monogenic class $Free(G)$ contains none of the classes $Free(C_4)$, $Free(C_5)$, $Free(\overline{C}_3)$, $Free(\overline{C}_4)$, $Free(\overline{C}_5)$, then G contains none of the graphs C_3, C_4, C_5, $\overline{C}_3$, $\overline{C}_4$, in which case it is not hard to check that G is a P_4 or one of its induced subgraphs, and therefore $Free(G)$ is wqo, according to the result of Damaschke [8]. □

For the family of bigenic classes, i.e. classes defined by two forbidden induced subgraphs, the situation is somewhat similar to Theorem 1, in the sense that there are only finitely many minimal bad classes in this family. This fact was proved in [11]. However, in the case of more than two forbidden induced subgraphs, the situation changes dramatically.

Theorem 2 *For every $k \geq 3$, the set $\mathcal{Y}_k$ contains infinitely many minimal bad classes.*

Proof Let us consider the class $Free(K_{1,3}, C_3, C_t)$ for any $t \geq 4$. This class is bad, since it contains infinitely many cycles. Assume that it is not a minimal bad class in $\mathcal{Y}_3$, and let $X \in \mathcal{Y}_3$ be a proper subclass of $Free(K_{1,3}, C_3, C_t)$ which is bad. Then the set of forbidden induced subgraphs for X contains a graph G which is a proper induced subgraph of one of $K_{1,3}, C_3, C_t$. If G is a proper induced subgraph of $K_{1,3}$ or C_3, then either G is an induced subgraph of P_4, in which case X must be good, or G consists of three isolated vertices, in which case X is good too, because it is finite (every graph in X is $(\overline{K}_3, K_3)$-free and hence has at most five vertices, by Ramsey's theorem).

Assume now that G is a proper induced subgraph of C_t with $t \geq 4$. Then $X \subseteq Free(K_{1,3}, C_3, P_t)$. We claim that:

> *For any natural t, the class $Free(K_{1,3}, C_3, P_t)$ is well-quasi-ordered*. Indeed, since $K_{1,3}$ is forbidden and C_3 is forbidden, the degree of each vertex of any graph in this class is at most 2, and since P_t is forbidden, every connected graph in this class has at most t vertices. It is known (see e.g. [8, 11]) that a class of graphs is well-quasi-ordered if and only if the set of connected graphs in the class is well-quasi-ordered. Since $Free(K_{1,3}, C_3, P_t)$ contains finitely many connected graphs, it is well-quasi-ordered.

Thus, the class $Free(K_{1,3}, C_3, C_t)$ contains no proper subclass from $\mathcal{Y}_3$ which is bad, i.e. $Free(K_{1,3}, C_3, C_t)$ is a minimal bad class for all $t \geq 4$.

For $k > 3$, the proof is similar, i.e. we consider the class $Free(K_{1,3}, C_3, \ldots, C_k, C_t)$ and show that it is a minimal bad class for any $t > k$. □

The finiteness of the number of minimal bad classes in the family $\mathcal{Y}_1 \cup \mathcal{Y}_2$ implies, in particular, that the problem of deciding whether a class in this family is good or bad is polynomial-time solvable. For larger values of k, this approach does not work, as is shown by Theorem 2. In the attempt to overcome this difficulty, we introduce in the next section the notion of a boundary class of graphs, which is a helpful tool for investigating finitely defined classes of graphs.

3 Boundary Properties of Graphs

The notion of boundary properties of graphs was introduced in [1] to study the maximum independent set problem in hereditary classes. Later this notion was applied to some other graph problems of both algorithmic [2, 3, 12] and combinatorial [13] nature. Now we employ it for the problem of determining whether a finitely defined class of graphs is well-quasi-ordered by the induced subgraph relation or not.

Definition 1 We will say that X is a LIMIT CLASS if X is the intersection of any sequence $X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots$ of bad classes.

To illustrate this definition, consider the sequence of graph classes $Free(C_3, \ldots, C_k)$, $k \geq 3$. Clearly, each class in this sequence is bad, because each of them contains infinitely many cycles. The intersection of this sequence is the class of graphs without cycles, i.e. the class of forests. Therefore, the class of forests is a limit class.

If X is the limit class of a sequence $X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots$, we say that the sequence *converges* to X. Observe that we do not require classes in the sequence $X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots$ to be distinct, which means that every bad class is a limit class. The converse is generally not true. Indeed, consider the sequence $Y_3 \supseteq Y_4 \supseteq Y_5 \supseteq \ldots$ with Y_k being the class of $(C_3, C_4, \ldots, C_k, H_1, H_2, \ldots, H_k)$-free graphs (where H_i denotes the graph represented in Fig. 1). Clearly, each class in this sequence is bad, since each of them contains infinitely many cycles. On the other hand, the limit class defined by this sequence contains no cycles and no graphs of the form H_i, i.e. in the limit class, every graph is a forest each connected component of which is a subdivision of a star. It is not difficult to see that this class is monotone. According to [9], every monotone class containing only finitely many cycles and finitely many H-graphs is well-quasi-ordered by the induced subgraph relation. Therefore, this example shows that a limit class is not necessarily bad.

This example also shows that the class of forests is not a minimal limit class, since it contains the smaller limit class, namely, the class of graphs each connected component of which is a subdivision of a star. Is this new limit class minimal? No. To see this, consider the following sequence: $Free(K_{1,3}, C_3, C_4, \ldots, C_k)$, $k \geq 3$. Since each class of this sequence contains infinitely many cycles, each of them is bad. The limit class of this sequence is the class of $K_{1,3}$-free forests, i.e. the class of graphs every connected component of which is a path. Sometimes this class is called the class of *linear forests*. Obviously, linear forests constitute a proper subclass of the limit class from the above example. Is it a minimal limit class? Yes, as we shall see at the end of this section.

The above discussion supposes that there exist limit classes which are minimal in the sense that they do not contain any proper limit subclasses. Moreover, it is correct to assume that every bad class contains a minimal limit class. To show this, we need to prove a number of lemmas about limit classes.

Lemma 1 *A finitely defined class is a limit class if and only if it is bad.*

Proof Every bad class is a limit class by definition. Now let $X = Free(G_1, \ldots, G_k)$ be a limit class and let $X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots$ be a sequence of bad classes converging to X. Obviously, there must exist a number n such that X_n is $(G_1, \ldots, G_k)$-free. But then for each $i \geq n$, we have $X_i = X$ and therefore X is bad. □

Lemma 2 *If a class Y contains a limit class X, then Y also is a limit class.*

Proof Let $X_1 \supseteq X_2 \supseteq X_3 \ldots$ be a sequence of bad classes converging to X. Then the sequence $(X_1 \cup Y) \supseteq (X_2 \cup Y) \supseteq (X_3 \cup Y) \ldots$ consists of bad classes and it converges to Y. □

Lemma 3 *If a sequence $X_1 \supseteq X_2 \supseteq X_3 \ldots$ of limit classes converges to a class X, then X also is a limit class.*

Proof Let $\{G_1, G_2, \ldots\}$ be the set of minimal forbidden induced subgraphs of X. For each natural k, define $X^{(k)}$ to be the class $Free(G_1, \ldots, G_k)$. Clearly, for every k, there must exist an n such that X_n does not contain $G_1, \ldots, G_k$, implying that $X_n \subseteq X^{(k)}$. Therefore, by Lemma 2, $X^{(k)}$ is a limit class, and by Lemma 1, $X^{(k)}$ is bad. This is true for all natural k, and therefore, $X^{(1)} \supseteq X^{(2)} \supseteq X^{(3)} \ldots$ is a sequence of bad classes converging to X, i.e. X is a limit class. □

Lemma 4 *Every bad class X contains a minimal limit class Y.*

Proof Let X be a bad class. To reveal a minimal limit class contained in X, let us fix an arbitrary enumeration $\mathcal{L}$ of the set of all graphs and let us define a sequence $X_1 \supseteq X_2 \supseteq \ldots$ of graph classes as follows. We define X_1 to be equal X. For $i > 1$, let G be the first graph in the enumeration $\mathcal{L}$ such that G belongs to X_{i-1} and $X_{i-1} \cap Free(G)$ is a limit class. If there is no such graph G, we define $X_i := X_{i-1}$. Otherwise, $X_i := X_{i-1} \cap Free(G)$.

Denote by Y the intersection of classes $X_1 \supseteq X_2 \supseteq X_3 \ldots$. Clearly, $Y \subseteq X$. By Lemma 3, Y is a limit class. Let us show that Y is a minimal limit class contained in X. By contradiction, assume there exists a limit class Z which is properly contained in Y. Let H be a graph in Y which does not belong to Z. Then $Z \subseteq Y \cap Free(H) \subseteq X_k \cap Free(H)$ for each k. Therefore, by Lemma 2, $X_k \cap Free(H)$ is a limit class for each k. For some k, the graph H becomes the first graph in the enumeration $\mathcal{L}$ such that $X_k \cap Free(H)$ is a limit class. But then $X_{k+1} := X_k \cap Free(H)$, and H belongs to no class X_i with $i > k$, which contradicts the fact that H belongs to Y. □

Lemma 4 motivates the following key definition.

Definition 2 A minimal limit class will be called a BOUNDARY CLASS.

The importance of this notion is due to the following theorem.

Theorem 3 *A finitely defined class is good if and only if it contains no boundary class.*

Proof From Lemma 4, we know that every bad class contains a boundary class. To prove the converse, consider a finitely defined class X containing a boundary class. Then, by Lemma 2, X is a limit class, and therefore, by Lemma 1, X is bad. □

A helpful minimality criterion is given by the following lemma.

Lemma 5 *A limit class $X = Free(M)$ is minimal (i.e. boundary) if and only if for every graph $G \in X$ there is a finite set $T \subseteq M$ such that $Free(\{G\} \cup T)$ is good.*

Proof Suppose X is a boundary class, and assume for contradiction that there is a graph $G \in X$ such that for every finite set $T \subseteq M$, the class $Free(\{G\} \cup T)$ is bad. Let $M := \{m_1, m_2, \ldots\}$ and $Z_i := Free(G, m_1, m_2, \ldots, m_i)$. Then, according to our assumption, Z_i is bad for each i and therefore $Z := \cap_i Z_i$ is a limit class. It contains no element from M and it does not contain G. Therefore, it is a proper subset of X, contradicting the minimality of X.

Conversely, assume that for every graph $G \in X$, there is a finite set $T \subseteq M$ such that $Free(\{G\} \cup T)$ is good, and suppose for contradiction that there exists a limit class Z which is properly contained in X. Since Z is a limit class, there exists a sequence $Z_1 \supseteq Z_2 \supseteq \ldots$ of bad classes converging to Z. Pick any graph $G \in X \setminus Z$ and a finite set $T \subseteq M$ such that $Free(\{G\} \cup T)$ is good. Then there must exist an n such that Z_n is $(\{G\} \cup T)$-free, in which case Z_n is good. This contradiction finishes the proof. □

Theorem 4 *The class of linear forests is a boundary class.*

Proof Let F be a linear forest. Without loss of generality, we may assume that $F = P_t$, since every linear forest is an induced subgraph of P_t for some value of t. Clearly, the class $Free(P_t, K_{1,3}, C_3, C_4, \ldots, C_t)$ is a subclass of linear forests, and obviously, the class of linear forests is well-quasi-ordered. Therefore, by Lemma 5, the class of linear forests is a minimal limit class. □

In the proof of Theorem 4 we observed that the class of linear forests is well-quasi-ordered by the induced subgraph relation. This is no wonder, because any boundary class must be good.

Lemma 6 *Every boundary class is well-quasi-ordered.*

Proof If a boundary class X is bad, it must contain an infinite antichain $G_1, G_2, \ldots$. Then for any G_i, the class $Free(G_i) \cap X$ is a proper limit subclass of X, contradicting the minimality of X. □

4 On the Number of Boundary Classes

The previous section not only introduces the notion of a boundary class, but also reveals one them, i.e. the class of linear forests. Throughout the paper we denote this class by $\mathcal{F}$. Are there other boundary classes? Yes, because for any boundary class X, the class of complements of graphs in X is also boundary. Therefore, the complements of linear forests form a boundary class; we denote this class by $\overline{\mathcal{F}}$. As we shall see later, there are many other boundary classes. Moreover, in this section we show that the family of boundary classes is infinite. To this end, for any natural number $k \geq 1$, we define the following graph operation. Given a graph G, we subdivide each edge of G by exactly k 'new' vertices and then create a clique on the set of 'old' vertices. Let us denote the graph obtained in this way by $G^{(k)}$. Also, for an arbitrary hereditary class X we define $X^{(k)}$ to be the class of all induced subgraphs of the graphs $G^{(k)}$ formed from graphs $G \in X$.

It is not difficult to see that classes $\mathcal{F}^{(k)}$ for various values of k are pairwise incomparable, i.e. none of them is a subclass of another. We will show that for any $k \geq 3$, the class $\mathcal{F}^{(k)}$ is a boundary class. To this end, let us prove a few auxiliary results.

Throughout the section we denote by $\mathcal{D}$ the class of graphs of vertex degree at most 2. Clearly, this is a hereditary class. The set of minimal forbidden induced subgraphs for this class consists of four graphs (each of them has a vertex of degree 3 and the three neighbours of that vertex induce all possible graphs on three vertices). We will show that for the class $\mathcal{D}^{(k)}$ the situation is similar in the sense that the set of minimal forbidden induced subgraphs for it is finite, regardless of the value of k.

Lemma 7 *For each $k \geq 3$, the set of minimal forbidden induced subgraphs for the class $\mathcal{D}^{(k)}$ is finite.*

Proof First of all, let us observe that the class $\mathcal{D}^{(k)}$ is a subclass of the class $\mathcal{M}^{[k]}$ of graphs whose vertices can be partitioned into a clique A and a set B of vertices inducing a P_{k+1}-free linear forest (i.e. a graph every connected component of which is a path on at most k vertices). $\mathcal{M}^{[k]}$ is a wider class than $\mathcal{D}^{(k)}$, since by definition we do not specify what is happening between the two parts A and B for graphs in $\mathcal{M}^{[k]}$, while for graphs in $\mathcal{D}^{(k)}$ there are severe restrictions on the edges between A and B (these restrictions are described below). Therefore, the set of minimal forbidden induced subgraphs for $\mathcal{D}^{(k)}$ is a subset of the set $M \cup D$, where M is the set of minimal forbidden induced subgraphs for $\mathcal{M}^{[k]}$ and D is the set of graphs from $\mathcal{M}^{[k]}$ that restrict the behavior of edges between A and B. We will show that both sets M and D are finite.

For the finiteness of M we refer the reader to [17], where the following result was proved: Let P and Q be two hereditary classes of graphs such that both P and Q are defined by finitely many forbidden induced subgraphs, and there is a constant bounding the size of a maximum clique for all graphs in P and the size of a maximum independent set for all graphs in Q. Then the class of all graphs whose vertices can be partitioned into a set inducing a graph from P and a set inducing a graph from Q has a finite characterization in terms of forbidden induced subgraphs. For the class $\mathcal{M}^{[k]}$, we have $Q = Free(\overline{K}_2)$ is the class of complete graphs, in which case the the size of a maximum independent set is 1, and $P = Free(K_{1,3}, C_3, \ldots, C_{k+1}, P_{k+1})$, in which case the size of a maximum clique is at most 2. Therefore, M is a finite set.

In order to show that the size of D is bounded, let us describe the restrictions on the behavior of edges connecting vertices of A to the vertices of B in graphs in the class $\mathcal{D}^{(k)}$.

(1) Every vertex of B has at most one neighbour in A;
(2) Only an end-vertex of a path in B can have a neighbour in A;
(3) If both end-vertices of a path in B have neighbours in A, then these neighbours are different and the path has exactly k vertices;
(4) Let P and P' be two paths in B such that each contains exactly k vertices and both end-vertices in both paths have neighbours in A. Then the pair of neighbours of P in A and the pair of neighbours of P' in A are different, i.e. they share at most one vertex.
(5) Every vertex of A has at most two neighbours in B.

It is not difficult to see that a graph $G \in \mathcal{M}^{[k]}$ belongs to $\mathcal{D}^{(k)}$ if an only if G satisfies restrictions (1)–(5). The first four restrictions are common for any graph $G^{(k)}$ (or an induced subgraph of $G^{(k)}$) and they completely specify the behavior of the edges connecting 'new' vertices to 'old' ones. Restriction (5) is specific for graphs in $\mathcal{D}^{(k)}$.

Now we translate restrictions (1)–(5) to the language of forbidden induced subgraphs. We denote by Φ and T the two graphs represented in Fig. 2. Also, C''_{k+2} stands for the graph consisting of two cycles C_{k+2} sharing an edge, and *diamond* stands for K_4 without an edge. It is a routine task to verify that the graphs *diamond*, $K_{1,4}, C_4, \ldots, C_{k+1}, C''_{k+2}, \Phi, T$ belong to $\mathcal{M}^{[k]}$ but do not belong to $\mathcal{D}^{(k)}$ (for $k \geq 3$). Moreover, they are minimal graphs that do not belong to $\mathcal{D}^{(k)}$.

Now let G be a graph in $\mathcal{M}^{[k]}$ ($k \geq 3$) which is free of *diamond*, $K_{1,4}, C_4, \ldots, C_{k+1}, C''_{k+2}, \Phi, T$. We may assume that

- every vertex of B has at least one non-neighbour in A, since otherwise this vertex can be moved to A,
- A contains at least three vertices, because there are finitely many connected $K_{1,4}$-free graphs in $\mathcal{M}^{[k]}$ with $|A| \leq 2$, and a minimal graph in $\mathcal{M}^{[k]}$ which does not belong to $\mathcal{D}^{(k)}$ must be connected.

Under these assumptions, the *diamond*-freeness of G guarantees that (1) is satisfied.

Suppose that the part B of G contains a path in which a non-end-vertex v has a neighbour x in A. Since $|A| \geq 3$, there must exist two other vertices y, z in A, and these vertices must be non-adjacent to v, by (1). Since v is a non-end-vertex of a path in B, it must have two distinct neighbours in the path, say u and w, with u being non-adjacent to w. By (1), each of u and w has at most one neighbour among x, y, z. If one of them is adjacent to x, then the forbidden graph Φ arises. If u or w is adjacent to y or z, then a C_4 arises, and if the vertices u, w have no neighbours among x, y, z, then the graph T arises. This discussion shows that restriction (2) is satisfied.

Assume both end-vertices of a path P in B have neighbours in A. Together with (2) this gives rise to a chordless cycle C consisting of P and its neighbours in A. If P has less than k vertices, then C is of size at most $k+1$, which is forbidden. If P has exactly k vertices and just one neighbour in A, then the size of C is $k+1$, which is impossible. Therefore, P has k vertices and two neighbours in A. Therefore, restriction (3) is satisfied.

Let P and P' be two paths in B such that each contains exactly k vertices and both end-vertices in both paths have neighbours in A. If the neighbours of P in A coincide with the neighbours of P' in A, then G contains the forbidden graph C''_{k+2}. Therefore, restriction (4) is satisfied.

Finally, if a vertex x of A has at least three neighbours in B, say u, v, w, then from the previous discussion, we know that u, v, w belong to different connected components of B, and therefore, x, u, v, w together with any vertex $y \in A$ different from x induce a $K_{1,4}$. This shows that restriction (5) is satisfied.

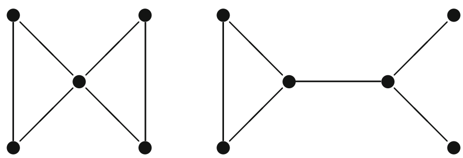


Fig. 2 The graphs Φ (*left*) and T (*right*)

From the above discussion, we conclude that D must be finite, which completes the proof of the lemma. □

Lemma 8 *Let G be a graph with at least four vertices, and let $G^{(k)}$ be an induced subgraph of $H^{(k)}$. Then G is a subgraph of H.*

Proof Observe that in the graphs $G^{(k)}$ and $H^{(k)}$, every 'new' vertex has degree 2, while every 'old' vertex has degree at least 3. Therefore, if $G^{(k)}$ is an induced subgraph of $H^{(k)}$, then 'new' vertices of $G^{(k)}$ are mapped to 'new' vertices of $H^{(k)}$ and 'old' vertices of $G^{(k)}$ are mapped to 'old' vertices of $H^{(k)}$. Let U be the set of vertices whose deletion from $H^{(k)}$ results in $G^{(k)}$. If U contains a 'new' vertex v subdividing an edge e of H, then U must contain all new vertices subdividing e, since otherwise a pendant vertex appears, which is not possible for $G^{(k)}$. Obviously, deletion of all new vertices subdividing e from $H^{(k)}$ is equivalent to deletion of the edge e from H. Also, if U contains an 'old' vertex v of H, then U must contain all new vertices subdividing all edges incident to v (in H), since otherwise again a pendent vertex appears. Clearly, deleting from $H^{(k)}$ an 'old' vertex v together with all new vertices subdividing all edges incident to v (in H) is equivalent to deleting from H vertex v together with all edges incident to v. Therefore, if $G^{(k)}$ is an induced subgraph of $H^{(k)}$, then G is a subgraph of H. □

Theorem 5 *For any natural number $k \geq 3$, the class $\mathcal{F}^{(k)}$ is a boundary class.*

Proof As before, $\mathcal{D}$ is the class of graphs of vertex degree at most 2. First, we show that $\mathcal{F}^{(k)}$ is a limit class. To this end, define the sequence $\mathcal{F}_3^{(k)}, \mathcal{F}_4^{(k)}, \ldots$ of graph classes by $\mathcal{F}_i^{(k)} := Free(C_3^{(k)}, C_4^{(k)}, \ldots, C_i^{(k)}) \cap \mathcal{D}^{(k)}$. It is not difficult to see that the sequence $\mathcal{F}_3^{(k)}, \mathcal{F}_4^{(k)}, \ldots$ converges to $\mathcal{F}^{(k)}$. Also, for each i, the class $\mathcal{F}_i^{(k)}$ contains an infinite antichain, namely $C_{i+1}^{(k)}, C_{i+2}^{(k)}, \ldots$, which follows from Lemma 8 and the obvious observation that cycles form an antichain with respect to the subgraph relation.

The proof of minimality of $\mathcal{F}^{(k)}$ is similar to that of Theorem 4. We consider a graph G in $\mathcal{F}^{(k)}$ and without loss of generality assume that $G = P_t^{(k)}$, since every graph in $\mathcal{F}^{(k)}$ is an induced subgraph of $P_t^{(k)}$ for some t. Then the class $Free(P_t^{(k)}, C_3^{(k)}, C_4^{(k)}, \ldots, C_t^{(k)}) \cap \mathcal{D}^{(k)}$ is a subclass of $\mathcal{F}^{(k)}$. By Lemma 7, this class is finitely defined, and since $\mathcal{F}^{(k)}$ is well-quasi-ordered, this class is well-quasi-ordered too. Therefore, by Lemma 5, $\mathcal{F}^{(k)}$ is a minimal limit class. □

5 Boundary Properties for Finitely Defined Classes

In the previous section we showed that there are infinitely many boundary properties. However, we need to know only two of them to determine well-quasi-orderability of monogenic classes. These are the class $\mathcal{F}$ of linear forests and the class $\overline{\mathcal{F}}$ of the complements of linear forests.

Theorem 6 *A monogenic class of graphs is wqo if and only if it contains neither $\mathcal{F}$ nor $\overline{\mathcal{F}}$.*

Proof Let $X = Free(G)$ be a monogenic class of graphs. If X contains $\mathcal{F}$ or $\overline{\mathcal{F}}$, then X is not wqo, by Theorem 3. Assume now that X contains neither $\mathcal{F}$ nor $\overline{\mathcal{F}}$, i.e. G belongs both to $\mathcal{F}$ and to $\overline{\mathcal{F}}$. In this case, G does not contain C_3, C_4, C_5, $\overline{C}_3$, $\overline{C}_4$, and thus X does not contain $Free(C_3)$, $Free(C_4)$, $Free(C_5)$, $Free(\overline{C}_3)$, $Free(\overline{C}_4)$. Therefore, X is wqo by Theorem 1. □

In what follows, we show that for larger values of k, the situation is similar to the monogenic case, in the sense that for any $k \geq 1$, the set of boundary classes essential for determining well-quasi-orderability of classes in $\mathcal{Y}_k$ is finite.

We start from stating Higman's Lemma [10] in the form provided in [11]. For an arbitrary set M, denote by M^* the set of all finite sequences of elements of M. If $\leq$ is a partial order on M, the elements of M^* can be partially ordered by the following relation: $(a_1, \ldots, a_m) \preceq (b_1, \ldots, b_n)$ if and only if there is an order-preserving injection $f : \{a_1, \ldots, a_m\} \to \{b_1, \ldots, b_n\}$ with $a_i \leq f(a_i)$ for each $i = 1, \ldots, m$. The Higman's lemma states:

Lemma 9 *If $(M, \leq)$ is a wqo, then $(M^*, \preceq)$ is a wqo.*

The following lemma is a corollary of Higman's lemma.

Lemma 10 *Let A be a class of graphs which is wqo under the induced subgraph relation. Let $\mathcal{B}$ be a family of graph classes such that each $B \in \mathcal{B}$ is obtained by forbidding a finite set of graphs from A. Then $\mathcal{B}$ is wqo under the subclass inclusion.*

Proof Let $B_1 = Free(G_1, \ldots, G_q)$ and $B_2 = Free(H_1, \ldots, H_r)$ be two classes in $\mathcal{B}$. Then it is not hard to see that $B_1 \subseteq B_2$ if and only if there is a mapping f from $\{H_1, \ldots, H_r\}$ to $\{G_1, \ldots, G_q\}$ such that $f(H_i)$ is an induced subgraph of H_i. Let us relax the partial order $\subseteq$ by making B_1 and B_2 incomparable if f is not injective. Under this new partial order $\mathcal{B}$ is a wqo according to Higman's Lemma. Therefore, the subclass inclusion $\subseteq$ is a wqo too, since any antichain in this partial order is also an antichain in the relaxed partial order. □

Theorem 7 *For any natural k, there is a finite set $\mathcal{B}_k$ of boundary classes such that a class $X = Free(G_1, \ldots, G_k)$ is wqo if and only if it contains none of the boundary classes from the set $\mathcal{B}_k$.*

Proof We prove the theorem by induction on k. For $k = 1$, the result follows from Theorem 6.

To make the inductive step, we assume that the theorem is true for $k - 1$. Let $\mathcal{C}$ be the set of graph classes $Free(G_1, \ldots, G_k)$ such that

- each of the graphs $G_1, \ldots, G_k$ belongs to at least one of the boundary classes in $\mathcal{B}_{k-1}$,
- $Free(G_1, \ldots, G_k)$ is not wqo.

Since the set $\mathcal{B}_{k-1}$ is finite and each class in this set is well-quasi-ordered (Lemma 6), their union $\bigcup_{B \in \mathcal{B}} B$ is well-quasi-ordered too. This implies by Lemma 10 that $\mathcal{C}$ is well-quasi-ordered by the subclass inclusion, and thus the set of minimal classes in $\mathcal{C}$ is finite; we denote this set by $\mathcal{C}^*$.

For each class in $\mathcal{C}^*$, we arbitrarily choose a boundary class contained in it (such a boundary class must exist by Theorem 3), and denote the set of boundary classes chosen in this way by $\mathcal{B}$. Since $\mathcal{C}^*$ is finite, $\mathcal{B}$ is finite too. Now we claim that the theorem holds with $\mathcal{B}_k = \mathcal{B}_{k-1} \cup \mathcal{B}$. To see this, consider a class of graphs $X = Free(G_1, \dots, G_k)$. If it is wqo, then it does not contain any boundary class from $\mathcal{B}_k$, since it contains no boundary classes at all, by Theorem 3.

Suppose now that $X = Free(G_1, \dots, G_k)$ is not wqo. If each of the graphs $G_1, \dots, G_k$ belongs to one of the boundary classes in $\mathcal{B}_{k-1}$, then it must contain a class from $\mathcal{C}^*$ by definition of $\mathcal{C}^*$ and therefore it must contain a boundary class from $\mathcal{B} \subseteq \mathcal{B}_k$. If one of the forbidden graphs, say G_i, does not belong to any class in $\mathcal{B}_{k-1}$, then we consider the class $Free(G_1, \dots, G_{i-1}, G_{i+1}, \dots, G_k)$. By induction, it contains a boundary class from $\mathcal{B}_{k-1}$. But then X contains the same boundary class. □

6 Concluding Remarks and Open Problems

In this paper, we introduced the notion of boundary properties of the family of well-quasi-ordered classes of graphs. This is a helpful tool to study well-quasi-orderability of finitely defined classes. We proved that for each k, there is a finite collection of boundary properties that allow us to determine whether a class of graphs defined by k forbidden induced subgraphs is wqo or not. This conclusion is in a sharp contrast with the fact that the number of boundary properties is generally infinite, which was also proved in this paper. The proof of this fact is obtained with the help of a simple graph operation applied to linear forests. More graph operations (complementation, "bipartite" complementation, etc.) can produce more boundary classes related to the class of linear forests. However, the authors are not aware of any boundary properties that are not derived from linear forests. Identifying such properties is a natural open question.

To formulate one more open problem related to this topic, let us extend the induced subgraph relation to a more general notion known as the labelled-induced subgraph relation. Assume $(W, \leq)$ is an arbitrary well-quasi-order. We call G a *labelled graph* if each vertex $v \in V(G)$ is equipped with an element $l(v) \in W$ (the label of v), and we say that a graph G is a *labelled-induced subgraph* of H if G is isomorphic to an induced subgraph of H and the isomorphism maps each vertex $v \in G$ to a vertex $w \in H$ with $l(v) \leq l(w)$.

It is interesting to observe that the class of linear forests, although well-quasi-ordered by the induced subgraph relation, is not well-quasi-ordered by the labelled-induced subgraph relation. On the other hand, all finitely defined classes which are known to be well-quasi-ordered by induced subgraphs also are well-quasi-ordered by the labelled-induced subgraph relation. This observation motivates the following conjecture.

Conjecture *Let X be a hereditary class which is well-quasi-ordered by the induced subgraph relation. Then X is well-quasi-ordered by the labelled-induced subgraph relation if and only if the set of minimal forbidden induced subgraphs for X is finite.*

Finally, we observe that the notion of boundary properties can also be applied to the study of other partial orders that are generally not well-quasi-orders. For instance, recently there was a considerable attention to the pattern containment relation on permutations (see e.g. [4, 5, 14, 16]). The problem of deciding whether a permutation class given by a finite set of "forbidden" permutations is wqo or not was proposed in [6], and we believe that the notion of boundary properties can be helpful in finding an answer to this question.

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